MTH U341 QUIZ 3 Name: ____________________
Fall 2008

Show all your work in the space provided. No credit for unjustified answers.

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function

$$f(x, y) = \frac{x^2}{2} + xy - 2x - \frac{y^3}{3}.$$

a.(12 points) Find all critical points of f

Solve $\nabla f(x, y) = 0$, that is $f_x(x, y) = f_y(x, y) = 0$. Compute

$$f_x(x, y) = x + y - 2, f_y(x, y) = x - y^2$$

So, $f_y = 0$ gives $x = y^2$, putting this into $f_x = 0$ gives

$$y^2 + y - 2 = 0$$

Factoring gives $(y+2)(y-1) = y^2 + y - 2 = 0$ so $y = -2, 1$. Since $x = y^2$, this gives the two critical points

$$(x, y) = (4, -2), (1, 1).$$

b.(12 points) For each critical point, determine if it is a local maximum, local minimum or saddle point.

Apply the 2nd derivative test. $D = f_{xx}f_{yy} - (f_{xy})^2$. We have

$$f_{xx} = 1,\ f_{yy} = -2y,\ f_{xy} = 1$$

so $D(x, y) = -2y - 1$. Thus $D(1, 1) = -3 < 0$, so f has a saddle point at $(1, 1)$. $D(4, -2) = 3 > 0$, and $f_{xx}(4, -2) = 1 > 0$, so f has a local minimum at $(4, -2)$.

c.(6 points) Does f have an absolute maximum? Does f have an absolute minimum? Explain your answer.

f does not have an absolute maximum. One reason is that f does not have a local maximum, and the domain of f, $\mathbb{R}^2$, has no boundary. Another reason is that, if we take $y = 0$ then $f(x, 0) = x^2/2 - 2x$. We can make $f(x, 0)$ as large as we like by making x to be a large negative number.

f has no absolute minimum. Taking $x = 0$, $f(0, y) = -y^3/3$. By taking y to be a large positive number, we can make f an arbitrarily large negative number.

2. Let D be the square $\{(x, y), 0 \le x \le 2, 0 \le y \le 2\}$. Let $f : D \to \mathbb{R}$ be the function

$$f(x, y) = x^2 - y^2 + xy + 1$$

a.(10 points) Find all critical points of f on D

Solve $\nabla f(x, y) = 0$, that is $f_x(x, y) = f_y(x, y) = 0$. Compute

$$f_x(x, y) = 2x + y,\ f_y(x, y) = -2y + x$$

so we solve $2x + y = 0$, $-2y + x = 0$, giving the single critical point $(x, y) = (0, 0)$.

b.(20 points) Find the absolute maximum and absolute minimum of f on D, and the points where they occur.

We have the candidate critical point $(0, 0)$. Look on the boundary, this consists of four edges. Edge I: $x = 0$, $0 \le y \le 2$, edge II, $y = 0$, $0 \le x \le 2$, edge III, $x = 2$, $0 \le y \le 2$ and edge IV: $y = 2$, $0 \le x \le 2$.

I: $f(0, y) = -y^2 + 1$, $f' = 2y = 0$ implies $y = 0$. This gives the point $(0, 0)$, we have the other endpoint $(0,2)$.

II: $f(x,0) = x^2 + 1$, $f' = 2x = 0$ implies $x = 0$. This gives the point $(0,0)$, we have the other endpoint $(2,0)$.

III: $f(2,y) = 4 - y^2 + 2y + 1$, $f' = -2y + 2 = 0$ implies $y = 1$, giving the point $(2,1)$. We also have the endpoints $(2,0)$, $(2,2)$.

IV: $f(x,2) = x^2 - 4 + 2x + 1$, $f' = 2x + 2 = 0$ implies $x = -1$. This is not in the interval $0 \leq x \leq 2$, so we ignore it. We have the endpoints $(0,2)$ and $(2,2)$.

This gives us the list of candidates $(0,0)$, $(0,2)$, $(2,0)$, $(2,2)$ and $(2,1)$. We have

$$f(0,0) = 1,\ f(2,0) = 5,\ f(0,2) = -3,\ f(2,2) = 5,\ f(2,1) = 6$$

so the maximum of 6 occurs at $(x,y) = (2,1)$ and the minimum of -3 occurs at $(x,y) = (0,2)$.

3.(25 points) Using the method of Lagrange multipliers, find the minimum of the function $x^2 + xy + 4y^2$ on the curve $xy = 18$, and determine at which point(s) of the curve the minimum occurs.

Let $f(x,y) = x^2 + xy + 4y^2$ and $g(x,y) = xy$. The Lagrange mulitplier equation is $\nabla f = \lambda \nabla g$. This vector equation is the same as the two equations $f_x = \lambda g_x$ and $f_y = \lambda g_y$. Computing f_x, f_y, g_x, g_y, we have the equations

$$2x + y = \lambda y,\ x + 8y = \lambda x.$$

If $\lambda = 0$, then $x = y = 0$ is the only solution, which is not on the constraint curve $xy = 18$, so $\lambda \neq 0$. Eliminating λ by cross-mulitplication and dividing by λ gives the equation

$$x(2x + y) = y(x + 8y) \Longrightarrow 2x^2 + xy = xy + 8y^2 \Longrightarrow 2x^2 = 8y^2 \Longrightarrow x^2 = 4y^2$$

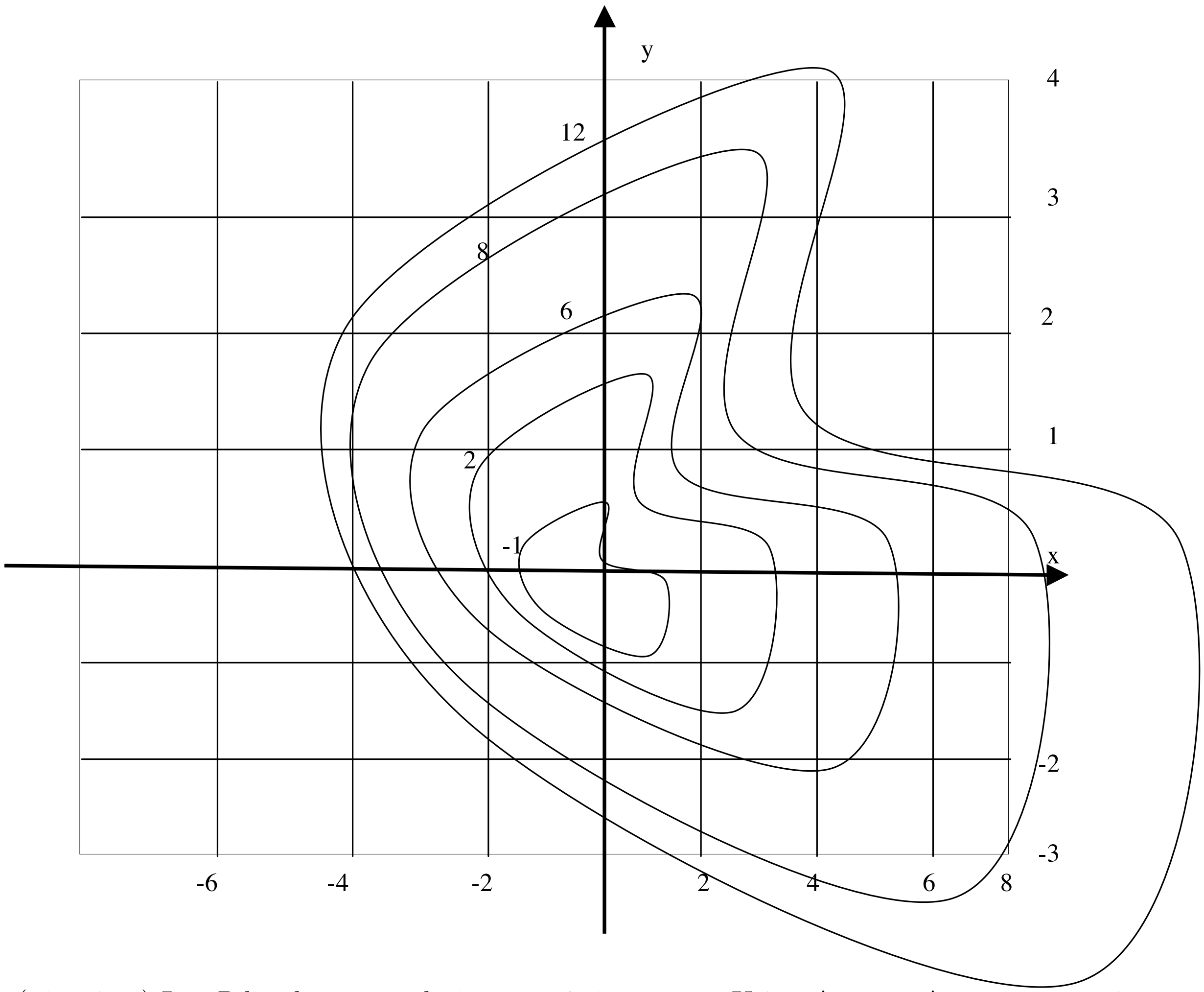
or $x = \pm 2y$. For $x = 2y$, we put this into the constraint $xy = 18$, giving $2y^2 = 18$, or $y = \pm 3$. Since $x = 2y$, this gives us the two points $(x,y) = (6,3), (-6,-3)$. If $x = -2y$, we put this into the constraint $xy = 18$, giving $-2y^2 = 18$, which has no solution.

Evaluating f at the two points gives

$$f(6,3) = f(-6,-3) = 36 + 18 + 36 = 90.$$

Because $f(x,y)$ can be made very large by taking x to be large and $y = 18/x$, or y large and $x = 18/y$, f gets very large as we move along along the hyperbola $xy = 18$ going off to infinity. Thus f has a minimum on $xy = 18$, and is must be given by the points we found. Thus, the minimum is 90, occurring at $(x,y) = (6,3), (-6,-3)$.

4. Here is the contour graph of a function $g(x, y)$:



a.(10 points) Let R be the rectangle $0 \leq x \leq 4$, $0 \leq y \leq 3$. Using $\Delta x = 2$, $\Delta y = 1$, approximate $\iint_R g(x, y)dA$, taking all sample points on the contour curves. Mark your sampling points on the graph.

There are many different answers, I will give one possibility. The region R is outlined in the picture below, and it is subdivided into 6 subrectangles. I have picked one sample point in each subrectangle, choosing each sample point to be on a contour curve. The Riemann sum for my choice of sample points is

$$(f(x_{11}^*, y_{11}^*) + f(x_{21}^*, y_{21}^*) + f(x_{12}^*, y_{12}^*) + f(x_{22}^*, y_{22}^*) + f(x_{13}^*, y_{13}^*) + f(x_{23}^*, y_{23}^*))\Delta x \cdot \Delta y$$
$$= (2 + 6 + 2 + 12 + 6 + 8)2 \cdot 1 = 72.$$

Since the Riemann sum approximates the integral, this gives us

$$\iint_R g(x, y)dA \approx 72.$$

b.(5 points) What is the average value of g over R (approximately)?

The average value of g over a rectangle R is $(\int_R g(x, y)dA)/\text{area}(R)$, so the average value is $\approx 72/12 = 6$.