MTH U341 QUIZ 1 Name: ____________________
Fall 2008

Show all your work in the space provided. Use the back of the page if necessary. No credit for unjustified answers.

1.(24 points) Let ℓ be the line passing through the points $(1, 2, -6)$ and $(-1, 3, -4)$.
a. Find the parametric equations and symmetric equations for ℓ.

The vector $\vec{v} = \langle -1, 3, -4\rangle - \langle 1, 2, -6\rangle = \langle -2, 1, 2\rangle$ is parallel to ℓ. The vector equation is thus

$$\langle x, y, z\rangle = \langle 1, 2, -6\rangle + t\vec{v} = \langle 1, 2, -6\rangle + t\langle -2, 1, 2\rangle = \langle 1 - 2t, 2 + t, -6 + 2t\rangle$$

giving

Parametric equations: $x = 1 - 2t$, $y = 2 + t$, $z = -6 + 2t$.

Symmetric equations

$$\frac{x-1}{-2} = \frac{y-2}{1} = \frac{z+6}{2}$$

b. Find the point of intersection of ℓ with the x-y plane.

Use the parametric equations. The x-y plane is $z = 0$, so the intersection is at $-6 + 2t = 0$, or $t = 3$. This gives $x = -5$, $y = 5$, so the intersection is at the point $(-5, 5, 0)$.

c. Find the angle of ℓ with the vector $\langle -1, -2, 2\rangle$ (you may use inverse trig functions to express your answer).

This is the same as the angle θ that the parallel vector $\vec{v}$ makes with $\vec{w} = \langle -1, -2, 2\rangle$. We have

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|}.$$

Computing gives

$$\begin{aligned}
\vec{v}\cdot\vec{w} &= \langle -2, 1, 2\rangle \cdot \langle -1, -2, 2\rangle = 2 - 2 + 4 = 4\\
|\vec{v}| &= |\langle -2, 1, 2\rangle| = \sqrt{(-2)^2 + 1^2 + 2^2} = 3\\
|\vec{w}| &= |\langle -1, -2, 2\rangle| = \sqrt{1^2 + (-2)^2 + 2^2} = 3\\
\cos\theta &= \frac{4}{3\cdot 3} = \frac{4}{9}
\end{aligned}$$

So $\theta = \cos^{-1}(4/9)$ (or $\cos^{-1}(-4/9)$ if you used the parallel vector pointing in the opposite direction).

2.(20 points) a. Find the equation of the plane through the points $P = (1, 0, -1)$, $Q = (0, 3, 2)$ and $R = (2, 1, 0)$.

Find a normal vector to the plane by taking $\vec{n} = \vec{PQ} \times \vec{PR}$.

$$\vec{PQ} = \langle -1, 3, 3\rangle,\ \vec{PR} = \langle 1, 1, 1\rangle$$

so

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 3 & 3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 3 \\ 1 & 1 \end{vmatrix} \vec{i} - \begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix} \vec{j} + \begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix} \vec{k}$$

$$= 0\vec{i} + 4\vec{j} - 4\vec{k} = \langle 0, -4, 4 \rangle$$

Using the point-normal form gives the equation

$$-4y + 4(z+1) = 0$$

or $y - z = 1$.

b. Let ℓ be the line given by the equations

$$x - 1 = \frac{1}{4}(y-2) = \frac{1}{3}(z+3).$$

Find the equation of the plane passing through $(1, 3, 2)$ and perpendicular to ℓ.

By looking at the symmetric equations for ℓ, we see that $\vec{v} = \langle 1, 4, 3 \rangle$ is a parallel vector. If you don't like this, you need to find two points P, Q on ℓ and take $\vec{v} = \vec{PQ}$. There is one obvious points, where each equation is 0: $P = (1, 2, -3)$. If each equation is say 1, you get the second point $Q = (2, 6, 0)$, giving the same $\vec{v}$ as before.

Now use the point-normal form:

$$1(x-1) + 4(y-3) + 3(z-2) = 0$$

or $x + 4y + 3z = 19$.

3.(20 points) Let P be the parallelogram with vertices A, B, C and D, where $A = (1, 1)$, $B = (0, 3)$, $C = (4, 6)$.
a. Find the coordinates of D.

Actually, there are three correct answers here, depending on how you complete the parallelogram. If the parallelogram has sides AB, BD DC and CA, then AB is parallel to CD, so $\vec{AB} = \vec{CD}$, giving

$$\vec{OD} = \vec{OC} + \vec{CD} = \vec{OC} + \vec{AB} = \langle 4, 6 \rangle + \langle -1, 2 \rangle = \langle 3, 8 \rangle$$

so $D = (3, 8)$. If you make the parallelogram with BC, CD, DA and AB, then $\vec{AD} = \vec{BC}$, so

$$\vec{OD} = \vec{OA} + \vec{AD} = \vec{OA} + \vec{BC} = \langle 1, 1 \rangle + \langle 4, 3 \rangle = \langle 5, 4 \rangle.$$

and $D = (-5, 4)$. Making the parallelogram with sides AC, CB, BD and DA gives $\vec{CB} = \vec{AD}$, so

$$\vec{OD} = \vec{OA} + \vec{AD} = \vec{OA} + \vec{CB} = \langle 1, 1 \rangle + \langle -4, -3 \rangle = \langle -3, -2 \rangle.$$

and $D = (-3, -2)$.
b. Find the area of P

We use the first case $D = (3, 8)$, the answer is the same in all three cases.

$$\text{Area} = \left| \vec{AB} \times \vec{AC} \right|.$$

We compute

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 0 \\ 3 & 5 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 0 \\ 5 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} -1 & 0 \\ 3 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} \vec{k}$$

$$= 0\vec{i} + 0\vec{j} - 11\vec{k} = \langle 0, 0, -11 \rangle$$

so the area $= 11$.

4.(16 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} x^2 - y^2 + 5 & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

a. Compute the limits:

$\lim_{(x,y)\to(1,1)} f(x,y) = 5$; $\lim_{(x,y)\to(0,0)} f(x,y) = 5$.

The function $g(x,y) = x^2 - y^2 + 5$ is a polynomial, so g is continuous, and thus the $\lim_{(x,y)\to(a,b)} g(x,y) = g(a,b)$ for all $(a,b) \in \mathbb{R}^2$, Taking $(a,b) = (1,1)$ or $(0,0)$ gives the answer.

b. Is f continuous at $(1,1)$? at $(0,0)$? Explain.

At $(1,1)$: $f(1,1) = 5 = \lim_{(x,y)\to(1,1)} f(x,y)$, so by definition of continuity, f is continuous at $(1,1)$.

At $(0,0)$: $f(0,0) = 0 \neq \lim_{(x,y)\to(0,0)} f(x,y)$, so by definition of continuity, f is not continuous at $(0,0)$.

5.(20 points) Sketch the contours (level curves) of the function $f(x,y) = 4x^2 + y^2 - 1$ at level -2, -1, 0, and 3. Label each contour with its level.

The equation for the contour at level k is $f(x,y) = k$, so we have the equations

$$\begin{aligned} \text{level -2} : 4x^2 + y^2 - 1 = -2 &\implies 4x^2 + y^2 = -1, \text{ empty} \\ \text{level -1} : 4x^2 + y^2 - 1 = -1 &\implies 4x^2 + y^2 = 0, \quad \text{the point } (0,0) \\ \text{level 0} : 4x^2 + y^2 - 1 = 0 &\implies 4x^2 + y^2 = 1, \quad \text{an ellipse} \\ \text{level 3} : 4x^2 + y^2 - 1 = 3 &\implies 4x^2 + y^2 = 4, \quad \text{an ellipse.} \end{aligned}$$