MTH U341 QUIZ 4 Name: ______________________

Fall 2008

Show all your work in the space provided. No credit for unjustified answers.

1. Compute the integrals

a. (10 points) $\int_0^3 \int_0^2 3x^2 + 2xy\,dx\,dy$

$$\begin{aligned}\int_0^3 \int_0^2 3x^2 + 2xy\,dx\,dy &= \int_0^3 \left[\int_0^2 3x^2 + 2xy\,dx\right] dy\\ &= \int_0^3 x^3 + x^2y|_{x=0}^{x=2}\,dy\\ &= \int_0^3 8 + 4y\,dy\\ &= 8y + 2y^2|_0^2\\ &= 16 + 8 = 24\end{aligned}$$

b. (10 points) $\iint_R x - 3y^2 dA$, where R is the rectangle $0 \le x \le 4$, $0 \le y \le 2$.

$$\begin{aligned}\iint_R x - 3y^2 dA &= \int_0^4 \left[\int_0^2 x - 3y^2 dy\right] dx\\ &= \int_0^4 xy - y^3|_{y=0}^{y=2}\,dx\\ &= \int_0^4 2x - 8\,dx\\ &= x^2 - 8x|_0^4\\ &= 16 - 32 = -16\end{aligned}$$

c. (10 points) $\int_0^2 \int_0^x \sqrt{1+2x^2}\,dy\,dx$

$$\begin{aligned}\int_0^2 \int_0^x \sqrt{1+2x^2}\,dy\,dx &= \int_0^2 y\sqrt{1+2x^2}\,|_{y=0}^{y=x}\,dx\\ &= \int_0^2 x\sqrt{1+2x^2}\,dx. \text{ Let } u = 1+2x^2,\ du = 4x\,dx.\\ &\qquad x = 0 \Rightarrow u = 1, x = 2 \Rightarrow u = 9\\ &= \int_1^9 (1/4)u^{1/2}\,du\\ &= (1/4)(2/3)u^{3/2}|_1^9\\ &= (1/6)(9^{3/2} - 1^{3/2}) = (1/6)(27-1) = 13/3.\end{aligned}$$

2. (15 points) Find $\iint_R x\,dA$ where R is the region bounded by the curves $y = 4 - x^2$, $y - x = 2$.

First, find where the curves intersect: solve the 2nd equation for y, giving $y = x + 2$. The curves intersect where $4 - x^2 = x + 2$, or $x^2 + x - 2 = 0$. This factors as $(x+2)(x-1) = 0$, so $x = -2, 1$, giving the intersection points $(x, y) = (-2, 0)$, $(1, 1)$. $y = 4 - x^2$ is a parabola opening downward with vertex at $(40, 4)$ and $y = x + 2$ is a line joining the points $(-2, 1)$, $(1, 1)$, so the $y = 4 - x^2$ gives the top of the region and the line $y = x + 2$ is the bottom.

Writing the integral as an iterated integral gives

$$\begin{aligned}
\iint_R x\,dA &= \int_{-2}^{1} \left[\int_{x+2}^{4-x^2} x\,dy \right] dx \\
&= \int_{-2}^{1} \left[xy|_{y=x+2}^{y=4-x^2} \right] dx \\
&= \int_{-2}^{1} 4x - x^3 - (x^2 + 2x)\,dx \\
&= \int_{-2}^{1} -x^3 - x^2 + 2x\,dx \\
&= -x^4/4 - x^3/3 + x^2|_{-1}^{2} \\
&= -4 - 8/3 + 4 - (-1/4 + 1/3 + 1) \\
&= -15/4
\end{aligned}$$

3. (20 points) Let D be the region defined by $1 \le x^2 + y^2 \le 4$, $x \ge 0$. If D is a metal plate with density $\sqrt{x^2 + y^2}$ gms./cm^2, find the mass of D. Use polar coordinates.

The mass is $m = \iint_D dm$, with $dm = \rho(x, y)dA$, ρ the density. In other words

$$m = \iint_D \sqrt{x^2 + y^2}\,dA.$$

In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ and $dA = rdrd\theta$. Thus

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2.$$

The region D has $1 \le r^2 \le 4$, that is, $1 \le r \le 2$. Since $x \ge 0$, this has $pi/2 \le \theta \le \pi/2$ (or if you want to start at $\theta = 3\pi/2$, you can write this as $3\pi/2 \le \theta \le 5\pi/2$). Call this region in the r-θ plane D'.

Thus, the integral computing the mass is

$$\begin{aligned}
m &= \iint_D \sqrt{x^2 + y^2}\,dA \\
&= \iint_{D'} \sqrt{r^2}\,r\,dr\,d\theta \\
&= \iint_{D'} r^2\,dr\,d\theta \\
&= \int_{-\pi/2}^{\pi/2} \left[\int_1^2 r^2\,dr \right] d\theta \\
&= \int_{-\pi/2}^{\pi/2} r^3/3|_1^2\,d\theta \\
&= \int_{-\pi/2}^{\pi/2} 7/3\,d\theta \\
&= (7/3)\theta|_{-\pi/2}^{\pi/2} \\
&= 7\pi/3.
\end{aligned}$$

4. Let E be the solid region bounded by the surfaces $z = 4 - y$, $y = x^2$ and the x-y plane.
a. (15 points) Find the volume of E

E is the region with top the plane $z = 4 - y$ and bottom the x-y-plane ($z = 0$). These meet along the line $4 - y = 0$, that is, $y = 4$. Thus, the base of E is the region R in the plane with top the line $y = 4$ and bottom the parabola $y = x^2$. These two curves meet at the points $(-2, 4)$ and $(2, 4)$. The volume of E is

$$\begin{aligned}
\mathrm{vol}(E) &= \in \int_R 4 - y\, dA \\
&= \int_{-2}^{2} \left[\int_{x^2}^{4} 4 - y\, dy \right] dx \\
&= \int_{-2}^{2} 4y - y^2/2|_{y=x^2}^{y=4}\, dx \\
&= \int_{-2}^{2} 16 - 8 - (4x^2 - x^4/2)\, dx \\
&= \int_{-2}^{2} 8 - 4x^2 + (x^4)/2\, dx \\
&= 8x - (4/3)x^3 + (x^5)/10|_{-2}^{2} \\
&= 16 - (4/3)8 + (32/10) - (-16 + (4/3)8 - (32/10)) \\
&= 32 - 64/3 + 32/5 = 32/3 + 32/5 = 256/15.
\end{aligned}$$

Note: you can also integrate in the other order:

$$\begin{aligned}
\mathrm{vol}(E) &= \in \int_R 4 - y\, dA \\
&= \int_{0}^{4} \left[\int_{-\sqrt{y}}^{\sqrt{y}} 4 - y\, dx \right] dy
\end{aligned}$$

b. (20 points) Compute $\iiint_E y\, dV$.
Use the same limits, except that z goes from the bottom 0 to the top $4 - y$.

$$\begin{aligned}
\iiint_E y\, dV &= \int_{-2}^{2} \left[\int_{x^2}^{4} \left(\int_{0}^{4-y} y\, dz \right) dy \right] dx \\
&= \int_{-2}^{2} \left[\int_{x^2}^{4} (yz|_{z=0}^{z=4-y})\, dy \right] dx \\
&= \int_{-2}^{2} \left[\int_{x^2}^{4} 4y - y^2\, dy \right] dx \\
&= \int_{-2}^{2} \left[2y^2 - y^3/3|_{y=x^2}^{y=4} \right] dx \\
&= \int_{-2}^{2} 32 - 64/3 - (2x^4 - x^6/3)\, dx \\
&= \int_{-2}^{2} 32/3 - 2x^4 + x^6/3\, dx \\
&= 32x/3 - (2/5)x^5 + (1/21)x^7|_{-2}^{2} \\
&= 64/3 - 64/5 + 128/21 - (-64/3 + 64/5 - 128/21) \\
&= 2(128/15 + 128/21) = 1024/35.
\end{aligned}$$