MTH U341 Surface Integrals, Stokes theorem, the divergence theorem
To be turned in Wed., Dec. 10.

1. Let S_a be the sphere of radius a, $x^2 + y^2 + z^2 = a^2$.
a. Use spherical coordinates (with $\rho = a$) to parametrize S_a.
b. Use the parametrization from (a) to compute the area of S_a as $\iint_{S_a} dS$. Be sure to substitute for dS!
2. Let S be the surface $z = x^2 + 3y^2$, $0 \le x \le 2$, $0 \le y \le 3$.
a. Parametrize S using the standard parametrization of a graph.
b. Let $\vec{r}(x, y)$ be the parametrization from (a). Compute the unit normal

$$\vec{n}(x, y) = \frac{1}{|\vec{r}_x \times \vec{r}_y|} \vec{r}_x \times \vec{r}_y$$

explicitly. What is $\vec{n}(0, 0)$?
c. Let $\vec{F}$ be the vector field $\vec{F}(x, y, z) = y\vec{i} + x^2\vec{j} + (4z + y^2 - x^2)\vec{k}$. Compute $\iint_S \vec{F} \cdot d\vec{S}$, where S is oriented using the unit normal from (b).

d. Let C be the boundary of S, with orientation induced from the orientation $\vec{n}$ of S given in (b). Check Stokes' theorem by computing $\int_C \vec{F} \cdot d\vec{r}$ and $\iint_S \text{curl}\vec{F} \cdot d\vec{S}$ and seeing that you get the same number for both. *Hint*: You can parametrize C by using the parametrization of S in (a). If R is the plane region corresponding to S by this parametrization, then C is parametrized by the boundary of R.

3. Let $\vec{F}$ be the vector field

$$\vec{F}(\vec{x}) = \frac{\vec{x}}{|x|^3}.$$

a. Compute $\text{div}\vec{F}$.
b. Compute $\iint_{S_a} \vec{F} \cdot d\vec{S}$, where S_a is as in (1) the sphere of radius a, center $(0, 0, 0)$, and with orientation the outward normal vector.
c. (You might want to wait until after class on Monday for this one, but try it before if you are a thrill-seeker) Let S be the ellipsoid $5x^2 + 11y^2 + 17z^2 = 123$, oriented with the outward normal. Use the divergence theorem and (b) to compute $\iint_S \vec{F} \cdot d\vec{S}$. *Hint*: Take a small enough so that S_a is inside of S and let D be the solid region between S and S_a.

4.(You might want to wait until after class on Monday for this one, but try it before if you are a thrill-seeker) Let S_1 be the paraboloid $z = 100 - x^2 - y^2$, $z \ge 0$, oriented with the "upward" normal. Let S_2 be the paraboloid $z = 200 - 2x^2 - 2y^2$, $z \ge 0$, also oriented with the "upward" normal.
a. Let $\vec{F}$ be a vector field on $\mathbb{R}^3$. Use Stokes' theorem to show that

$$\iint_{S_1} \text{curl}\vec{F} \cdot d\vec{S} = \iint_{S_2} \text{curl}\vec{F} \cdot d\vec{S}.$$

Hint: Note that S_2 and S_1 both have the same boundary curve, $x^2 + y^2 = 100$, $z = 0$.
b. Let $\vec{F}$ be a vector field on $\mathbb{R}^3$. Use the divergence theorem to show that

$$\iint_{S_1} \text{curl}\vec{F} \cdot d\vec{S} = \iint_{S_2} \text{curl}\vec{F} \cdot d\vec{S}.$$

Hint: $\text{div}(\text{curl}\vec{F}) = 0$. Note that S_2 lies above S_1, meeting along their boundary curve, and consider the solid region D between S_1 and S_2.