

MTH U341, Practice Final'08

Problem 1. Find the limit $\lim_{(x,y) \rightarrow (0,0)} \frac{x}{\sqrt{x^2+y^2}}$ if it exists or show that the limit does not exist. **Answer:** no limit

Problem 2. Find the tangent plane to the surface

$$\vec{r}(u, v) = \langle u \sin(v), v \cos(u), u^2 + v^2 \rangle$$

at the point $(u, v) = (\pi/4, \pi/4)$.

$$\text{Answer: } \left(-\frac{\pi^2}{8\sqrt{2}} - \frac{\pi}{2\sqrt{2}}\right)\left(x - \frac{\pi}{4\sqrt{2}}\right) + \left(\frac{\pi^2}{8\sqrt{2}} - \frac{\pi}{2\sqrt{2}}\right)\left(y - \frac{\pi}{4\sqrt{2}}\right) + \left(\frac{1}{2} + \frac{\pi^2}{32}\right)\left(z - \frac{\pi^2}{8}\right) = 0$$

Problem 3. Find the linear approximation of the function $f(x, y) = x^2y + \sqrt{xy}$ at the point $(1, 2)$ and use it to approximate $f(1.5, 1.5)$.

$$\text{Answer: } L(x, y) = (2 + \sqrt{2}) + \left(4 + \frac{1}{\sqrt{2}}\right)(x - 1) + \left(1 + \frac{1}{2\sqrt{2}}\right)(y - 2)$$

$$f(1.5, 1.5) \sim (2 + \sqrt{2}) + \left(4 + \frac{1}{\sqrt{2}}\right)0.5 + \left(1 + \frac{1}{2\sqrt{2}}\right)(-0.5)$$

Problem 4.

- (1) Find the rate of change of $f(x, y) = ye^{x+y}$ at the point $(-1, 1)$ in the direction of the vector $\langle 1, 2 \rangle$;

$$\text{Answer: } \sqrt{5}$$

- (2) Find the maximal rate of change of $f(x, y) = ye^{x+y}$ at the point $(-1, 1)$ and the direction in which it occurs.

$$\text{Answer: } \text{maximal rate } \sqrt{5}, \text{ direction of the maximal rate } \langle 1/\sqrt{5}, 2/\sqrt{5} \rangle.$$

Problem 5. Find the local maximums, minimums, and the saddle points (if any) of the function $f(x, y) = x^3 + y^3 - xy - 2$.

$$\text{Answer: } (0, 0) - \text{saddle point, } (1/3, 1/3) - \text{local minimum}$$

Problem 6. Find the minimum and the maximum of the function $f(x, y) = x + y^2$ subject to the constraint $x^2 + y^2 = 1$.

$$\text{Answer: } \max f = 5/4, \min f = -1$$

Problem 7. Find the volume under the surface $z = xy$ over the region bounded by $y = 2x$, $y = \frac{1}{2}x$, and $x = 1$.

$$\text{Answer: } \frac{15}{32}$$

Problem 8. Evaluate the double integral $\iint_D y \, dA$ where D is the region bounded by $y = x^2$ and the line $x + y = 2$.

$$\text{Answer: } \frac{36}{5}$$

Problem 9. Find the mass of a lamina with density $\rho(x, y) = \sqrt{x^2 + y^2}$ that lies in the first quadrant and is bounded by the circle $x^2 + y^2 = 4$ and the coordinate lines $x = 0$ and $y = 0$.

Answer: $\frac{4}{3}\pi$

Problem 10. Evaluate the integral $\iiint_E x \, dV$ where E is the solid tetrahedron bounded by the coordinate planes $x = 0$, $y = 0$, $z = 0$, and the plane $x + y + 2z = 3$.

Answer: $\frac{27}{16}$

Problem 11. Evaluate the integral $\iiint_E y \, dV$ where E lies above the plane $z = 0$, below the plane $z = y$, and inside the cylinder $x^2 + y^2 = 4$.

Answer: 2π

Problem 12. Evaluate the integral $\iiint_E y\sqrt{x^2 + y^2 + z^2} \, dV$ where E is bounded by the xz -plane and the two hemispheres $\{x^2 + y^2 + z^2 = 1, y \geq 0\}$ and $\{x^2 + y^2 + z^2 = 4, y \geq 0\}$.

Answer: $\frac{31}{5}\pi$

Problem 13. Let C be the straight-line path from the point $A = (2, 0)$ to the point $B = (0, 4)$. Let \vec{F} be the force field $\vec{F}(x, y) = \langle x^2 + 2y, x - y^2 \rangle$. Find the work done by \vec{F} on a particle moving along C .

Answer: -28

Problem 14. Consider the vector field, $\vec{F}(x, y) = \langle \sqrt{y}, \frac{x}{2\sqrt{y}} + 2y \rangle$, $y > 0$.

- (1) Determine if \vec{F} is conservative; **Answer:** conservative
- (2) Find a function $f(x, y)$ such that $\vec{F} = \nabla f$; **Answer:** $f = x\sqrt{y} + y^2$
- (3) Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ where C is the straight interval connecting $A(1, 1)$ with $B(3, 2)$. **Answer:** $2 + 3\sqrt{2}$

Problem 15. Use Green's theorem to evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle e^x + y^2, e^{y^2} \rangle$ where C is the rectangle with corners $(2, 2)$, $(-2, 2)$, $(2, 0)$ and $(-2, 0)$, oriented counterclockwise.

Answer: -16

Problem 16. Let S be the part of the paraboloid $z = 1 + x^2 + y^2$ lying above the rectangle $0 \leq x \leq 1$, $-1 \leq y \leq 0$ and oriented by the upward normal. Compute the surface integral $\iint_S \vec{F} \cdot \vec{n} \, dS$ where $\vec{F}(x, y, z) = \langle xz, xy, yz \rangle$.

Answer: $-\frac{457}{180}$

Problem 17. Use the Divergence theorem to evaluate the integral $\iint_S \vec{F} \cdot \vec{n} \, dS$

where $\vec{F} = \langle x + \cos z, y, 2z + e^x y \rangle$ and S is the surface of the solid bounded by the planes $z = 0$, $z = 3$, and the cylinder $x^2 + y^2 = 9$ (oriented by the outward normal).

Answer: 108π

Problem 18. Use Stokes' Theorem to calculate $\int_C \vec{F} \cdot d\vec{r}$, where

$$\vec{F} = x^2z \vec{i} + xy^2 \vec{j} + z^2 \vec{k}$$

and C is the curve of intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 9$. (C is oriented counterclockwise when viewed from above.)

Answer: $\frac{81\pi}{2}$