MTH U341 QUIZ 5 Name: ____________________

Fall 2008

Show all your work in the space provided. No credit for unjustified answers.

1. Let C be the curve $x^2 + y^2 = 4$, $x \geq 0$, $y \geq 0$.

a. (10 points) Suppose C represents a wire made of a material of varying density $\rho(x, y) = xy^2$ gms/cm., with x and y in cm. Compute the mass of C.

$$\begin{aligned} \text{Mass } &= \int_C \rho(x, y)\, ds \\ &= \int_C xy^2\, dx \end{aligned}$$

Parametrize C by $x = 2\cos t$, $y = 2\sin t$, $dx = -2\sin t\, dt$, $dy = 2\cos t\, dt$, $0 \leq t \leq \pi/2$. Then

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{(-2\sin t)^2 + (2\cos t)^2}\, dt = \sqrt{4(\sin^2 t + \cos^2 t)}\, dt = 2\, dt.$$

and thus

$$\begin{aligned} \int_C xy^2\, dx &= \int_0^{\pi/2} (2\cos t)(2\sin t)^2 2\, dt \\ &= 16 \int_0^{\pi/2} \cos t \sin t\, dt \text{ Let } u = \sin t, du = \cos t\, dt \\ &= 16 \int_0^1 u^2\, du \\ &= (16/3)u^3|_0^1 = 16/3 \text{ grams.} \end{aligned}$$

b. (10 points) Suppose C represents the path followed by a particle P as P moves from the point $(2, 0)$ to the point $(0, 2)$ in the plane. If the particle moves through a force field $\vec{F}$ given by $\vec{F}(x, y) = <2 - y, x>$ kg. m/sec^2, and x and y are in meters, compute the work done by $\vec{F}$ on P.

$$\begin{aligned} \text{Work } &= \int_C \vec{F} \cdot d\vec{r} \\ &= \int_C <2 - y, x> \cdot <dx, dy> \\ &= \int_C (2 - y)dx + xdy \end{aligned}$$

Parametrize C as in (1a). Thus

$$\begin{aligned} \int_C (2 - y)dx + xdy &= \int_0^{\pi/2} (2 - 2\sin t)(-2\sin t) + (2\cos t)(2\cos t)\, dt \\ &= 4\int_0^{\pi/2} -\sin t + \sin^2 t + \cos^2 t\, dt = 4\int_0^{\pi/2} 1 - \sin t\, dt \\ &= 4(t + \cos t)|_0^{\pi/2} = 4(\pi/2 - 1) = 2\pi - 4 \text{ Joules.} \end{aligned}$$

c. (10 points) Compute the integral $\int_{C'} y\, dx - x^2\, dy$, where C' is the straight line path from $(1, 0)$ to $(0, 2)$.

Parametrize C', for example $x = 1 - t$, $y = 2t$, $dx = -dt$, $dy = 2dt$, $0 \le t \le 1$. Then

$$\begin{aligned}\int_{C'} y\,dx - x^2\,dy &= \int_0^1 2t\,(-dt) - (1-t)^2(2dt) \\ &= \int_0^1 -2t - 2 + 4t - 2t^2\,dt = \int_0^1 -2 + 2t - 2t^2\,dt \\ &= -2t + t^2 - (2/3)t^3|_0^1 \\ &= (-2 + 1 - 2/3) = -5/3.\end{aligned}$$

2. Let $\vec{F_1}$, $\vec{F_2}$ and $\vec{F_3}$ be vector fields on all of $\mathbb{R}^2$ defined by $\vec{F_1}(x, y) = <2y + xy, x^2>$, $\vec{F_2}(x, y) = <2xy + y^2, 2xy + x^2 + 2y>$, $\vec{F_3}(x, y) = <xy - y^2, 2xy + x^2>$.

a. (10 points) Without explicitly finding a potential function, determine which of the vector fields $\vec{F_1}$, $\vec{F_2}$ and $\vec{F_3}$ are conservative on $\mathbb{R}^2$ (at least one is conservative).

For each vector field $<P, Q>$, we check to see if $\partial P/\partial y = \partial Q/\partial x$; this is enough since $\mathbb{R}^2$ is simply connected.

$\vec{F_1}$: $\partial(2y + xy)/\partial y = 2 + x$, $\partial(x^2)/\partial x = 2x$, so $\vec{F_1}$ is not conservative.

$\vec{F_2}$: $\partial(2xy + y^2)/\partial y = 2x + 2y$, $\partial(2xy + x^2 + 2y)/\partial x = 2y + 2x$, so $\vec{F_2}$ is conservative.

$\vec{F_3}$: $\partial(xy - y^2)/\partial y = x - 2y$, $\partial(2xy + x^2)/\partial x = 2y + 2x$, so $\vec{F_3}$ is not conservative.

b. (10 points) Choose ONE conservative vector field $\vec{F}$ from part (1), and find a potential function f for $\vec{F}$.

$\vec{F} = \vec{F_2} = <2xy + y^2, 2xy + x^2 + 2y>$. If $\nabla f = \vec{F}$, then $f_x = 2xy + y^2$ and $f_y = 2xy + x^2 + 2y$.

$$f_x = 2xy + y^2 \Longrightarrow f = \int 2xy + y^2\,dx = x^2y + xy^2 + g(y).$$

Put this into the second equation:

$$f_y = 2xy + x^2 + 2y \Longrightarrow x^2 + 2xy + g'(y) = 2xy + x^2 + 2y \Longrightarrow g'(y) = 2y \Longrightarrow g(y) = y^2 + c.$$

Thus

$$f(x, y) = x^2y + xy^2 + y^2$$

is a potential function for $\vec{F}$.

c. (10 points) With $\vec{F}$ the vector field you chose for (2), find $\int_C \vec{F} \cdot d\vec{r}$, where C is the curve parametrized by $x = t^{3/2} + 2t^2$, $y = \sqrt{t} - e^{t^3}(1 - t) - 3t$, $0 \le t \le 1$.

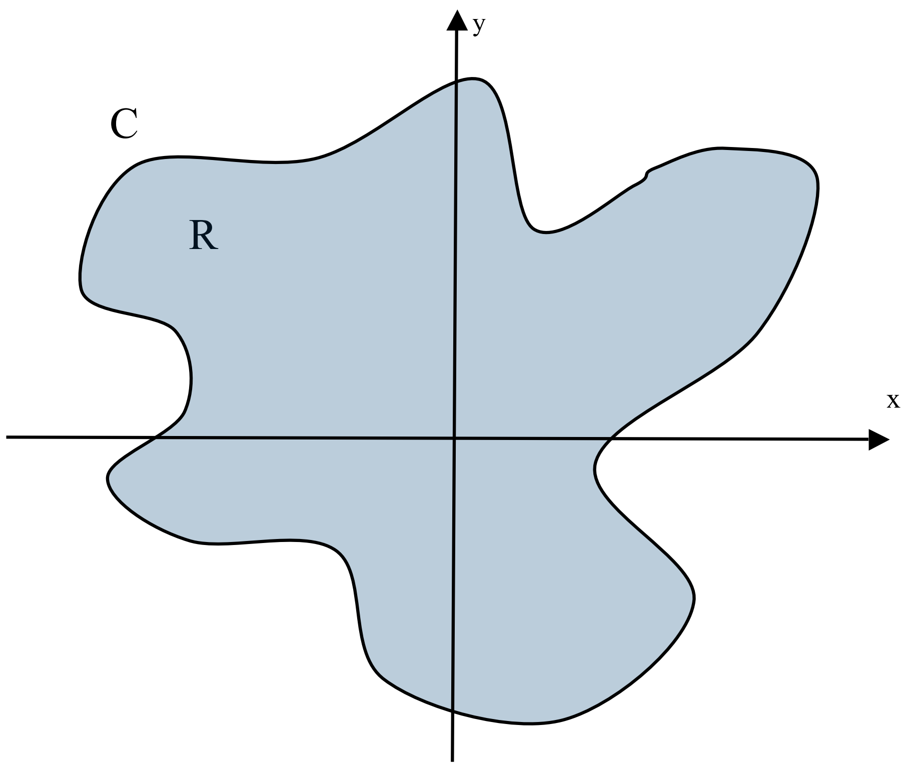
Use the fundamental theorem for gradient vector fields. The starting point for C is $\vec{r}(0) = (0, -1)$ and the endpoint for C is $\vec{r}(1) = (3, -2)$. Thus

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_C \nabla(x^2y + xy^2 + y^2) \cdot d\vec{r} \\ &= x^2y + xy^2 + y^2|_{(0,-1)}^{(3,-2)} \\ &= (3^2(-2) + 3(-2)^2 + (-2)^2) - (0 + 0 + (-1)^2) = -18 + 12 + 4 - 1 = -3.\end{aligned}$$

3. (10 points) In the picture below, C is oriented in the counter-clockwise direction and the area

of R is 7. Use Green's theorem to compute $\int_C 3y\,dx + x\,dy$. *Warning:* $\partial(3y)/\partial y \neq \partial(x)/\partial x$.

$$\begin{aligned}
\int_C 3y\,dx + x\,dy &= \int_{\partial R} 3y\,dx + x\,dy \\
&= \iint_R \partial(x)/\partial x - \partial(3y)/\partial y\,dA \\
&= \iint_R 1 - 3\,dA = -2\iint_R dA \\
&= (-2)\text{area}(R) = -14.
\end{aligned}$$



4. You are given the following data: $\vec{F}(x,y) = <P(x,y), Q(x,y)>$ is a vector field defined for $(x,y) \neq (0,0)$ and the functions P and Q satisfy the equation

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

In addition, you are given that

$$\int_C P\,dx + Q\,dy = 5,$$

where C is the unit circle $x^2 + y^2 = 1$, oriented in the counter-clockwise direction.

a. (10 points)Let $D = \{(x, y) \text{ in } \mathbb{R}^2, (x, y) \neq (0, 0)\}$, that is, D is the plane with the origin $(0, 0)$ removed. Is $\vec{F}$ conservative on D? Explain.

$\vec{F}$ is not conservative on D: C is a closed path in D and $\int_C P\,dx + Q\,dy = 5 \neq 0$. Thus, the line integrals of $\vec{F}$ are not path-independent, and thus $\vec{F}$ is not conservative.

b. (10 points) Let $D' = \{(x, y), y > 0\}$. Is $\vec{F}$ conservative on D'? Explain.

$\vec{F}$ is conservative on D': D' is an open, simply-connected region in the plane. Since $\partial P/\partial y = \partial Q/\partial x$, it follows that $\vec{F}$ is conservative on D'.

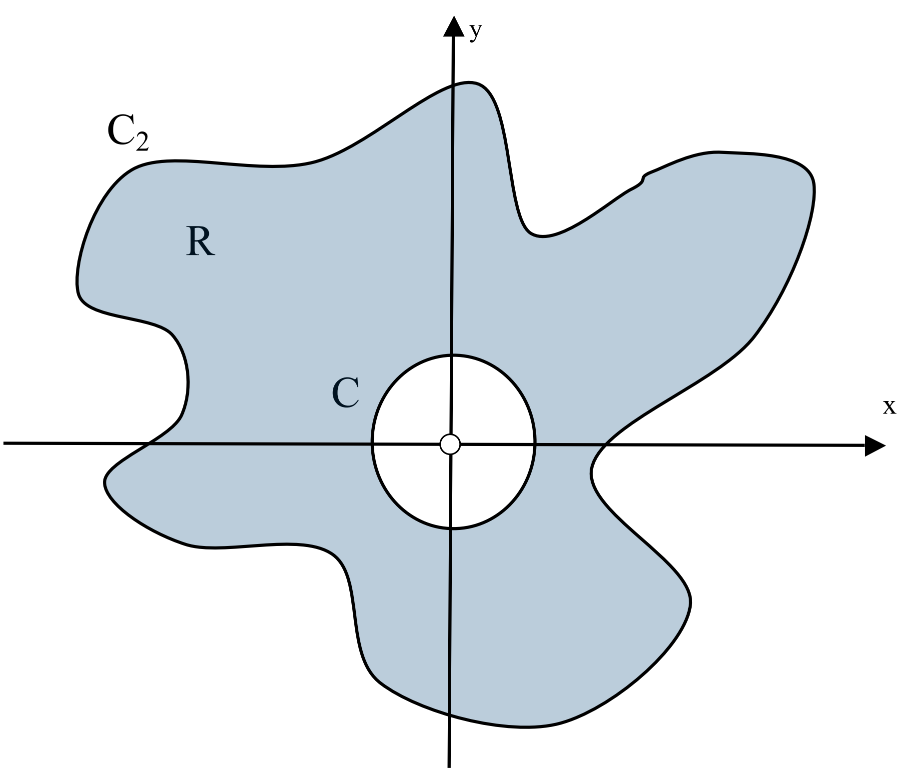
c. (10 points) In the picture below, C is the unit circle $x^2 + y^2 = 1$, oriented in the counter-clockwise direction. Find $\int_{C_2} P\,dx + Q\,dy$, where C_2 is oriented in the counter-clockwise direction. Show how you got your answer.

The boundary of R is $C_2 - C$. By Green's theorem, we have

$$\int_{C_2} \vec{F} \cdot d\vec{r} - \int_C \vec{F} \cdot d\vec{r} = \int_{\partial R} \vec{F} \cdot d\vec{r}$$
$$= \iint_R \partial Q/\partial x - \partial P/\partial y\,dA = 0, \text{since } \partial P/\partial y = \partial Q/\partial x$$

Thus

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} = 5.$$



$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

$$\int_C P\,dx + Q\,dy = 5,$$