MTH U341 Surface Integrals, Stokes theorem, the divergence theorem
To be turned in Wed., Dec. 10.

1. Let S_a be the sphere of radius a, $x^2 + y^2 + z^2 = a^2$.
a. Use spherical coordinates (with $\rho = a$) to parametrize S_a.

$$\vec{r}(\phi, \theta) = <a \sin\phi \cos\theta, a \sin\phi \sin\theta, a \cos\phi>.$$

b. Use the parametrization from (a) to compute the area of S_a as $\iint_{S_a} dS$. Be sure to substitute for dS!

Recall that $dS = |\vec{r}_\phi \times \vec{r}_\theta|$.

$$\vec{r}_\phi = <a \cos\phi \cos\theta, a \cos\phi \sin\theta, -a \sin\phi>$$
$$\vec{r}_\theta = <-a \sin\phi \sin\theta, a \sin\phi \cos\theta, 0>,$$

so

$$\begin{aligned}\vec{r}_\phi \times \vec{r}_\theta &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix} \\ &= a^2 \sin^2\phi \cos\theta\, \vec{i} + a^2 \sin^2\phi \sin\theta\, \vec{j} + a^2 \cos\phi \sin\phi\, \vec{k} \\ &= a^2 \sin\phi <\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi>.\end{aligned}$$

Thus

$$\begin{aligned}|\vec{r}_\phi \times \vec{r}_\theta| &= |a^2 \sin\phi <\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi>| \\ &= a^2 \sin\phi |<\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi>| \\ &= a^2 \sin\phi \sqrt{\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \cos^2\phi} \\ &= a^2 \sin\phi \sqrt{\sin^2\phi(\cos^2\theta + \sin^2\theta) + \cos^2\phi} \\ &= a^2 \sin\phi \sqrt{\sin^2\phi + \cos^2\phi} \\ &= a^2 \sin\phi.\end{aligned}$$

We can now compute the integral:

$$\begin{aligned}\iint_{S_a} dS &= \int_0^{2\pi} \int_0^{\pi} a^2 \sin\phi \, d\phi \, d\theta \\ &= \int_0^{2\pi} -a^2 \cos\phi|_0^\pi, d\theta \\ &= \int_0^{2\pi} -a^2(\cos(\pi) - \cos(0)), d\theta \\ &= \int_0^{2\pi} 2a^2 \, d\theta = 4\pi a^2.\end{aligned}$$

2. Let S be the surface $z = x^2 + 3y^2$, $0 \le x \le 2$, $0 \le y \le 3$.
a. Parametrize S using the standard parametrization of a graph.

$$\vec{r}(x, y) = <x, y, x^2 + 3y^2>$$

that is, $x = x$, $y = y$, $z = x^2 + 3y^2$.

b. Let $\vec{r}(x, y)$ be the parametrization from (a). Compute the unit normal

$$\vec{n}(x, y) = \frac{1}{|\vec{r}_x \times \vec{r}_y|}\vec{r}_x \times \vec{r}_y$$

explicitly. What is $\vec{n}(0, 0)$?

$\vec{r}_x = <1, 0, 2x>$, $\vec{r}_y = <0, 1, 6y>$ and

$$\begin{aligned}\vec{r}_x \times \vec{r}_y &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2x \\ 0 & 1 & 6y \end{vmatrix} \\ &= <-2x, -6y, 1>.\end{aligned}$$

Thus

$$\begin{aligned}|<-2x, -6y, 1>| &= \sqrt{4x^2 + 36y^2 + 1} \\ \vec{n}(x, y) &= \frac{1}{\sqrt{4x^2 + 36y^2 + 1}}<-2x, -6y, 1>\end{aligned}$$

and

$$\vec{n}(0, 0) = <0, 0, 1>.$$

c. Let $\vec{F}$ be the vector field $\vec{F}(x, y, z) = y\vec{i} + x^2\vec{j} + (4z + y^2 - x^2)\vec{k}$. Compute $\iint_S \vec{F} \cdot d\vec{S}$, where S is oriented using the unit normal from (b).

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_R \vec{F}(\vec{r}(x, y)) \cdot \vec{r}_x \times \vec{r}_y \, dA$$

where R is the rectangle $0 \leq x \leq 2$, $0 \leq y \leq 3$. Also

$$\vec{F}(\vec{r}(x, y)) = <y, x^2, 4(x^2 + 3y^2) + y^2 - x^2> = <y, x^2, 3x^2 + 13y^2>,$$

so from (2b) we get

$$\begin{aligned}\iint_R \vec{F}(\vec{r}(x, y)) \cdot \vec{r}_x \times \vec{r}_y \, dA &= \int_0^3 \int_0^2 <y, x^2, 3x^2 + 13y^2> \cdot <-2x, -6y, 1> dx\, dy \\ &= \int_0^3 \int_0^2 -2xy - 6x^2y + 3x^2 + 13y^2 \, dx\, dy \\ &= \int_0^3 [-x^2y - 2x^3y + x^3 + 13y^2x]|_0^2 \, dy \\ &= \int_0^3 26y^2 - 20y + 8 \, dy \\ &= (26/3)y^3 - 10y^2 + 8y|_0^3 \\ &= 168.\end{aligned}$$

d. Let C be the boundary of S, with orientation induced from the orientation $\vec{n}$ of S given in (b). Check Stokes' theorem by computing $\int_C \vec{F} \cdot d\vec{r}$ and $\iint_S \text{curl}\vec{F} \cdot d\vec{S}$ and seeing that you get the same number for both. *Hint*: You can parametrize C by using the parametrization of S in (a). If R is the plane region corresponding to S by this parametrization, then C is parametrized by the boundary of R.

First compute the flux integral.

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & x^2 & 4z + y^2 - x^2 \end{vmatrix}$$

$$= (\partial(4z + y^2 - x^2)/\partial y - \partial(x^2)/\partial z)\,\vec{i} - (\partial(4z + y^2 - x^2)/\partial x - \partial(y)/\partial z)\,\vec{j} + (\partial(x^2)/\partial x - \partial(y)/\partial y)\,\vec{k}$$

$$= <2y, 2x, 2x - 1>.$$

Let R be the rectangle $0 \le x \le 2$, $0 \le y \le 3$. From (2b) we get

$$\begin{aligned}\iint_S \text{curl}\vec{F} \cdot d\vec{S} &= \iint_R \text{curl}\vec{F}(\vec{r}(x,y)) \cdot r_x \times r_y \, dA \\ &= \iint_R <2y, 2x, 2x-1> \cdot <-2x, -6y, 1> dA \\ &= \int_0^3 \int_0^2 -4xy - 12xy + 2x - 1 \, dx\, dy \\ &= \int_0^3 -8x^2y + x^2 - x|_{x=0}^{x=2} \, dy \\ &= \int_0^3 -32y + 2 \, dy \\ &= -16y^2 + 2y|_0^3 = -138.\end{aligned}$$

Now, we compute the line integral. Since S is parametrized by the rectangle R using the function $\vec{r}(x,y)$, the boundary ∂S is parametrized by ∂R, using the restriction of $\vec{r}(x,y)$ to the edges of R. This gives 4 separate pieces to ∂S as we go counter-clockwise around R, starting at $(0,0)$:

C_1, parametrized by $\vec{r}_1(x) = \vec{r}(x,0) =< x, 0, x^2>$, x goes from 0 to 2
C_2, parametrized by $\vec{r}_2(y) = \vec{r}(2,y) =< 2, y, 4 + 3y^2>$, y goes from 0 to 3
C_3, parametrized by $\vec{r}_3(x) = \vec{r}(x,3) = <x, 3, x^2 + 12>$, x goes from 2 to 0
C_4, parametrized by $\vec{r}_4(y) = \vec{r}(0,y) = <0, y, 3y^2>$, y goes from 3 to 0

We do the integrals one at a time and add the results:

$$\begin{aligned}\int_{C_1} \vec{F} \cdot d\vec{r} &= \int_0^2 \vec{F}(r_1(x)) \cdot r_1'(x)\, dx \\ &= \int_0^2 <0, x^2, 3x^2> \cdot <1, 0, 2x> dx \\ &= \int_0^2 6x^3 \, dx \\ &= (6/4)x^4|_0^2 = 24.\end{aligned}$$

$$\begin{aligned}\int_{C_2} \vec{F}\cdot d\vec{r} &= \int_0^3 \vec{F}(r_2(y))\cdot r_2'(y)\,dy\\ &= \int_0^3 <y,4,12+13y^2>\cdot<0,1,6y>\,dy\\ &= \int_0^3 4+72y+78y^3\,dy\\ &= 4y+36y^2+(39/2)y^4|_0^3 = 1915.5.\end{aligned}$$

$$\begin{aligned}\int_{C_3} \vec{F}\cdot d\vec{r} &= \int_2^0 \vec{F}(r_3(x))\cdot r_3'(x)\,dx\\ &= \int_2^0 <3,x^2,3x^2+117>\cdot<1,0,2x>\,dx\\ &= \int_2^0 3+6x^3+234x\,dx\\ &= 3x+(6/4)x^4+117x^2|_2^0\\ &= -498.\end{aligned}$$

$$\begin{aligned}\int_{C_4} \vec{F}\cdot d\vec{r} &= \int_3^0 \vec{F}(r_4(y))\cdot r_4'(y)\,dy\\ &= \int_3^0 <y,0,13y^2>\cdot<0,1,6y>\,dy\\ &= \int_3^0 78y^3\,dy\\ &= (39/2)y^4|_3^0 = -1579.5.\end{aligned}$$

Thus

$$\int_{\partial S} \vec{F}\cdot d\vec{r} = 24+1915.5-498-1579.5 = -138.$$

3. Let $\vec{F}$ be the vector field

$$\vec{F}(\vec{x}) = \frac{\vec{x}}{|x|^3}.$$

a. Compute div$\vec{F}$.

Explicitly, $\vec{x} = <x,y,z>$, $|\vec{x}| = (x^2+y^2+z^2)^{1/2}$. Thus,

$$\vec{F}(\vec{x}) = <\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}}>,$$

and

$$\text{div}\vec{F} = \frac{\partial}{\partial x}\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\right] + \frac{\partial}{\partial y}\left[\frac{y}{(x^2+y^2+z^2)^{3/2}}\right] + \frac{\partial}{\partial z}\left[\frac{z}{(x^2+y^2+z^2)^{3/2}}\right].$$

We calculate the partial derivatives using the quotient rule:

$$\begin{aligned}
\frac{\partial}{\partial x}\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\right] &= \frac{(x^2+y^2+z^2)^{3/2}\cdot 1-(3/2)(x^2+y^2+z^2)^{1/2}2x\cdot x}{(x^2+y^2+z^2)^3} \\
&= \frac{(x^2+y^2+z^2)^{3/2}-3(x^2+y^2+z^2)^{1/2}x^2}{(x^2+y^2+z^2)^3} \\
&= \frac{(x^2+y^2+z^2)^{1/2}((x^2+y^2+z^2)-3x^2)}{(x^2+y^2+z^2)^3} \\
&= \frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}}.
\end{aligned}$$

Switching x with y and x with z gives

$$\begin{aligned}
\frac{\partial}{\partial y}\left[\frac{y}{(x^2+y^2+z^2)^{3/2}}\right] &= \frac{x^2-2y^2+z^2}{(x^2+y^2+z^2)^{5/2}} \\
\frac{\partial}{\partial z}\left[\frac{z}{(x^2+y^2+z^2)^{3/2}}\right] &= \frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}}
\end{aligned}$$

and thus

$$\begin{aligned}
\mathrm{div}\vec{F} &= \frac{x^2-2y^2+z^2}{(x^2+y^2+z^2)^{5/2}}+\frac{x^2-2y^2+z^2}{(x^2+y^2+z^2)^{5/2}}+\frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}} \\
&= \frac{x^2-2y^2+z^2+x^2-2y^2+z^2+x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}} \\
&= 0.
\end{aligned}$$

b. Compute $\iint_{S_a}\vec{F}\cdot d\vec{S}$, where S_a is as in (1) the sphere of radius a, center $(0,0,0)$, and with orientation the outward normal vector.

Since the radial vector $\vec{x}$ is perpendicular to the sphere, we have

$$\vec{n}=\frac{\vec{x}}{|\vec{x}|}$$

Since $|\vec{x}| = a$ for $\vec{x}$ in S_a, we have

$$\begin{aligned}
\iint_{S_a} \vec{F}\cdot d\vec{S} &= \iint_{S_a} \vec{F}\cdot \vec{n} dS \\
&= \iint_{S_a} \frac{\vec{x}}{|\vec{x}|^3}\cdot \frac{\vec{x}}{|\vec{x}|} dS \\
&= \iint_{S_a} \frac{\vec{x}\cdot\vec{x}}{|\vec{x}|^4} dS \\
&= \iint_{S_a} \frac{|\vec{x}|^2}{|\vec{x}|^4} dS \\
&= \iint_{S_a} \frac{|\vec{x}|^2}{|\vec{x}|^4} dS \\
&= \iint_{S_a} \frac{1}{a^2} dS \\
&= \frac{1}{a^2}\iint_{S_a} dS \\
&= \frac{1}{a^2}4\pi a^2 \quad \text{(from (1b))} \\
&= 4\pi.
\end{aligned}$$

c. (You might want to wait until after class on Monday for this one, but try it before if you are a thrill-seeker) Let S be the ellipsoid $5x^2 + 11y^2 + 17z^2 = 123$, oriented with the outward normal. Use the divergence theorem and (b) to compute $\iint_S \vec{F}\cdot d\vec{S}$. *Hint*: Take a small enough so that S_a is inside of S and let D be the solid region between S and S_a.

Use the divergence theorem: The boundary of D is

$$\partial D = S - S_a$$

where $-S_a$ means the sphere of radius a with the *inward* pointing normal. We have

$$\begin{aligned}
\iint_S \vec{F}\cdot d\vec{S} - \iint_{S_a} \vec{F}\cdot d\vec{S} &= \iint_{S-S_a} \vec{F}\cdot d\vec{S} \\
&= \iint_{\partial D} \vec{F}\cdot d\vec{S} \\
&= \iiint_D \text{div } \vec{F}\cdot dV \quad \text{(the divergence theorem)} \\
&= 0 \quad (\text{since div } \vec{F} = 0).
\end{aligned}$$

Thus

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{S_a} \vec{F}\cdot d\vec{S} = 4\pi.$$

4.(You might want to wait until after class on Monday for this one, but try it before if you are a thrill-seeker) Let S_1 be the paraboloid $z = 100 - x^2 - y^2$, $z \geq 0$, oriented with the "upward" normal. Let S_2 be the paraboloid $z = 200 - 2x^2 - 2y^2$, $z \geq 0$, also oriented with the "upward" normal.

a. Let $\vec{F}$ be a vector field on $\mathbb{R}^3$. Use Stokes' theorem to show that

$$\iint_{S_1} \text{curl}\vec{F}\cdot d\vec{S} = \iint_{S_2} \text{curl}\vec{F}\cdot d\vec{S}.$$

Hint: Note that S_2 and S_1 both have the same boundary curve, $x^2 + y^2 = 100$, $z = 0$.

Let C be the common boundary curve of S_1 and S_2. By Two applications of Stokes' theorem

$$\iint_{S_1} \text{curl}\vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} = \iint_{S_2} \text{curl}\vec{F} \cdot d\vec{S}.$$

b. Let $\vec{F}$ be a vector field on $\mathbb{R}^3$. Use the divergence theorem to show that

$$\iint_{S_1} \text{curl}\vec{F} \cdot d\vec{S} = \iint_{S_2} \text{curl}\vec{F} \cdot d\vec{S}.$$

Hint: $\text{div}(\text{curl}\vec{F}) = 0$. Note that S_2 lies above S_1, meeting along their boundary curve, and consider the solid region D between S_1 and S_2.

Let $-S_1$ be the surface S_1 oriented with the downward pointing normal. Since S_2 is the "top" part of D and S_1 is the "bottom", we have

$$\partial D = S_2 - S_1,$$

because the normal vector on ∂D pointing away from D is the upward normal on S_2, but the downward one on S_1. Following the hint, we use the divergence theorem, using also that $\text{div}(\text{curl}\vec{F}) = 0$:

$$\begin{aligned} \iint_{S_2} \text{curl}\vec{F} \cdot d\vec{S} - \iint_{S_1} \text{curl}\vec{F} \cdot d\vec{S} &= \iint_{\partial D} \text{curl}\vec{F} \cdot d\vec{S} \\ &= \iiint_D \text{div}\ (\text{curl}\vec{F})\, dV = 0 \end{aligned}$$

and thus

$$\iint_{S_1} \text{curl}\vec{F} \cdot d\vec{S} = \iint_{S_2} \text{curl}\vec{F} \cdot d\vec{S}.$$