MTH U341 QUIZ 2 Name: ____________________
Fall 2008

Show all your work in the space provided. No credit for unjustified answers.

1. Let $f = f(x, y)$, where x and y are the functions of t: $x = 3t^2 - 2t$, $y = 6/t$. Suppose that

$$f(8,3) = 5,\ f_x(8,3) = 7,\ f_y(8,3) = -2,\ f(2,3) = 4,\ f_x(2,3) = -1,\ f_y(2,3) = 9.$$

a. (8 points) Compute df/dt at $t = 2$.

Ans. At $t = 2$,

$$x = 3 \cdot 2^2 - 2 \cdot 2 = 8; \quad y = 6/2 = 3.$$

Also $x'(t) = 6t - 2$, $y'(t) = -6/t^2$, so

$$\frac{dx}{dt}(2) = 10; \ \frac{dy}{dt}(2) = -\frac{3}{2}.$$

By the chain rule

$$\begin{aligned} \frac{df}{dt}(2) &= \frac{\partial f}{\partial x}(8,3) \cdot \frac{dx}{dt}(2) + \frac{\partial f}{\partial y}(8,3) \cdot \frac{dy}{dt}(2) \\ &= 7 \cdot 10 + (-2) \cdot (-\frac{3}{2}) \\ &= 73 \end{aligned}$$

b. (8 points) Let $\vec{u} = <3/5, 4/5>$. Compute $D_{\vec{u}} f(8,3)$.

Ans.

$$\begin{aligned} D_{\vec{u}} f(8,3) &= \frac{3}{5} \cdot f_x(8,3) + \frac{4}{5} \cdot f_y(8,3) \\ &= \frac{3}{5} \cdot 7 + \frac{4}{5} \cdot (-2) \\ &= \frac{13}{5} \end{aligned}$$

c. (8 points) Approximate $f(1.98, 3.05)$.

Ans. The linear approximation formula is

$$f(x + \Delta x, y + \Delta y) \approx f(x, y) + f_x(x, y)\Delta x + f_y(x, y)\Delta y.$$

Take $(x, y) = (2, 3)$, so $\Delta x = -0.02$, $\Delta y = 0.05$ and

$$f(1.98, 3.05) \approx 4 + (-1) \cdot (-0.02) + 9 \cdot (0.05) = 4 + 0.02 + -0.45 = 4.47.$$

2. Let $f(x, y) = x^3 y - 2xy^2 + y^2$.
a. (6 points) Compute $f_x(x, y)$ and $f_y(x, y)$.

$f_x(x, y) = 3x^2 y - 2y^2; \qquad f_y(x, y) = x^3 - 4xy + 2y$

b. (8 points) Find the equation of the tangent plane to the graph of f at the point $(1, 2, -2)$.

Ans. The tangent plane to the graph $z = f(x, y)$ at the point $(a, b, f(a, b))$ has equation

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

In this case, we have

$$f_x(1,2) = -2,\ f_y(1,2) = -3$$

this gives the equation

$$z + 2 = -2(x-1) - 3(y-2),$$

or

$$z + 2x + 3y = 6.$$

c. (8 points) In what direction (unit vector) from the point $(1,2)$ does f increase most rapidly?

Ans. In the direction of the gradient $\nabla f(1,2) = <f_x(1,2), f_y(1,2)> = <-2,-3>$. To make this a direction vector, divide by the length $\sqrt{(-2)^2 + (-3)^2} = \sqrt{13}$, so:

In the direction$<-\dfrac{2}{\sqrt{13}}, -\dfrac{3}{\sqrt{13}}>$

3. (14 points) Let $f(x,y) = \sqrt{x^3 - y^2 + 5}$. Use the linear approximation of f near $(x,y) = (2,3)$ to approximate $\sqrt{1.98^3 - 3.01^2 + 5}$.

Ans. Use the linear approximation formula

$$f(1.98, 3.01) \approx f(2,3) + f_x(2,3)\Delta x + f(2,3)\Delta y$$

with $\Delta x = -0.02$, $\Delta y = 0.01$. Compute

$$f(2,3) = \sqrt{2^3 - 3^2 + 5} = \sqrt{4} = 2$$

$$f_x(x,y) = \frac{1}{2}(x^3 - y^2 + 5)^{-1/2}(3x^2); \quad f_y(x,y) = \frac{1}{2}(x^3 - y^2 + 5)^{-1/2}(-2y)$$

so

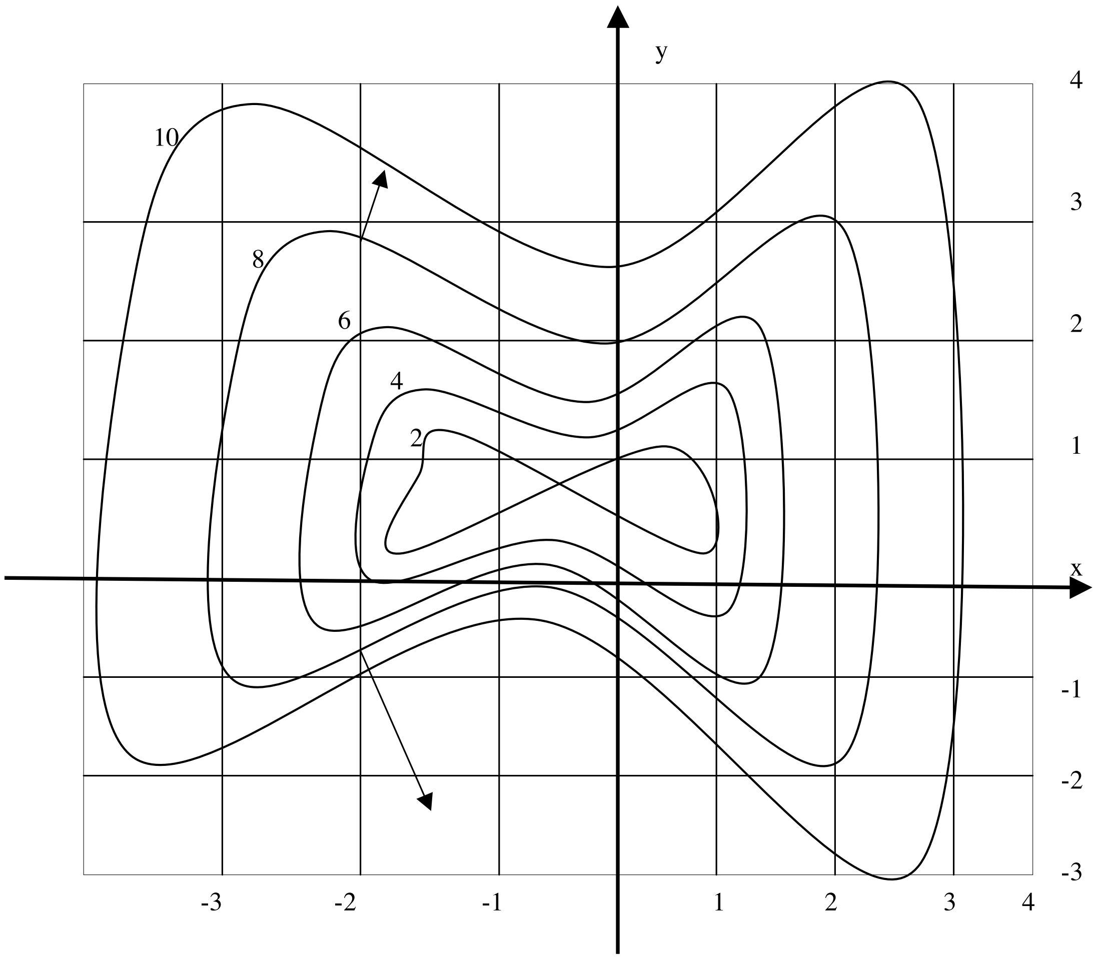
$$f_x(2,3) = \frac{12}{2\sqrt{4}} = 3; \quad f_y(2,3) = \frac{-6}{2\sqrt{4}} = -\frac{3}{2}.$$

Thus,

$$\sqrt{1.98^3 - 3.01^2 + 5} \approx 2 + 3 \cdot (-0.02) - \frac{3}{2} \cdot (0.01) = 1.925.$$

(actual value $= 1.924134091\ldots$).

4. (12 points) Here is the contour graph of a function $g(x,y)$:



Use the contour graph to estimate $g_x(1,3)$ and $g_y(1,3)$

Ans. From the graph, we see that

$$g(1,3) \approx 10,\ g(1\frac{2}{3},3) \approx 8,\ g(1,2.5) \approx 8$$

This gives

$$g_x(1,3) \approx \frac{g(1\frac{2}{3},3) - g(1,3)}{1\frac{2}{3} - 1} = \frac{-2}{\frac{2}{3}} = -3$$

and

$$g_y(1,3) \approx \frac{g(1,2.5) - g(1,3)}{2.5 - 3} = \frac{-2}{-0.5} = 4.$$

(any reasonable values, with work shown, were accepted).

5. (8 points) On the contour graph on the previous page, sketch in the gradient of g at the two points of intersection of the contour of level 8 with the vertical line $x = -2$. Note that one of the gradient vectors should be longer than the other one; be sure your sketch shows this.

Ans. The two vectors are perpendicular to the level curve at the given points, and point outward toward the level curve of level 10. The gradient vector at the bottom is longer (see graph).

6. Let S be the surface formed by revolving the graph $z = x^2 + 1$ about the x-axis.
a. (10 points) Write down a parametric equation for S.

$$<x, y, z> = <x, (x^2 + 1)\cos t, (x^2 + 1)\sin t>.$$

b. (10 points) Let T be the surface given by the parametric equations

$$\vec{r}(u, v) = <u^2 - 2v, u - v, 3u - v^2>.$$

Find the equation of the tangent plane to T at the point with $(u, v) = (2, 3)$.

To find the normal vector $\vec{n}$, take

$$\vec{n} = \vec{r}_u(2, 3) \times \vec{r}_v(2, 3)$$

We compute $\vec{r}_u = <2u, 1, 3>$, $\vec{r}_v = <-2, -1, -2v>$, so

$$\vec{r}_u(2,3) \times \vec{r}_v(2,3) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 1 & 3 \\ -2 & -1 & -6 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3 \\ -1 & -6 \end{vmatrix} \vec{i} - \begin{vmatrix} 4 & 3 \\ -2 & -6 \end{vmatrix} \vec{j} + \begin{vmatrix} 4 & 1 \\ -2 & -1 \end{vmatrix}$$

$$= -3\vec{i} + 18\vec{j} - 2\vec{k}$$

$$= <-3, 18, -2>.$$

We have $\vec{r}(2, 3) = <-2, -1, -3>$, giving the equation

$$-3(x + 2) + 18(y + 1) - 2(z + 3) = 0$$

or

$$-3x + 18y - 2z = -6.$$