THE HOMOTOPY CONIVEAU FILTRATION

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Abstract. We examine the "homotopy coniveau tower" for a general cohomology theory on smooth k-schemes, satisfying some natural axioms, and give a new proof that the layers of this tower for K-theory agree with motivic cohomology.

We show how these constructions lead to a tower of functors on the Morel-Voevodsky stable homotopy category, and identify this stable homotopy coniveau tower with Voevodsky's slice filtration. We also show that the 0th layer for the motivic sphere spectrum is the motivic cohomology spectrum, which gives the layers for a general $\mathbb{P}^1$-spectrum the structure of a module over motivic cohomology. This recovers and extends recent results of Voevodsky on the 0th layer of the slice filtration, and yields a spectral sequence that is reminiscent of the classical Atiyah-Hirzebruch spectral sequence.

Contents

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0. Introduction

The original purpose of this paper was to give an alternative argument for the technical underpinnings of the papers [3, 6], in which the constructruction of a spectral sequence from motivic cohomology to K-theory is given. As in the method used by Suslin [17] to analyze the Grayson spectral sequence, we rely on localization properties of the relevant spectra.

Having done this, it becomes clear that the method applies more generally to a functor from smooth schemes over a given base field k to spectra, satisfying certain conditions. We therefore give a general discussion for a functor $E : \mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{Spt}$ from smooth k-schemes to spectra, which is homotopy invariant and satisfies Nisnevic excision.

For such a functor, and an X in $\mathbf{Sm}_k$, we construct the *homotopy coniveau tower*

$$\ldots \to E^{(p+1)}(X,-) \to E^{(p)}(X,-) \to \ldots \to E^{(0)}(X,-) \sim E(X)$$

where the $E^{(p)}(X,-)$ are simplicial spectra with n-simplices the limit of the spectra with support $E^W(X\times\Delta^n)$, where W is a closed subset of codimension $\geq p$ in "good position". This is just the evident extension of the tower used by Friedlander-Suslin in [6]. One can consider this tower as the algebraic analog of the one in topology formed by applying a cohomology theory to the skeletal filtration of a CW complex. The main objects of our study are the layers $E^{(p/p+1)}(X,-)$ in this tower.

We begin by verifying the basic properties of the $E^{(p)}(X,-)$: homotopy invariance, localization, and functoriality. This latter is accomplished by a generalization of the classical Chow's moving lemma, analogous to the procedure used for Bloch's higher cycle complexes; in order to use this method, we require that E can be delooped at least twice (in the sense of $\mathbb{P}^1$-spectra), so that some form of push-forward map is available.

We apply the method used by Kahn in [11] to replace the total spectra $|E^{(p)}(X,-)|$ with functors

$$E^{(p)} : \mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{Spt},$$

and similarly for the layers $E^{(p/p+1)}$.

We iterate these functors, and thereby find a simple description of the layers. In this, a crucial role is played by the 0th layers $(\Omega_T^p E)^{(0/1)}$ of the T-loop space

$$(\Omega_T^p E)(X) := E^{X\times 0}(X\times\mathbb{A}^p).$$

Indeed, for W smooth, the restriction map

$$E^{(0/1)}(W) \to E^{(0/1)}(k(W))$$

is a weak equivalence, which enables us to, roughly speaking, extend the functor $E^{(0/1)}$ to all k-schemes as a locally constant sheaf for the Zariski topology. One can then identify $E^{(p/p+1)}(X)$ with the simplicial spectrum $E^{(p/p+1)}_{\text{s.l.}}(X,-)$ having n-simplices

$$E^{(p/p+1)}_{\text{s.l.}}(X,n) = \coprod_{x\in X^{(p)}(n)} (\Omega^p_T E)^{(0/1)}(k(x)).$$

Here $X^{(p)}(n)$ is the set of codimension p points of $X \times \Delta^n$, with closure in good position.

Once one has this description of the layers, it is easy to compute the layers for K-theory, as one can easily show that $\Omega^p_T K = K$ and that $K^{(0/1)}(F)$ is canonically a $K(K_0(F), 0) = K(\mathbb{Z}, 0)$ for F a field. This gives a direct identification of $K^{(p/p+1)}_{\text{s.l.}}(X,n)$ with Bloch's higher cycle group $z^p(X,n)$, and thus the weak equivalence $K^{(p/p+1)}(X,-)$ with $z^p(X,-)$.

After this, we turn to the $\mathbb{P}^1$-stable theory. The localization property for the $E^{(p)}$ allows one to define a $\mathbb{P}^1$-spectrum $\phi_p\mathcal{E}$ for a $\mathbb{P}^1$-Ω-spectrum $\mathcal{E} := (E_0, E_1, \ldots)$, by the formula

$$(\phi_p\mathcal{E})_n := \begin{cases} E_n^{(n+p)} & \text{for } n+p \geq 0 \\ E_{n+p} & \text{for } n+p < 0. \end{cases}$$

Here we should note that a $\mathbb{P}^1$-Ω-spectrum $\mathcal{E} := (E_0, E_1, \ldots)$ is, roughly speaking, a sequence of functors $E_n : \mathbf{Sm}_k^{\text{op}} \to \mathbf{Spt}$ satisfying homotopy and Nisnevic excision, together with weak equivalences $E_n \to \Omega_{\mathbb{P}^1} E_{n+1}$, where the operation $\Omega_{\mathbb{P}^1}$ is given by

$$(\Omega_{\mathbb{P}^1} E)(X) := \text{fib}(E(X \times \mathbb{P}^1) \to E(X \times \infty)).$$

This gives the stable homotopy coniveau tower

$$\ldots \to \phi_{p+1}\mathcal{E} \to \phi_p\mathcal{E} \to \ldots \to \phi_0\mathcal{E} \to \phi_{-1}\mathcal{E} \to \ldots \to \mathcal{E}.$$

We examine this tower for suspension spectra, and then for $\mathcal{E}$ in the subcategory $\Sigma^d_{\mathbb{P}^1}\mathcal{SH}^{\text{eff}}(k)$ of $\mathcal{SH}(k)$. Our main results here are

(1) For $\mathcal{E}$ in $\Sigma^d_{\mathbb{P}^1}\mathcal{SH}^{\text{eff}}(k)$, the maps in the tower

$$\phi_d\mathcal{E} \to \phi_{d-1}\mathcal{E} \to \ldots \to \mathcal{E}.$$

are all weak equivalences.

(2) For $\mathcal{E}$ in $\mathcal{SH}(k)$, $\phi_d\mathcal{E}$ is in $\Sigma^d_{\mathbb{P}^1}\mathcal{SH}^{\text{eff}}(k)$.

This allows us to make a comparison with Voevodsky's slice filtration. In [20], Voevodsky constructs the tower

$$\ldots \to f_{p+1}\mathcal{E} \to f_p\mathcal{E} \to \ldots \to f_0\mathcal{E} \to f_{-1}\mathcal{E} \to \ldots \to \mathcal{E}.$$

where the map $f_d\mathcal{E} \to \mathcal{E}$ is universal for maps $\mathcal{F} \to \mathcal{E}$ with $\mathcal{F}$ in $\Sigma^d_{\mathbb{P}^1}\mathcal{SH}^{\mathrm{eff}}(k)$. We show that the canonical map

$$\phi_d\mathcal{E} \to f_d\mathcal{E}$$

given by (2) is a weak equivalence, thus identifying the slice tower with the stable homotopy coniveau tower.

Finally, we compute of the 0th layer σ_0 in the homotopy coniveau tower for the motivic sphere spectrum $\mathbb{S}$, assuming that the base-field k is perfect. The idea here is that the cycle-like description of $E^{(p/p+1)}_{\mathrm{s.l.}}(X,-)$ enables one to define a "reverse cycle map"

$$\mathrm{rev} : \mathcal{H}\mathbb{Z} \to \sigma_0\mathbb{S}.$$

It is then rather easy to show that rev induces a weak equivalence after applying the 0th layer functor σ_0 again. However, since motivic cohomology is already the 0th layer of K-theory, applying σ_0 leaves $\mathcal{H}\mathbb{Z}$ unchanged, and similarly for $\sigma_0\mathbb{S}$, giving the desired weak equivalence $\mathcal{H}\mathbb{Z} \sim \sigma_0\mathbb{S}$. The analogous statement for the slice filtration in characteristic zero has been recently proven by Voevodsky [22] by a different method.

In any case, for a $\mathbb{P}^1$-spectrum $\mathcal{E}$, each layer $\sigma_p\mathcal{E}$ is a $\mathcal{H}\mathbb{Z}$-module. We thus have the objects $\pi^\mu_p\mathcal{E}$ of $\mathcal{DM}(k)$ whose Eilenberg-Maclane $\mathbb{P}^1$-spectrum satisfies

$$\mathcal{H}((\pi^\mu_p\mathcal{E})[p]) \cong \sigma_p\mathcal{E}$$

as $\mathcal{H}\mathbb{Z}$-modules. Here $\mathcal{H}$ is the the "Eilenberg-Maclane functor" from $\mathcal{DM}(k)$ to $\mathcal{SH}(k)$. For $E : \mathbf{Sm}_k \to \mathbf{Spt}$ the 0th S^1-spectrum of $\mathcal{E}$ and $X \in \mathbf{Sm}_k$, the spectral sequence associated to the homotopy coniveau tower can be expressed as

$$E_2^{p,q} := \mathbb{H}^p(X, \pi^\mu_{-q}\mathcal{E}) \Longrightarrow \hat{E}_{-p-q}(X).$$

Here $\hat{E}$ is the completion of E with respect to the homotopy coniveau tower. Under certain connectivity properties of E, one has $E = \hat{E}$.

Using the bi-graded homotopy groups of a $\mathbb{P}^1$-spectrum, this gives the weight-shifted spectral sequence

$$E_2^{p,q} = \mathbb{H}^p(X, (\pi^\mu_{-q}\mathcal{E}) \otimes \mathbb{Z}(b)) \Longrightarrow \hat{\mathcal{E}}^{p+q,b}(X).$$

For the K-theory $\mathbb{P}^1$-spectrum

$$\mathcal{K} := (K, K, \ldots),$$

our computation of the layers $K^{(p/p+1)}$ gives

$$\pi^\mu_p\mathcal{K}=\mathbb{Z}(p)[p];\quad p=0,\pm1,\pm2,\ldots,$$

and we recover the Bloch-Lichtenbaum, Friedlander-Suslin spectral sequence

$$E_2^{p,q}:=\mathbb{H}^{p-q}(X,\mathbb{Z}(-q))\Longrightarrow K_{-p-q}(X).$$

We have had a great deal of help in developing the techniques that went in to this paper. Fabien Morel played a crucial role in numerous discussions on the $\mathbb{A}^1$-stable homotopy category and related topics. Conversations with Bruno Kahn, Jens Hornbostel and Marco Schlichting were very helpful, as were the lectures of Bjorn Dundas and Vladamir Voevodsky at the Sophus Lie Summer Workshop in $\mathbb{A}^1$-stable homotopy theory. I would also like to thank Paul Arne Ostvar for his detailed comments on an earlier version of this manuscript. Finally, I am very grateful to the Humboldt Foundation and the Universitität Essen for their support of this reseach, especially my colleagues Hélène Esnault and Eckart Viehweg.

1. The homotopy coniveau tower

1.1. **Presheaves of simplicial sets.** Let **sSets** denote the category of simplicial sets, and **sSets**$*$ the category of pointed simplicial sets. For a category $\mathcal{C}$, we have the category $s\mathcal{C}$ of functors $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{sSets}$ (presheaves of simplicial sets), and $s_*\mathcal{C}$ of functors $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{sSets}*$ (presheaves of pointed simplicial sets).

We give **sSets** and **sSets**$*$ the standard model structures: cofibrations are (pointed) monomorphisms, weak equivalences are weak equivalences on the geometric realization, and fibrations are detemined by the RLP with respect to trivial cofibrations; the fibrations are then exactly the Kan fibrations. We let $|A|$ denote the geometric realization, and $[A, B]$ the homotopy classes of (pointed) maps $|A| \to |B|$.

We give $s\mathcal{C}$ and $s_*\mathcal{C}$ the model structure of functor categories described by Bousfield-Kan [4]. That is, the cofibrations and weak equivalences are the pointwise ones, and the fibrations are determined by the RLP with respect to trivial cofibrations. We let $\mathcal{H}s\mathcal{C}$ and $\mathcal{H}s_*\mathcal{C}$ denote the associated homotopy cateogies.

1.2. **Spectra.** Let **Spt** denote the category of spectra. To fix ideas, a spectrum will be a sequence of pointed simplicial sets $E_0, E_1, \ldots$ together with maps of pointed simiplicial sets $\epsilon_n : S^1 \wedge E_n \to E_{n+1}$. Maps of spectra are maps of the underlying simplicial sets which are compatible with the attaching maps ϵ_n. The stable homotopy groups $\pi_n^s(E)$ are defined by

$$\pi_n^s(E) := \lim_{m\to\infty} [S^{m+n}, E_m].$$

The category **Spt** has the following model structure: Cofibrations are maps $f : E \to F$ such that $E_0 \to F_0$ is a cofibration, and for each $n \geq 0$, the map

$$E_{n+1} \coprod_{S^1\wedge E_n} S^1 \wedge F_n \to F_{n+1}$$

is a cofibration. Weak equivalences are the stable weak equivalences, i.e., maps $f : E \to F$ which induce an isomorphism on π_n^s for all n. Fibrations are characterized by having the RLP with respect to trivial cofibrations.

Let $\mathcal{C}$ be a category. We say that a natural transformation $f : E \to E'$ of functors $\mathcal{C}^{\mathrm{op}} \to \mathbf{Spt}$ is a weak equivalence if $f(X) : E(X) \to E'(X)$ is a stable weak equivalence for all X.

We use the following model structure on the category of functors $\mathcal{C}^{\mathrm{op}} \to \mathbf{Spt}$ (see [9]): Cofibrations and weak equivalences are given pointwise, and fibrations are characterized by having the RLP with respect to trivial cofibrations. We denote this model category by $\mathbf{Spt}(\mathcal{C})$, and the associated homotopy category by $\mathcal{H}\mathbf{Spt}(\mathcal{C})$. We write $\mathcal{SH}$ for the homotopy category of **Spt**.

1.3. **Simplicial spectra.** For a spectrum E, we have the Postnikov tower

$$\begin{array}{ccccc} \cdots \longrightarrow \tau_{\geq N}E & & \longrightarrow & & \tau_{\geq N-1}E \longrightarrow \cdots \\ & \searrow & & \swarrow & \\ & & E & & \end{array}$$

with $\tau_{\geq N}E \to E$ the $N-1$-connected cover of E, i.e., $\tau_{\geq N}E \to E$ is an isomorphism on homotopy groups π_n for $n \geq N$, and $\pi_n(\tau_{\geq N}E) = 0$ for $n < N$. One can make this tower functorial in E, so we can apply the construction $\tau_{\geq N}$ to functors $E : \mathcal{C} \to \mathbf{Spt}$.

We have the category **Ord** with objects the finite ordered sets $[n] := \{0 < \ldots < n\}$, $n = 0, 1, \ldots$, and maps order-preserving maps of sets. Let $\mathbf{Ord}_{\leq N}$ be the full subcategory with objects $[n]$, $0 \leq n \leq N$.

Let $E : \mathbf{Ord}^{\mathrm{op}} \to \mathbf{Spt}$ be a simplicial spectrum. We have the N-truncated simplicial spectrum $E_{\leq N} : \mathbf{Ord}^{\mathrm{op}}_{\leq N} \to \mathbf{Spt}$, the associated total spectrum $|E_{\leq N}|$, and the tower of spectra

$$|E_{\leq 0}| \to \ldots \to |E_{\leq N}| \to \ldots \to |E| \tag{1.3.1}$$

Since taking the total spectrum commutes with filtered colimits, we have the natural weak equivalences

$$\begin{array}{ccc} \operatorname{hocolim}_M |E_{\leq M}| & \xrightarrow{\sim} & |E| \\ \sim\uparrow & & \uparrow\sim \\ \operatorname{hocolim}_{N,M} \tau_{\geq -N}|E_{\leq M}| & \xrightarrow[\sim]{} & \operatorname{hocolim}_N \tau_{\geq -N}|E|. \end{array}$$

When the context makes the meaning clear, we will often omit the separate notation for the total spectrum, and freely pass between a simplicial spectrum and its associated total spectrum.

1.4. **The homotopy coniveau tower.** We fix a base scheme S, and let $\mathbf{Sm}_S$ denote the category of smooth S-schemes, of finite type over S. We will assume that S is noetherian, of finite Krull dimension and separated. We often write $\mathbf{Spt}(S)$ and $\mathcal{H}\mathbf{Spt}(S)$ for $\mathbf{Spt}(\mathbf{Sm}_S)$ and $\mathcal{H}\mathbf{Spt}(\mathbf{Sm}_S)$.

We have the cosimplicial scheme Δ^*, with

$$\Delta^r = \operatorname{Spec}\,(\mathbb{Z}[t_0, \ldots, t_r]/\sum_j t_j - 1).$$

The *vertices* of Δ^r are the closed subschemes v_i^r defined by $t_i = 1$, $t_j = 0$ for $j \neq i$. A *face* of Δ^r is a closed subscheme defined by equations of the form $t_{i_1} = \ldots = t_{i_s} = 0$.

Let $E : \mathbf{Sm}_S^{\mathrm{op}} \to \mathbf{Spt}$ be a functor. For X in $\mathbf{Sm}_S$ with closed subscheme W and open complement $j : X \setminus W \to X$, we let $E^W(X)$ denote the homotopy fiber of $j^* : E(X) \to E(X \setminus W)$ (in $\mathbf{Spt}$). If we have a chain of closed subsets $W' \subset W \subset X$, we have a natural map $i_{W',W*} : E^{W'}(X) \to E^W(X)$ and a natural weak equivalence

$$\text{(1.4.1)} \qquad \mathrm{cofib}(i_{W',W*} : E^{W'}(X) \to E^W(X)) \sim E^{W\setminus W'}(X \setminus W').$$

Here "cofib" means homotopy cofiber in the category of spectra.

For X in $\mathbf{Sm}_S$, we let $\mathcal{S}_X^{(p)}(r)$ denote the set of closed subsets W of $X \times \Delta^r$ such that

$$\mathrm{codim}_{X\times F}(W \cap (X \times F)) \geq p$$

for all faces F of Δ^r. Clearly, sending r to $\mathcal{S}_X^{(p)}(r)$ defines a simplicial set $\mathcal{S}_X^{(p)}(-)$. We let $X^{(p)}(r)$ be the set of codimension p points x of $X \times \Delta^r$ with closure $\bar{x} \in \mathcal{S}_X^{(p)}(r)$.

We let $E^{(p)}(X, r)$ denote the (filtered) limit

$$E^{(p)}(X, r) = \operatorname*{hocolim}_{W \in \mathcal{S}_X^{(p)}(r)} E^W(X \times \Delta^r).$$

Sending r to $E^{(p)}(X, r)$ defines a simplicial spectrum $E^{(p)}(X, -)$. Since $\mathcal{S}_X^{(p+1)}(r)$ is a subset of $\mathcal{S}_X^{(p)}(r)$, we have the tower of simplicial spectra

$$\text{(1.4.2)} \qquad \ldots \to E^{(p+1)}(X, -) \to E^{(p)}(X, -) \to \ldots \to E^{(0)}(X, -),$$

which we call the *homotopy coniveau tower*. We let $E^{(p/p+1)}(X, -)$ denote the cofiber of the map $E^{(p+1)}(X, -) \to E^{(p)}(X, -)$.

Our first axioms are:

A1. E is *homotopy invariant*: For each X in $\mathbf{Sm}_S$, the map $p^* : E(X) \to E(X \times \mathbb{A}^1)$ is a weak equivalence.

A2. E satisfies *Nisnevic excision*: Let $f : X' \to X$ be an étale morphism in $\mathbf{Sm}_S$, and $W \subset X$ a closed subset. Let $W' = f^{-1}(W)$, and suppose that f restricts to an isomorphism $W' \to W$. Then $f^* : E^W(X) \to E^{W'}(X')$ is a weak equivalence.

Definition 1.4.1. Let X be in $\mathbf{Sm}_S$. The *weight-completed* spectrum $\hat{E}(X)$ is

$$\hat{E}(X) = \operatorname*{holim}_p E^{(0/p)}(X, -).$$

Proposition 1.4.2. *Suppose that E satisfies axiom 1. Then there is a weakly convergent spectral sequence*

$$E_1^{p,q} = \pi_{-p-q}(E^{(p/p+1)}(X,-)) \Longrightarrow \hat{E}_{-p-q}(X). \tag{1.4.3}$$

If $E = \tau_{\geq N}E$ for some N, then $\hat{E} \sim E$ and the above spectral sequence is strongly convergent.

Proof. The spectral sequence is constructed by the standard process of linking the long exact sequences of homotopy groups arising from the homotopy cofiber sequences $E^{(p+1)}(X,-) \to E^{(p)}(X,-) \to E^{(p/p+1)}(X,-)$. The first assertion then follows from the general theory of homotopy limits (see [4]).

For the second, suppose $E = \tau_{\geq N}E$. We first show that the sequence is strongly convergent.

By (1.4.1) and a limit argument, we have

$$\pi_m(E^{(p/p+1)}(X,r)) = \lim_{\substack{\to \\ W'\subset W}} \pi_m\big(E^{W\setminus W'}(X\times\Delta^r\setminus W')\big),$$

where the limit is over $W' \in \mathcal{S}_X^{(p+1)}(r)$, $W \in \mathcal{S}_X^{(p)}(r)$. It follows that $\pi_m(E^{(p/p+1)}(X,r)) = 0$ for $m < N$.

From the tower (1.3.1), we thus have the the strongly convergent spectral sequence

$$E_1^{a,b} = \pi_{-a}(E^{(p/p+1)}(X,-b)) \Longrightarrow \pi_{-a-b}(E^{(p/p+1)}(X,-)).$$

Since $\mathcal{S}_X^{(p)}(r) = \emptyset$ for $p > \dim X + r$, this implies that

$$\pi_{-p-q}E^{(p/p+1)}(X,-) = 0$$

for $p > -p-q+\dim X + N$, from which it follows that the spectral sequence (1.4.3) is strongly convergent.

Similarly, it follows that the natural map $E^{(0)}(X,-) \to \hat{E}(X)$ is a weak equivalence. The simplicial spectrum $E^{(0)}(X,-)$ is just the simplicial spectrum $E(X\times\Delta^*)$, i.e., $r \mapsto E(X\times\Delta^r)$. Since E is homotopy invariant, the natural map

$$E(X) \to E(X\times\Delta^*)$$

is a weak equivalence, completing the proof. □

1.5. **First properties.** We give a list of elementary properties of the spectra $E^{(p)}(X,-)$

(1) Sending X to $E^{(p)}(X,-)$ is functorial for equi-dimensional (e.g. flat) maps $Y \to X$ in $\mathbf{Sm}_S$.

(2) The pull-back $p_1^* : E^{(p)}(X, -) \to E^{(p)}(X \times \mathbb{A}^1, -)$ is a weak equivalence. The proof is the same as that for Bloch's cycle complexes, given in [1].
(3) Sending E to $E^{(p)}(X, -)$ is functorial in E.
(4) The functor $E \mapsto E^{(p)}(X, -)$ sends weak equivalences to weak equivalences, and send homotopy (co)fiber sequences to homotopy (co)fiber sequences.

Exactly the same properties hold for the layers $E^{(p/p+r)}$.

2. Localization

We now show that the simplicial spectra $E^{(p)}(X, -)$ behave well with respect to localization.

2.1. **Stable homology of spectra.** For a simplicial set S, we have the simplicial abelian group $\mathbb{Z}S$, with n-simplices $\mathbb{Z}S_n$ the free abelian group on S_n. Let $E = \{E_n, \phi_n : \Sigma E_n \to E_{n+1}\}$ be a spectrum; we take the E_n to be pointed simplicial sets, and the ϕ_n to be maps of pointed simplicial sets. Form the spectrum $\mathbb{Z}E$ by taking $(\mathbb{Z}E)_n = \mathbb{Z}E_n$, where $\mathbb{Z}\phi_n : \Sigma(\mathbb{Z}E)_n \to (\mathbb{Z}E)_{n+1}$ is the map induced by ϕ_n, composed with the natural map $\Sigma(\mathbb{Z}E)_n \to \mathbb{Z}(\Sigma E_n)$. The natural maps $E_n \to \mathbb{Z}E_n$ give a natural map $E \to \mathbb{Z}E$ of spectra; one shows that this construction respects weak equivalence and taking homotopy cofibers (hence also homotopy fibers). The *stable homology* $H_n(E)$ is defined by $H_n(E) = \pi_n(\mathbb{Z}E)$. Using the Dold-Thom theorem, one has the formula for $H_n(E)$ as

$$H_n(E) = \lim_{\to} \tilde{H}_{n+m}(E_m),$$

where the maps $\tilde{H}_{n+m}(E_n) \to \tilde{H}_{n+m+1}(E_{m+1})$ are given by the composition

$$\tilde{H}_{n+m}(E_n) \cong \tilde{H}_{n+m+1}(\Sigma E_n) \xrightarrow{\phi_{n*}} \tilde{H}_{n+m+1}(E_{n+1}).$$

The Hurewicz theorem for simplicial sets gives the following analogous result for spectra:

Proposition 2.1.1. *Let E be a spectrum which is N-connected for some $N \in \mathbb{Z}$. Then $\pi_n(E) = 0$ for all n if and only if $H_n(E) = 0$ for all n.*

Proof. Since both π_n and H_n respect weak equivalence, and are compatible with suspension of spectra, we may assume that $N \geq 1$, and that E is an Ω-spectrum, i.e., the natural maps $E_n \to \Omega E_{n+1}$ are weak equivalences. Then $\pi_n(E) = \pi_{n+m}(E_m)$ for all m. Suppose $H_n(E) = 0$ for all n; we prove by induction that $\pi_{n+m}(E_m) = 0$ for all n and m.

By assumption $\pi_{N+m}(E_m) = 0$ for all m, with $N \geq 1$. We may therefore proceed by induction on n to show that $\pi_{n+m}(E_m) = 0$ for all n and m. Supposing that $\pi_{n+m-1}(E_m) = 0$ for all m, the Hurewizc theorem implies that the Hurewicz map $\pi_{n+m}(E_m) \to \tilde{H}_{n+m}(E_m)$ is an isomorphism for all m, and one easily checks that the Hurewicz map is compatible with the limits defining H_n and π_n. Thus, the maps $\tilde{H}_{n+m}(E_m) \to \tilde{H}_{n+m+1}(E_{m+1})$ are isomorphisms for all m; since the limit is zero by assumption, this implies that $\tilde{H}_{n+m}(E_m) = 0$ for all m, whence $\pi_{n+m}(E_m) = 0$ for all m.

The proof that $\pi_n(E) = 0$ for all n implies $H_n(E) = 0$ for all n is similar, but easier, and is left to the reader. □

2.2. **The localization theorem.** In this section, E will be a functor $E : \mathbf{Sm}_S^{\mathrm{op}} \to \mathbf{Spt}$ satisfying axiom 2.

Let X be smooth and essentially of finite type over S, and let $j : U \to X$ be an open subscheme, with complement $i : Z \to X$. We let $\mathcal{S}^{(p)}_{X,Z}(r)$ denote the subset of $\mathcal{S}^{(p)}_X(r)$ consisting of those W contained in $Z \times \Delta^r$. Let $\mathcal{S}^{(p)}_{U/X}(r)$ be the image of $\mathcal{S}^{(p)}_X(r)$ in $\mathcal{S}^{(p)}_U(r)$ under $(j \times \mathrm{id})^{-1}$. Taking the colimit of $E^W(X \times \Delta^r)$ over $W \in \mathcal{S}^{(p)}_{X,Z}(r)$ and varying r and p gives us the tower of simplicial spectra

$$\ldots \to E_Z^{(p+1)}(X, -) \to E_Z^{(p)}(X, -) \to \ldots \to E_Z^{(d)}(X, -) = E_Z^{(0)}(X, -),$$

where d is any integer satisfying $d \leq \mathrm{codim}_X Z_j$ for all irreducible components Z_j of Z. Similarly, taking the colimit over $W \in \mathcal{S}^{(p)}_{U/X}(r)$ for varying p and r gives the tower of simplicial spectra

$$\ldots \to E^{(p+1)}(U_X, -) \to E^{(p)}(U_X, -) \to \ldots \to E^{(0)}(U_X, -).$$

We have as well the natural maps

$$\begin{aligned} i_* &: E_Z^{(p)}(X, r) \to E^{(p)}(X, r), \ j^{*!} : E^{(p)}(X, r) \to E^{(p)}(U_X, r), \\ \iota &: E^{(p)}(U_X, r) \to E^{(p)}(U, r), \ j^* : E^{(p)}(X, r) \to E^{(p)}(U, r), \end{aligned}$$

with $j^* = \iota \circ j^{*!}$.

Let $E^{(p/p+s)}(-)$ denote the cofiber of the maps $E^{(p+s)}(-) \to E^{(p)}(-)$. Since E satisfies axiom 2, we have the homotopy fiber sequences

$$E_Z^{(p)}(X, r) \xrightarrow{i_*} E^{(p)}(X, r) \xrightarrow{j^{*!}} E^{(p)}(U_X, r)$$
$$E_Z^{(p/p+s)}(X, r) \xrightarrow{i_*} E^{(p/p+s)}(X, r) \xrightarrow{j^{*!}} E^{(p/p+s)}(U_X, r).$$

These give the homotopy fiber sequences of simplicial spectra

$$E_Z^{(p)}(X,-) \xrightarrow{i_*} E^{(p)}(X,-) \xrightarrow{j^{*!}} E^{(p)}(U_X,-)$$
$$E_Z^{(p/p+s)}(X,-) \xrightarrow{i_*} E^{(p/p+s)}(X,-) \xrightarrow{j^{*!}} E^{(p/p+s)}(U_X,-)$$

The localization theorem is

Theorem 2.2.1. *Suppose the base-scheme S is a scheme essentially of finite type over a semi-local DVR with infinite residue fields. Then the maps*

$$E^{(p)}(U_X,-) \to E^{(p)}(U,-)$$
$$E^{(p/p+s)}(U_X,-) \to E^{(p/p+s)}(U,-)$$

are weak equivalences

Proof. The second weak equivalence follows from the first by taking cofibers.

For the first map, this result follows by exactly the same method as used in the proof of [13, Theorem 8.10]. Indeed, to show that the map $E^{(p)}(U_X,-) \to E^{(p)}(U,-)$ is a weak equivalence, it suffices to prove the result with $E^{(p)}(-,n)$ replaced by $\tau_{\geq N}E^{(p)}(-,n)$ for all N, and thus we may assume that $E^{(p)}(-,n)$ is N-connected for some N. By the Hurewicz theorem (Proposition 2.1.1), it suffices to show that $E^{(p)}(U_X,-) \to E^{(p)}(U,-)$ is a homology isomorphism. This follows by applying [13, Theorem 8.2], just as in the proof of Theorem 8.10 (*loc. cit.*). □

3. Functoriality and Chow's moving lemma

Fix a field k.

3.1. T-loop spaces and $\mathbb{P}^1$-loop spaces.

Definition 3.1.1. For a functor $E : \mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{Spt}$, we let

$$\Omega_T E : \mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{Spt}$$

be the functor

$$\Omega_T E(X) := E^{X\times 0}(X \times \mathbb{P}^1) = \mathrm{fib}(E(X \times \mathbb{P}^1) \xrightarrow{\mathrm{res}} E(X \times (\mathbb{P}^1 \setminus 0)).$$

Define the functor $\Omega_{\mathbb{P}^1} E$ by

$$\Omega_{\mathbb{P}^1} E(X) := \mathrm{fib}(E(X \times \mathbb{P}^1) \xrightarrow{\mathrm{res}} E(X \times \infty)).$$

Remarks 3.1.2. (1) If E satisfies axioms 1 and 2, so do $\Omega_T E$ and $\Omega_{\mathbb{P}^1} E$.
(2) The commutative diagram

$$\begin{array}{ccc} E(X \times \mathbb{P}^1) & \xrightarrow{\text{res}} & E(X \times (\mathbb{P}^1 \setminus 0)) \\ \| & & \downarrow{\scriptstyle \text{res}} \\ E(X \times \mathbb{P}^1) & \xrightarrow{\text{res}} & E(X \times \infty) \end{array}$$

gives us the homotopy fiber sequence

$$\Omega_T E(X) \to \Omega_{\mathbb{P}^1} E(X) \to \mathrm{fib}(E(X \times (\mathbb{P}^1 \setminus 0)) \xrightarrow{\text{res}} E(X \times \infty)).$$

If E satisfies axiom 1, $\mathrm{fib}(E(X \times (\mathbb{P}^1 \setminus 0)) \xrightarrow{\text{res}} E(X \times \infty))$ is weakly contractible, hence the natural map $\Omega_T E \to \Omega_{\mathbb{P}^1} E$ is a weak equivalence.

Fix a scheme X in $\mathbf{Sm}_k$. We may restrict E to $\mathbf{Sm}_X$, giving the functor $E_X : \mathbf{Sm}_X^{\mathrm{op}} \to \mathbf{Spt}$. If we have a closed subset Z of X, we have the functor

$$(f : U \to X) \mapsto E^{f^{-1}(Z)}(U),$$

which we denote by E_X^Z. If $f : Y \to X$ is a morphism in $\mathbf{Sm}_k$, we have the pushforward $f_* : \mathbf{Spt}(Y) \to \mathbf{Spt}(X)$, defined by

$$f_* F(U \to X) := F(U \times_X Y).$$

Clearly, f_* preserves weak equivalences, hence descends to

$$f_* : \mathcal{H}\mathbf{Spt}(Y) \to \mathcal{H}\mathbf{Spt}(X).$$

Lemma 3.1.3. *Let $i : Z \to X$ be a codimension d closed embedding, with X and Z in $\mathbf{Sm}_k$. Suppose that $E : \mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{Spt}$ satisfies axioms 1 and 2, and that the normal bundle $N_{Z/X}$ is trivial. Then a choice of isomorphism $\phi : N_{Z/X} \cong Z \times \mathbb{A}^d$ determines a natural isomorphism in $\mathcal{H}\mathbf{Spt}(X)$,*

$$\omega_\psi : E_X^Z \to i_*(\Omega_T^d E_Z),$$

natural in (Z, X, ϕ).

Proof. By axiom 2, the inclusion $\mathbb{A}^d \to (\mathbb{P}^1)^d$ induces a natural weak equivalence

$$\Omega_T^d E_Z(Y) \to (E_Z)^{Y \times 0}(Y \times \mathbb{A}^d).$$

Let $s : Z \to N_{Z/X}$ be the zero-section. By taking a deformation to the normal bundle, as in [16], $E^Z(X)$ is naturally isomorphic to $E^{s(Z)}(N_{Z/X})$ in $\mathcal{H}\mathbf{Spt}$. The chosen isomorphism $\phi : N_{Z/X} \cong Z \times \mathbb{A}^d$ sends $s(Z)$ over to $Z \times 0$. As the deformation diagram is preserved by pullback with respect to a smooth $U \to X$, the result is proved. $\square$

This immediately yields

Proposition 3.1.4. *Let $E : \mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{Spt}$ be a functor satisfying axioms 1 and 2. Let $i : Z \to X$ be a codimension d closed embedding in $\mathbf{Sm}_k$ such that the normal bundle $N_{Z/X}$ is trivial. Then for all $p \geq 0$ we have isomorphisms in $\mathcal{SH}$:*

$$E_Z^{(p)}(X, -) \sim (\Omega_T^d E)^{(p-d)}(Z, -)$$
$$E_Z^{(p/p+1)}(X, -) \sim (\Omega_T^d E)^{(p-d/p-d+1)}(Z, -),$$

where, for $n < 0$, we set $(\Omega_T^d E)^{(n)} = (\Omega_T^d E)^{(0)}$ and $E^{(n/n+1)} = *$. The isomorphisms may depend on the choice of trivialization of $N_{Z/X}$, but are natural in the category of closed embeddings i with trivialization of N_i.

We also have

Corollary 3.1.5. *Let X be in $\mathbf{Sm}_k$, and let E be as in Proposition 3.1.4. For each $N \geq 0$, there is a spectral sequence*

$$E^1_{p,q}(E) := \oplus_{x \in X^{(p)}} \pi_{p+q} \Omega_T^p E^{(N-p/N-p+s)}(k(x), -) \Longrightarrow \pi_{p+q} E^{(N/N+s)}(X, -).$$

Proof. This follows from the localization property Theorem 2.2.1 and Proposition 3.1.4 by the usual limit process. □

3.2. **Push-forward.** It will be useful to know that the simplicial spectra $E^{(p)}(X, -)$ satisfy the analog of "Chow's moving lemma" for Bloch's cycle complexes $z^p(X, *)$. For this, we need an additional axiom.

A3. There is a functor $E_2 : \mathbf{Sm}_S^{\mathrm{op}} \to \mathbf{Spt}$ satisfying the axioms 1 and 2 and a natural weak equivalence

$$\sigma : E \to \Omega_T^2(E_2).$$

Remarks 3.2.1. (1) Suppose E satisfies axiom 3. Set $E_0 := E$, $E_1 := \Omega_T E_2$. Then we have weak equivalences

$$\epsilon^d : E \to \Omega_T^d(E_d); \quad d = 0, 1, 2,$$

and a diagram of weak equivalences (for $d = 2$)

$$\begin{array}{ccc} E & \xrightarrow{\epsilon^1} & \Omega_T^1 E_1 \\ & \searrow^{\epsilon^2} & \uparrow{\scriptstyle \rho_2} \\ & & \Omega_T^2 E_2 \end{array}$$

which is commutative in $\mathcal{SH}$. In addition, E_0 and E_1 satisfy axioms 1 and 2.

(2) Letting R_X be the ring of global functions on X, we have the action of $\mathrm{GL}_d(R_X)$ on $X \times \mathbb{A}^d$, fixing $X \times 0$. Via ϵ_X^d, this gives an action of $\mathrm{GL}_d(R_X)$ (in$\mathcal{H}\mathbf{Spt}(X)$) on E_X, natural in X and stable in d (for $d \leq 2$):

$$g \mapsto \Psi^g : E_X \to E_X.$$

In general, Ψ^g is not homotopically trivial.

(3) We may replace E, E_1 and E_2 with bifibrant models (in $\mathbf{Spt}(k)$). Then axiom 3 holds with "weak equivalence" replaced by "homotopy equivalence" the diagram in (1) is commutative, up to homotopy, and the map Ψ^g in (2) exists as a homotopy equivalence in $\mathbf{Spt}(X)$.

Let $s : T \to U$ be a closed embedding in $\mathbf{Sm}_k$. We let $E(U/T)$ denote the fiber of the restriction map

$$i^* : E(U) \to E(T).$$

In particular, if $W \subset U$ is a closed subset disjoint from T, then, as $U \setminus W \supset T$, we have the natural map

$$E^W(U) \to E(U/T).$$

Remark 3.2.2. Let $i : \mathbb{P}^{d-1} \to \mathbb{P}^d$ be the inclusion of a linear subspace. Let $i_0 : \mathbb{P}^0 \to \mathbb{P}^d$ be a complementary point and let $V = \mathbb{P}^d \setminus \mathbb{P}^0 \supset \mathbb{P}^{d-1}$. The linear projection from $\mathbb{P}^0$ identifies V with the space of the tautological line bundle $O(1)$ on $\mathbb{P}^{d-1}$; since E is homotopy invariant and satisfies Zariski Mayer-Vietoris, it follows that the restriction map $E(V) \to E(\mathbb{P}^{d-1})$ is a weak equivalence. Thus, the map

$$E^{\mathbb{P}^0}(\mathbb{P}^d) \to E(\mathbb{P}^d/\mathbb{P}^{d-1})$$

is a weak equivalence.

Proposition 3.2.3. *Let $f : Y \to X$ be a finite morphism in $\mathbf{Sm}_k$, which we factor as $f = p \circ i$, where $i : Y \to X \times \mathbb{A}^d$ is a closed embedding with $d \leq 2$, and $p : X \times \mathbb{A}^d \to X$ is the projection. Suppose that the normal bundle of i is trivial; let $\psi : N_i \to Y \times \mathbb{A}^d$ be an isomorphism. Then*

1. *there is natural transformation in $\mathcal{H}\mathbf{Spt}(X)$,*
$$f_* : f_*E_Y \to E_X.$$
2. *Let $Z \to X$ be smooth, and let $W \subset Y \times_X Z$ be a closed subset. Suppose that $p_2 : Y \times_X Z \to Z$ is étale on a neighborhood of W, and $W \to W' := p_2(W)$ is an isomorphism. The map*
$$f(Z)_*^W : E^W(Y \times_X Z) \to E^{W'}(Z)$$
induced by f_* is an isomorphism in $\mathcal{SH}$.

(3) *Suppose we factor p as $p_2 \circ p_1$, where $p_1 : X \times \mathbb{A}^2 \to X \times \mathbb{A}^1$ is a split projection of vector bundles over X and $p_2 : X \times \mathbb{A}^1 \to X$ is the projection. Suppose further that $p_1 \circ i$ is an embedding with trivialized normal bundle, and that the induced split surjection $N_i \to N_{p_1 \circ i}$ is compatible with the choice of trivializations. Then using i or $p_1 \circ i$ gives rise to the same natural transformation f_*.*

Proof. (1) Let $p : X \times \mathbb{A}^d \to X$ be the projection. The embedding i induces for each smooth $Z \to X$ an embedding $i_Z : Y \times_X Z \to Z \times \mathbb{A}^d$. Similarly, a fixed choice of trivialization $\phi : N_i \to Y \times \mathbb{A}^d$ induces a trivialization of N_{i_Z}, natural in Z. Thus, by Lemma 3.1.3, we have the isomorphism in $\mathcal{H}\mathbf{Spt}(X)$

$$p_*(\omega_\psi^{-1}) : f_*((\Omega_T^d E_d)_Y) \to p_*((E_d)^Y_{X \times \mathbb{A}^d}).$$

Composing with $f_* \epsilon^d$, we have the isomorphism in $\mathcal{H}\mathbf{Spt}(X)$

$$\Psi : f_* E_Y \to p_*((E_d)^Y_{X \times \mathbb{A}^d}).$$

Fix homogeneous coordinates $x_0, \ldots, x_d$ for $\mathbb{P}^d$, and identify $\mathbb{A}^d$ with the open subscheme $x_0 \neq 0$ via the coordinates $x_1/x_0, \ldots, x_d/x_0$. Let $\mathbb{P}^{d-1} \to \mathbb{P}^d$ be the complement, and let $\mathbb{P}^0 \to \mathbb{P}^d$ be the point $(1 : 0 : \ldots : 0)$. For each smooth $Z \to X$, we have the diagram (where $\sim$ denotes a weak equivalence)

$$\begin{aligned}(3.2.1)\quad (E_d)^{Y \times_X Z}(Z \times \mathbb{A}^d) &\xleftarrow{\sim} (E_d)^{Y \times_X Z}(Z \times \mathbb{P}^d) \\ &\to E_d(Z \times \mathbb{P}^d / Z \times \mathbb{P}^{d-1}) \xleftarrow{\sim} (E_d)^{Z \times \mathbb{P}^0}(Z \times \mathbb{P}^d) \\ &= \Omega_T^d(E_d)(Z) \underset{\epsilon^d}{\xleftarrow{\sim}} E(Z).\end{aligned}$$

Let $\Phi_Z : (E_d)^{Y \times_X Z}(Z \times \mathbb{A}^d) \to E(Z)$ be the resulting map in $\mathcal{SH}$. As the diagram (3.2.1) is natural with respect to X-morphisms $Z' \to Z$, we have the map in $\mathcal{H}\mathbf{Spt}(X)$,

$$\Phi : p_*((E_d)^Y_{X \times \mathbb{A}^d}) \to E_X,$$

with $\Phi(Z) = \Phi_Z$ for each $Z \to X$ in $\mathbf{Sm}_X$. Setting

$$f_* := \Phi \circ \Psi$$

gives the desired map.

Before proceeding to the proof of (2), consider a morphism $g : Z' \to Z$ in $\mathbf{Sm}_X$. Since f_* is a natural transformation, the diagram

$$
\begin{array}{ccc}
E(Y \times_X Z) & \xrightarrow{f_*(Z)} & E(Z) \\
{\scriptstyle (\mathrm{id}\times g)^*}\downarrow & & \downarrow{\scriptstyle g^*} \\
E(Y \times_X Z') & \xrightarrow[f_*(Z')]{} & E(Z')
\end{array}
\tag{3.2.2}
$$

commutes.

To prove (2), we may assume $Z = X$. Let $U \subset Y$ be a neighborhood of W over which f is étale, and let $\hat{W} \subset Y \times_X U$ be the image of W under the evident section $U \to Y \times_X U$. From (3.2.2), we have the commutative diagram

$$
\begin{array}{ccc}
E^W(Y) & \xrightarrow{f_*(U)^W} & E^{W'}(U) \\
{\scriptstyle (\mathrm{id}\times g)^*}\downarrow & & \downarrow{\scriptstyle g^*} \\
E^{\hat{W}}(Y \times_X U) & \xrightarrow[f_*(X)^{\hat{W}}]{} & E^W(U)
\end{array}
$$

Since $(\mathrm{id} \times g)^*$ and g^* are isomorphisms (by axiom 2), we may replace X with U; changing notation, we may assume that f admits a section $s : X \to Y$. Removing the complement of $s(X)$, we may assume that $f : Y \to X$ is an isomorphism, so we may assume $X = Y$ and $f = \mathrm{id}_X$.

Translating by $-i$ sends the embedding $i : X \to X \times \mathbb{A}^d$ to the zero-section. Since this translation sits in an $\mathbb{A}^1$-family of automorphisms of $X \times \mathbb{A}^d$, the automorphisms $f(i)_*$ and $f(0)_*$ of $E^W(X)$ agree (in $\mathcal{SH}$). Thus, we may assume that i is the zero-section. We thus have a canonical identification of N_i with $X \times \mathbb{A}^d$; composing with ψ defines an element $\gamma \in \mathrm{GL}_d(R_X)$.

Tracing through the definition of f_* in case $f = \mathrm{id}$, $i : X \to X \times \mathbb{A}^d$ is the zero-section, and ψ is a given trivialization of N_i, we find that $f_* = \Psi^\gamma$. As Ψ^γ is an isomorphism, (2) is proved.

The assertion (3) follows directly from the definitions. $\square$

Suppose for example that X is affine, $X = \mathrm{Spec}\,(R)$, and that Y is embedded as a closed codimension one subscheme of $X \times \mathbb{A}^1 = \mathrm{Spec}\,(R[t])$, with chosen defining equation $g(t)$. Assuming X and Y are smooth over k, the differential dg gives a trivalization of the conormal bundle of Y in $X \times \mathbb{A}^1$. We let $f(g)_*$ denote the push-forward defined using this embedding i and the induced trivialization of the normal bundle N_i.

3.3. **Chow's moving lemma.** We can now present the proof of the moving lemma. First we show

Lemma 3.3.1. *Let k be a field, X a smooth k-scheme of dimension d over k, and x a point of X. Then there is a neighborhood basis for $x \in X$ consisting of affine open subschemes U such that U admits a closed embedding in $\mathbb{A}_k^{d+2}$ with trivial normal bundle.*

Proof. We may assume that X is affine, so X admits a closed embedding in $\mathbb{A}_k^N$ for some N. By the technique of Noether normalization, X admits a finite birational morphism onto a hypersurface $\bar{X} \subset \mathbb{A}_k^{d+1}$ which is an isomorphism in a neighborhood of x; in particular, x admits a neighborhood basis consisting of affine open subschemes U of x which are isomorphic to a principal open subscheme $\bar{X}_f$ for some $f \in k[x_1, \ldots, x_{d+1}]$ (where $f = 0$ of course contains the singular locus of $\bar{X}$).

Note that $\operatorname{Spec}(k[x_1, \ldots, x_{d+1}][1/f])$ embeds in $\mathbb{A}_k^{d+2}$ as the hypersurface defined by $x_{d+2}f - 1 = 0$. Clearly, this closed subscheme has trivial normal bundle in $\mathbb{A}^{d+2}$. As $\bar{X}_f \subset \mathbb{A}_k^{d+1} \setminus \{f = 0\}$ has trivial normal bundle (the conormal bundle is generated by dg, if g is a defining equation for $\bar{X}$), it follows that $\bar{X}_f \subset \mathbb{A}_k^{d+2}$ has trivial normal bundle as well. □

Definition 3.3.2. Let $f : Y \to X$ be a morphism of smooth k-schemes. Let $\mathcal{S}_X^{(p)}(r)_f$ be the subset of $\mathcal{S}_X^{(p)}(r)$ consisting of closed subsets $W \subset X \times \Delta^r$ such that $(f \times \mathrm{id})^{-1}(W)$ is in $\mathcal{S}_Y^{(p)}(r)$.

Replacing $\mathcal{S}_X^{(p)}(r)$ with $\mathcal{S}_X^{(p)}(r)_f$, we have the tower of simplicial spectra

$$\ldots \to E_f^{(p+1)}(X, -) \to E_f^{(p)}(X, -) \to \ldots \to E_f^{(0)}(X, -) \sim E(X).$$

which maps to the tower (1.4.2). We let $E_f^{(p/p+r)}(X, -)$ denote the cofiber of $E_f^{(p+r)}(X, -) \to E_f^{(p)}(X, -)$.

We use the notations from [12, Moving Lemma]. We suppose that k is an infinite field. Let $X \subset \mathbb{A}_k^{d+2}$ be a smooth closed subscheme of dimension d over k. We suppose that X has trivial normal bundle in $\mathbb{A}^{d+2}$ and we fix a trivialization.

Let W be a cosimplicial closed subset of $X \times \Delta^{*\leq N}$, and let W_n be the component of W in $X \times \Delta^n$. We will always assume that $W_m = (\mathrm{id} \times g)^{-1}(W_n)$ for each face map $g : \Delta^m \to \Delta^n$, $m, n \leq N$. Thus, the complements $U_n := X \times \Delta^n \setminus W_n$ form an open N-truncated cosimplicial subscheme of $X \times \Delta^{*\leq N}$, and we may form the N-truncated simplicial spectrum with supports $E^W(X \times \Delta^{*\leq N})$ as the homotopy fiber of $E(X \times \Delta^{*\leq N}) \to E(U)$. We write $E^W(X, - \leq N)$ for this N-truncated simplicial spectrum.

Also, if W' is another cosimplicial closed subset of $X \times \Delta^{*\leq N}$, with open cosimplicial complement V, we write $E^{W\setminus W'}(X \setminus W', - \leq N)$ for the N-truncated simplicial spectrum $E^{W\setminus W'}(V)$.

We parametrize the surjective linear projections $\pi : \mathbb{A}^{d+2} \to \mathbb{A}^d$ by an open subscheme U of $\mathbb{A}^{d(d+2)}$.

Suppose that W_n is in $\mathcal{S}_X^{(p)}(n)$ for all $n \leq N$ (for some fixed p). There is an open subscheme U_W of U parametrizing those $\pi : \mathbb{A}^{d+2} \to \mathbb{A}^d$ with the following properties

(1) $\pi_{|X} : X \to \mathbb{A}^d$ is finite.
(2) Write $\pi \times \mathrm{id}_{\Delta^n}^{-1}(\pi \times \mathrm{id}_{\Delta^n}(W_n)) = V_n$, and let W'_n be the closure of $V_n \setminus W_n$. Then W'_n is in $\mathcal{S}_X^{(p)}(n)$ for all $n \leq N$.
(3) Set $W''_n := W_n \cap W'_n$. Then $\pi \times \mathrm{id}_{\Delta^n}^{-1}(\pi \times \mathrm{id}_{\Delta^n}(W''_n))$ is in $\mathcal{S}_X^{(p+1)}(n)$ for all $n \leq N$
(4) $\pi \times \mathrm{id}_{\Delta^n}$ is étale along $W_n \setminus W''_n$ and
$$W_n \setminus W''_n \xrightarrow{\pi \times \mathrm{id}_{\Delta^n}} \pi \times \mathrm{id}_{\Delta^n}(W_n \setminus W''_n)$$
is an isomorphism, for $n \leq N$.

For such a π, we let V and W' be the N-truncated cosimplicial closed subsets of $X \times \Delta^*$ with $V \cap X \times \Delta^n = V_n$, $W' \cap X \times \Delta^n = W'_n$, $n \leq N$. We will also write π for $\pi \times \mathrm{id}_{\Delta^*}$

Lemma 3.3.3. *Take $\pi : \mathbb{A}^{d+2} \to \mathbb{A}^d$ corresponding to a k-valued point of U_W and let W'' be an N-truncated cosimplicial subset of $X \times \Delta^*$. Suppose that*

(1) $W'' = \pi^{-1}(\pi(W''))$,
(2) π is étale along $W \setminus W''$
(3) $\pi : W \setminus W'' \to \pi(W \setminus W'')$ is an isomorphism (of reduced cosimplicial schemes).

Then the composition (which we denote by $\rho_{\pi,W}$)
$$\begin{aligned} &E^{W\setminus W''}(X \setminus W'', - \leq N) \\ &\qquad \xrightarrow{\pi_*} E^{\pi(W\setminus W'')}(\mathbb{A}^d \setminus \pi(W''), - \leq N) \\ &\qquad \xrightarrow{\pi^*} E^{V\setminus W''}(X \setminus W'', - \leq N) \\ &\qquad = E^{W\setminus W''}(X \setminus \bar{W}'', - \leq N) \oplus E^{V\setminus (W\cup W'')}(X \setminus \bar{W}'', - \leq N) \\ &\qquad \xrightarrow{p} E^{W\setminus W''}(X \setminus \bar{W}'', - \leq N) \end{aligned}$$
is an isomorphism in $\mathcal{SH}$. Here p is the projection, and the direct sum decomposition of $E^{V\setminus W''}(X\setminus W'', - \leq N)$ arises from the decomposition

of $V \setminus W''$ into disjoint closed subsets

$$V \setminus W'' = W \setminus W'' \coprod V \setminus (W \cup W'').$$

Proof. Note that Proposition 3.2.3, gives us the pushforward map

$$E^{W\setminus W''}(X \setminus W'', - \leq N) \xrightarrow{\pi_*} E^{\pi(W\setminus W'')}(\mathbb{A}^d \setminus \pi(W''), - \leq N).$$

The result then follows from Proposition 3.2.3(2) ☐

Note that, for $\pi \in U_W$, the closed cosimplicial subset W'' defined by

$$W''_n := \pi \times \mathrm{id}_{\Delta^n}^{-1}(\pi \times \mathrm{id}_{\Delta^n}(W_n \cap W'_n))$$

satisfies the conditions of Lemma 3.3.3.

Let π, W and W'' be as in Lemma 3.3.3. Let

$$E^{\pi(W\setminus W'')}(\mathbb{A}^d \setminus \pi(W''), - \leq N) \xrightarrow{\tilde{\pi}^*} E^{W\setminus W''}(X \setminus W'', - \leq N)$$

be the composition (in $\mathcal{SH}$)

$$\begin{aligned} &E^{\pi(W\setminus W'')}(\mathbb{A}^d \setminus \pi(W''), - \leq N) \\ &\qquad \xrightarrow{\pi^*} E^{W\setminus W''}(X \setminus W'', - \leq N) \\ &\qquad\qquad \xrightarrow{\rho_{\pi,W}^{-1}} E^{W\setminus W''}(X \setminus W'', - \leq N). \end{aligned}$$

Lemma 3.3.4. *Let π, W and W'' be as in Lemma 3.3.3. Then*

$$\mathrm{id} - \tilde{\pi}^* \circ \pi_* : E^{W\setminus W''}(X \setminus W'', - \leq N) \to E^{W\setminus W''}(X \setminus W'', - \leq N)$$

is the zero map (in $\mathcal{SH}$).

Proof. This follows directly from Lemma 3.3.3 and the definition of $\tilde{\pi}^*$. ☐

Theorem 3.3.5. *Let k be an infinite field. Suppose E satisfies axiom 3. Let $X \in \mathbf{Sm}_k$ be affine of dimension d over k. Suppose that X admits a closed embedding into $\mathbb{A}_k^{d+2}$ with trivial normal bundle. Then for every p, the map*

$$E_f^{(p)}(X, -) \to E^{(p)}(X, -)$$

is a weak equivalence.

Proof. Since E satisfies axiom 3, E also satisfies axioms 1 and 2. We replace [12, Lemma 9.3] with Lemma 3.3.4 and we can then repeat the argument of [12, §9], replacing K-theory with E, to prove the desired weak equivalence. ☐

3.4. **Norms in finite extensions.** In order to extend Theorem 3.3.5 to the case of a finite base field k, we will need a result of Morel in the $\mathbb{P}^1$-stable homotopy category. For this reason, the extension to finite fields will only be valid for a theory E that is the 0-spectrum of a $\mathbb{P}^1$-Ω-spectrum $\mathcal{E} = (E_0 = E, E_1, \ldots)$ (*cf.* §6 for this definition).

For a field F, we have the Grothendieck-Witt ring $\mathrm{GW}(F)$ with augmentation $\dim : \mathrm{GW}(F) \to \mathbb{Z}$ and augmentation ideal $I(F) := \ker \dim$.

Let $F \to L$ be a separable field extension of degree n; let $f : \operatorname{Spec} L \to \operatorname{Spec} F$ be the map of schemes. We fix an embedding $i : \operatorname{Spec} L \to \mathbb{A}^1$ and a defining equation g for $i(\operatorname{Spec} L)$. For $X \in \mathbf{Sm}_F$, we thus have the embedding $i_X : X_L \to X \times \mathbb{A}^1_F$ with defining equation g, giving the push-forward map $f_* : E(X_L) \to E(X)$ for each E satisfying axioms 1-3.

Lemma 3.4.1. *Let F be a field such that each element $x \in I(F)$ is nilpotent. Suppose that E is the 0-spectrum of a $\mathbb{P}^1$-Ω-spectrum $\mathcal{E} \in \mathcal{SH}(F)$. Then, for $X \in \mathbf{Sm}_F$, the map $f_*f^* : E(X)[1/n] \to E(X)[1/n]$ is an isomorphism in $\mathcal{SH}$.*

Proof. We use the notation from §6. For an element $u \in L^\times$, we have the quadratic form $t(u)$ on the F-vector space L defined by

$$(x, y) \mapsto \mathrm{Tr}_{L/F}(uxy).$$

This gives the class $[t(u)]$ in $\mathrm{GW}(F)$. In particular, the defining equation $g(x)$ for $i(\operatorname{Spec} L)$ gives the element $dg/dx \in L^\times$, and the class $[t(dg/dx)] \in \mathrm{GW}(F)$.

$\mathcal{E}$ is canonically a module for the sphere spectrum $\mathbb{S}$, so it suffices to prove the result for $\mathcal{E} = \mathbb{S}$. By results of Morel [15] there is a ring homomorphism $\rho : \mathrm{GW}(F) \to \mathrm{Hom}_{\mathcal{SH}(F)}(\mathbb{S}, \mathbb{S})$, and $f_*f^* : \mathbb{S} \to \mathbb{S}$ is given by $\rho([t(dg/dx)])$. Since $t(dg/dx)$ has dimension n, this element is invertible in $\mathrm{GW}(F)[1/n]$ by our assumption on F, hence f_*f^* is invertible in $\mathrm{Hom}_{\mathcal{SH}(F)}(\mathbb{S}, \mathbb{S})[1/n]$. □

Remark 3.4.2. For a finite field F, one has $I(F)^2 = 0$. Thus, via Lemma 3.4.1, one can extend Theorem 3.3.5 to the case of a finite base-field k, assuming that E is the 0-spectrum of a $\mathbb{P}^1$-Ω-spectrum $\mathcal{E} \in \mathcal{SH}(k)$. Indeed, one uses the standard trick of passing to an infinite pro-p extension $k(p)$ and an infinite pro-q extension $k(q)$ of k, for primes $p \neq q$. By Theorem 3.3.5 $E^{(n)}_f(X_{k(p)}, -) \to E^{(n)}(X_{k(p)}, -)$ and $E^{(n)}_f(X_{k(q)}, -) \to E^{(n)}(X_{k(q)}, -)$ are weak equivalences, for $X \in \mathbf{Sm}_k$. By Lemma 3.4.1, this implies $E^{(n)}_f(X, -) \to E^{(n)}(X, -)$ is also a weak equivalence.

3.5. **Nisnevic model structure.** Let $\mathcal{C}$ be a subcategory of $\mathbf{Sm}_k$; we assume that, if $f : Y \to X$ is a smooth morphism in $\mathbf{Sm}_k$ and X and Y are in $\mathcal{C}$, then f is in $\mathrm{Hom}_{\mathcal{C}}(Y, X)$. In particular, the Nisnevic topology is defined on $\mathcal{C}$. Also, if $E : \mathcal{C}^{\mathrm{op}} \to \mathbf{sSets}*$ is a functor (i.e. a presheaf of pointed simplicial sets on $\mathcal{C}$), we have the associated Nisnevic sheaf $\tilde{E} : \mathcal{C}^{\mathrm{op}} \to \mathbf{sSets}*$. Thus, if $E : \mathcal{C}^{\mathrm{op}} \to \mathbf{Spt}$ is a functor to spectra,

$$E := ((E_0, E_1, \ldots), \epsilon_n : S^1 \wedge E_n \to E_{n+1}),$$

we may define the "associated sheaf" of spectra on $\mathcal{C}$, $\tilde{E}$, by

$$\tilde{E} := (\widetilde{E_0}, \widetilde{E_1}, \ldots),$$

with $\epsilon_n : S^1 \wedge \widetilde{E_n} \to \widetilde{E_{n+1}}$ the map induced by ϵ_n. Finally, for a point $x \in X$, with $X \in \mathcal{C}$, and $E = (E_0, E_1, \ldots)$ a sheaf of spectra on $\mathcal{C}$, the *stalk* of E at x, E_x, is the spectrum $(E_{0x}, E_{1x}, \ldots)$, where E_{nx} is the stalk of the sheaf of simplicial sets E_n at x.

Let $\mathbf{Spt}_{\mathrm{Nis}}(\mathcal{C})$ denote the category of functors $E : \mathcal{C}^{\mathrm{op}} \to \mathbf{Spt}$ with the following model structure: a map $f : E \to F$ is a cofibration if $\tilde{f}_x : \tilde{E}_x \to \tilde{F}_x$ is a cofibration in $\mathbf{Spt}$ for all $x \in X \in \mathcal{C}$, a map $f : E \to F$ is a weak equivalence if $\tilde{f}_x : \tilde{E}_x \to \tilde{F}_x$ is a weak equivalence in $\mathbf{Spt}$ for all $x \in X \in \mathcal{C}$, and the fibrations are characterized by having the RLP with respect to trival cofibrations. We let $\mathcal{H}\mathbf{Spt}_{\mathrm{Nis}}(\mathcal{C})$ denote the associated homotopy category. For details, we refer the reader to [9].

Additionally, we will need to extend these constructions to categories of the form $\mathcal{C} \times \mathcal{S}$, where $\mathcal{S}$ is the category associated to a partially ordered set. We let $\mathbf{Spt}_{\mathrm{Nis}}(\mathcal{C} \times \mathcal{S})$ be the model structure on the category of functors $E : (\mathcal{C} \times \mathcal{S})^{\mathrm{op}} \to \mathbf{Spt}$ for which a map $f : E \to F$ is a cofibration (resp. weak equivalence) if $f(s) : E(s) \to F(s)$ is a cofibration (resp. weak equivalence) in $\mathbf{Spt}_{\mathrm{Nis}}(\mathcal{C})$ for all $s \in \mathcal{S}$. Fibrations are characterized by having the RLP with respect to trivial fibrations.

The categories $\mathcal{C}$ we will be mainly interested in are: $\mathbf{Sm}_k$, $\mathbf{Sm}_X$ for $X \in \mathbf{Sm}_k$, and the category $\mathbf{Sm}/\!/X$, which is the subcategory of $\mathbf{Sm}_X$ with the same objects as $\mathbf{Sm}_k$, and with morphisms $f : Y \to Z$ the smooth X-morphisms. The partially ordered set we will use is the natural numbers $\mathbb{N}$, with its standard order.

Fixing an natural number p, we have the inclusion functor $i_p : \mathcal{C} \to \mathcal{C} \times \mathbb{N}$, inducing the functor $i_p^* : \mathbf{Spt}_{\mathrm{Nis}}(\mathcal{C} \times \mathbb{N}) \to \mathbf{Spt}_{\mathrm{Nis}}(\mathcal{C})$.

Lemma 3.5.1. *The functor i_p^* preserves cofibrations, weak equivalences and fibrations.*

Proof. Since the cofibrations and weak equivalences are defined stalkwise, it is evident that i_p^* preserves both of these. To see that i_p^* preserves fibrations, let

$$i_{p*} : \mathbf{Spt}_{\mathrm{Nis}}(\mathcal{C}) \to \mathbf{Spt}_{\mathrm{Nis}}(\mathcal{C} \times \mathbb{N})$$

be the functor with

$$(i_{p*}E)(X, n) = \begin{cases} E(X) & \text{for } n \geq p \\ 0 & \text{for } n < p. \end{cases}$$

The necessary map $(i_{p*}E)(X, n+1) \to (i_{p*}E)(X, n)$ is the identity for $n \geq p$ and the 0 map for $n < p$. We make the evident definition of $i_{p*}(f)$ for a morphism f in $\mathbf{Spt}_{\mathrm{Nis}}(\mathcal{C})$. It is clear that i_{p*} is left adjoint to i_p^* and that i_{p*} preserves cofibrations and weak equivalences. Thus, i_p^* preserves fibrations. □

Let E be in $\mathbf{Spt}_{\mathrm{Nis}}(\mathcal{C} \times \mathcal{S})$, with $\mathcal{C}$ and $\mathcal{S}$ as above. Let Y be in $\mathcal{C}$, and assume that the operations $X \mapsto X \times_k Y$, $f \mapsto f \times \mathrm{id}_Y$ define a functor $\times Y : \mathcal{C} \to \mathcal{C}$. Set $E^Y := E \circ (\times Y \times \mathrm{id}_{\mathcal{S}})$, i.e., $E^Y(X, s) = E(X \times Y, s)$.

Lemma 3.5.2. *Let $\phi : E \to F$ be a fibration in $\mathbf{Spt}_{\mathrm{Nis}}(\mathcal{C} \times \mathcal{S})$*

1. *Let Y be in $\mathbf{Sm}_k$. Then $\phi^Y : E^Y \to F^Y$ is a fibration.*
2. *Let Y be in $\mathbf{Sm}_k$. If E is fibrant then E^Y is fibrant.*
3. *Let $f : Z \to Y$ be a monomorphism in $\mathbf{Sm}_k$.Then $\phi^Y \circ f^* : E^Y \to F^Z$ is a fibration.*

Proof. Taking $Z = \emptyset$ in (2), we see that (1) follows from (3); (2) follows from (1) taking $F = *$,. For (3), consider two presheaves of sets A on B on $\mathcal{C} \times \mathcal{S}$. Setting $B^X(Y, s) := B(X \times Y, s)$, we have the canonical isomorphism

$$\mathrm{Hom}(A, B^X) = \mathrm{Hom}(A \times X, B),$$

where $A \times X$ is the product of A with the presheaf represented by X. By naturality, this isomorphism extends to presheaves of spectra on $\mathcal{C} \times \mathcal{S}$ (with $\wedge$ replacing $\times$); let $\hat{g} : A \wedge X \to B$ denote the map corresponding to $g : A \to B^X$.

Now let $i : A \to B$ be a trivial cofibration,and suppose we have a commutative diagram

(3.5.1)
$$\begin{array}{ccc} A & \xrightarrow{g} & E^Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \phi^Z \circ f^*} \\ B & \xrightarrow[h]{} & F^Z \end{array}$$

Now, $A \wedge Z \to B \wedge Z$ and $A \wedge Y \to B \wedge Y$ are trivial cofibrations. As ϕ is a fibration, we have a lifting $\alpha : B \wedge Z \to E$ in the diagram

$$\begin{array}{ccc} A \wedge Z & \xrightarrow{\hat{g}\circ \mathrm{id}\wedge f} & E \\ {\scriptstyle i\wedge \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \phi} \\ B \wedge Z & \xrightarrow[\hat{h}]{} & F \end{array}$$

Also, since f is a monomorphism, $A \wedge Z \to A \wedge Y$ and $B \wedge Z \to B \wedge Y$ are cofibrations. Thus, we have the map $\hat{g} \coprod \alpha : A\wedge Y \coprod_{A\wedge Z} B\wedge Z \to E$ and the trivial cofibration

$$A \wedge Y \coprod_{A\wedge Z} B \wedge Z \to B \wedge Y.$$

Since $\phi : E \to F$ is a fibration, $\hat{g} \coprod \alpha$ extends to a map $B \wedge Y \to E$, which gives a lifting in the diagram (3.5.1). □

Remark 3.5.3. In the categories $\mathbf{Spt}(\mathcal{C} \times \mathcal{S})$ and $\mathbf{Spt}_{\mathrm{Nis}}(\mathcal{C} \times \mathcal{S})$, there are *functorial* fibrant replacement operations: $E \mapsto (\alpha_E : E \to E_{\mathrm{fib}})$. In addition, for Y in $\mathbf{Sm}_k$ and E in $\mathbf{Spt}(\mathcal{C}\times\mathcal{S})$ or $\mathbf{Spt}_{\mathrm{Nis}}(\mathcal{C}\times\mathcal{S})$, there is a natural arrow from $(E^Y)_{\mathrm{fib}}$ to $(E^Y)^{\mathrm{fib}}$:

$$(E, Y) \mapsto (\gamma_{E,Y} : (E^Y)_{\mathrm{fib}} \to (E^Y)^{\mathrm{fib}}).$$

Let X be in $\mathbf{Sm}_k$, $Z \subset X$ a closed subset. Let E be in $\mathbf{Spt}(\mathcal{C} \times \mathcal{S})$ for some $\mathcal{C}$ containing $\mathbf{Sm}/\!/X$, E_X the restriction to $\mathcal{C}/X \times \mathcal{S}$. We let $E_{X,Z} : (\mathcal{C}/X)^{\mathrm{op}} \times \mathcal{S} \to \mathbf{Spt}$ denote the functor $(f : Y \to X, s) \mapsto E(s)^{f^{-1}(Z)}(Y)$. Let $j : \mathbb{P}^1 \setminus 0 \to \mathbb{P}^1$ and $i : \infty \to \mathbb{P}^1$ be the inclusions.

Lemma 3.5.4. *Let $p : X \times \mathbb{P}^1 \to X$ be the projection.*
(1) There are maps

$$\iota^T_{X,Z} : E_{X\times\mathbb{P}^1,Z\times 0}) \to (\Omega_T E)_{X,Z}$$
$$\iota^{\mathbb{P}^1}_{X,Z} : p_*(E_{X\times\mathbb{P}^1,Z\times 0}) \to (\Omega_{\mathbb{P}^1} E)_{X,Z}.$$

which are natural in E and in (X, Z). In addition, if $q : (\Omega_T E)_{X,Z}$ $to(\Omega_{\mathbb{P}^1} E)_{X,Z}$ is the canonical map induced by the inclusion $i : \infty \to \mathbb{P}^1 \setminus 0$, then $\iota^T_{X,Z} = \iota^{\mathbb{P}^1}_{X,Z} \circ q$.

(2) If $E(s)$ is satisfies axiom 1 and 2 for each $s \in \mathcal{S}$, then $\iota^T_{X,Z}$ and $\iota^{\mathbb{P}^1}_{X,Z}$ are weak equivalences.

Proof. If we have defined $\iota^{\mathbb{P}^1}_{X,Z}$, we just set $\iota^T_{X,Z} = \iota^{\mathbb{P}^1}_{X,Z} \circ q$. We drop the $\mathcal{S}$ to simplify the notation.

For $f : Y \to X$, $p_*(E_{X\times\mathbb{P}^1, Z\times 0})(Y)$ is the homotopy fiber of the restriction map

$$E(Y\times\mathbb{P}^1) \to E(Y\times\mathbb{P}^1\setminus f^{-1}(Z)\times 0),$$

while $(\Omega_{\mathbb{P}^1}E)_{X,Z}(Y)$ is the double homotopy fiber over the square

$$\begin{array}{ccc} E(Y\times\mathbb{P}^1) & \longrightarrow & E((Y\setminus f^{-1}(Z))\times\mathbb{P}^1) \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle i^*} \\ E(Y\times\infty) & \longrightarrow & E((Y\setminus f^{-1}(Z))\times\infty) \end{array}$$

Thus, the canonical map

$$E(Y\times\mathbb{P}^1\setminus f^{-1}(Z)\times 0) \to E((Y\setminus f^{-1}(Z))\times\mathbb{P}^1)\times_{E((Y\setminus f^{-1}(Z))\times\infty)}E(Y\times\infty)$$

defines the map $\iota^{\mathbb{P}^1}_{X,Z}(Y)$.

For (2), if E satisfies Zariski Mayer-Vietoris, then

$$\{(Y\setminus f^{-1}(Z))\times\mathbb{P}^1, Y\times(\mathbb{P}^1\setminus 0)\}$$

is a Zariski open cover of $Y\times\mathbb{P}^1\setminus f^{-1}(Z)\times 0$. Thus, the canonical map

$$E(Y\times\mathbb{P}^1\setminus f^{-1}(Z)\times 0) \to \operatorname{fib}\left(\begin{array}{ccc} & & E((Y\setminus f^{-1}(Z))\times\mathbb{P}^1) \\ & & \downarrow{\scriptstyle j^*} \\ E(Y\times(\mathbb{P}^1\setminus 0)) & \longrightarrow & E((Y\setminus f^{-1}(Z))\times(\mathbb{P}^1\setminus 0)) \end{array}\right)$$

is a weak equivalence. Now suppose that E is $\mathbb{A}^1$-homotopy invariant (axiom 1). Then the inclusion $?\times\infty \to ?\times(\mathbb{P}^1\setminus 0)$ induces a weak equivalence $E(?\times(\mathbb{P}^1\setminus 0)) \to E(?\times\infty)$, hence a weak equivalence

$$E(Y\times\mathbb{P}^1\setminus f^{-1}(Z)\times 0) \to \operatorname{fib}\left(\begin{array}{ccc} & & E(Y\setminus f^{-1}(Z))\times\mathbb{P}^1) \\ & & \downarrow{\scriptstyle i^*} \\ E(Y\times\infty) & \longrightarrow & E((Y\setminus f^{-1}(Z))\times\infty). \end{array}\right)$$

□

3.6. **Straightening a homotopy limit.** Let I be a two-category. We let I_0 denote the underlying category. A *continuous* functor $X : I \to \mathbf{Spc}$ is an assignment $i \mapsto X(i)$, i in I, $X(i)$ in $\mathbf{Spc}$, plus for each pair of objects i, j, a map of simplicial sets

$$X : \mathcal{N}_*(\mathrm{Hom}_I(i,j)) \to \mathcal{H}om(X(i), X(j)),$$

compatible with the two composition laws for each triple i, jk in I.

For a 2-category I, and an object i of I we have the 2-category of objects under i, i/I, with objects the morphisms $i \to j$, morphisms $f : i \to j$ to $f' : i \to j'$ a pair (g, θ) with $g : j \to j'$ a morphism, $\theta : gf \to f'$ a 2-morphism, and 2-morphisms $\tau : (g, \theta) \to (g', \theta')$ a 2-morphism $\tau : g \to g'$ such that $\theta = \theta' \circ (\tau \circ f)$.

If i', i is a pair of objects of I, we may consider the category $\mathrm{Hom}_I(i', i)$ as a 2-category with trivial 2-morphisms, and we have the 2-functor

$$\circ : i/I \times \mathrm{Hom}_I(i', i) \to i'/I$$

defined as follows: send the pair of objects $(f : i \to j, g : i' \to i)$ to the object $fg : i' \to j$ of i'/I. Send the pair of morphisms $((h, \theta) : f \to f', \tau : g \to g')$ to $(h, (\theta \circ g) \circ (hf \circ \tau))$, and the pair of 2-morphisms (ρ, id) to ρ. Taking nerves gives us the map of bi-simplicial sets

$$\circ \mathcal{N}_{**}(i/I) \times \mathcal{N}_*(\mathrm{Hom}_I(i', i)) \to \mathcal{N}_{**}(i'/I),$$

where $\mathcal{N}_*(\mathrm{Hom}_I(i', i))$ is the bi-simplicial set which is constant in the second variable.

Suppose that I is a small 2-category. Noting that the nerve of a two-category is a bi-simplicial set, we may form the homotopy colimit of a continuous functor $X : I \to \mathbf{Spc}$ by adapting the formula for usual functors:

$$\operatorname*{hocolim}_I X := \coprod_{i \in I} \mathcal{N}_{**}(i/I) \times X(i)/\sim$$

where the relation $\sim$ is given as the coequalizer over the two maps

$$\begin{array}{ccc} \mathcal{N}_{**}(j/I) \times \mathcal{N}_*(\mathrm{Hom}_I(j, i)) \times X(i) & \xrightarrow{\circ \times \mathrm{id}} & \mathcal{N}_{**}(i/I) \times X(i) \\ \Big\downarrow{\scriptstyle \mathrm{id} \times \circ} & & \\ \mathcal{N}_{**}(j/I) \times X(j). & & \end{array}$$

Here we make everything into maps of simplicial sets by diagonalizing where necessary.

Let B be a small category. Form the 2-category $\mathrm{Fac}B$ with as follows: $\mathrm{Fac}B$ has the same objects as B. A morphism $b \to b'$ in $\mathrm{Fac}B$ consists of a sequence of composable morphisms

$$b = b_0 \xrightarrow{f_1} b_1 \xrightarrow{f_2} \ldots \xrightarrow{f_n} b_n = b'$$

where if some $f_i = \mathrm{id}$, we identify the sequence with the one obtained by deleting f_i (unless $n = 1$). Composition is just by concatenation. We write such a sequence as $f_n * \ldots * f_1$. Given such a sequence, we can form a new (shorter) sequence by changing some of the $*$'s to compositions in the category B; this operations defines a 2-morphism.

Thus, the category $\mathrm{Hom}_{\mathrm{Fac}B}(b,b')$ is a partially ordered set with final elements $\mathrm{Hom}_B(b,b')$.

Lemma 3.6.1. *Let $F:\mathrm{Fac}B\to\mathbf{Spc}$ be a continuous functor. Suppose that B has a final object $*$, and let $i:F(*)\to\mathrm{hocolim}_{\mathrm{Fac}B}F$ be the map induced by the inclusion of $*$ in $\mathrm{Fac}B$. Then i is a homotopy retract of $\mathrm{hocolim}_{\mathrm{Fac}B}F$; in particular, i is a trivial cofibration.*

Proof. For $b\in B$, we have the full sub 2-category $b/*$ of $b/\mathrm{Fac}B$, with objects the morphisms $b\to *$ in $\mathrm{Fac}B$. Let $i_b:b/*\to b/\mathrm{Fac}B$ be the inclusion. For each morphism $f:b'\to b$ in $\mathrm{Fac}B$, the functor $f^*:b/\mathrm{Fac}B\to b'/\mathrm{Fac}B$ sends $b/*$ to $b'/*$, so we have the subspace $\mathrm{hocolim}_{\mathrm{Fac}B}F/*$ of $\mathrm{hocolim}_{\mathrm{Fac}B}F$ generated by the subspaces $\mathcal{N}_{**}(b/*)\times F(b)$ of $\mathcal{N}_{**}(b/\mathrm{Fac}B)\times F(b)$.

Sending a morphism $b\to b'$ in $\mathrm{Fac}B$ to the composition $b\to b'\xrightarrow{p_{b'}} *$, where $p_{b'}:b'\to *$ is the canonical morphism in B, defines a functor $\pi_*:b/\mathrm{Fac}B\to b/*$. The morphism $p_{b'}$ defines a natural transformation from id to $i_b\circ\pi_*$. Thus, we have the retraction $r:\mathrm{hocolim}_{\mathrm{Fac}B}F\to\mathrm{hocolim}_{\mathrm{Fac}B}F/*$, right inverse to the inclusion $\tilde{i}:\mathrm{hocolim}_{\mathrm{Fac}B}F/*\to\mathrm{hocolim}_{\mathrm{Fac}B}F$, and a homotopy of $r\tilde{i}$ with the identity on $\mathrm{hocolim}_{\mathrm{Fac}B}F$.

Clearly $i:F(*)\to\mathrm{hocolim}_{\mathrm{Fac}B}F$ factors through $\mathrm{hocolim}_{\mathrm{Fac}B}F/*$; in fact, the map $F(*)\to\mathrm{hocolim}_{\mathrm{Fac}B}F/*$ is a homeomorphism. Indeed, the 2-category $b/*$ has only the trivial 2-morphisms, and the underlying category is $\mathrm{Hom}_{\mathrm{Fac}B}(b,*)$. Thus, $\mathrm{hocolim}_{\mathrm{Fac}B}F/*$ is the quotient of the disjoint union of the spaces $\mathcal{N}_*\mathrm{Hom}_{\mathrm{Fac}B}(b,*)\times F(b)$ by the evaluation maps

$$\mathcal{N}_*\mathrm{Hom}_{\mathrm{Fac}B}(b,*)\times F(b)\to F(*),$$

as well as by the diagrams

$$\begin{array}{ccc}\mathcal{N}_*\mathrm{Hom}_{\mathrm{Fac}B}(b,*)\times F(b') & \xrightarrow{\sigma^*\times\mathrm{id}} & \mathcal{N}_*\mathrm{Hom}_{\mathrm{Fac}B}(b',*)\times F(b') \\ \downarrow{\scriptstyle \mathrm{id}\times\sigma_*} & & \\ \mathcal{N}_*\mathrm{Hom}_{\mathrm{Fac}B}(b,*)\times F(b) & & \end{array}$$

for $\sigma:b'\to b$ a morphism in $\mathrm{Fac}B$. The functoriality of the evaluation maps shows that the relations of the second type are implied by those of the first type, hence the map $F(*)\to\mathrm{hocolim}_{\mathrm{Fac}B}F$ is a homeomorphism. □

Suppose we have a continuous functor $F:\mathrm{Fac}B\to\mathbf{Spc}$. Then, for each $b\in B$, we have the 2-category $\mathrm{Fac}(B/b)$, the continuous functor $F/b:\mathrm{Fac}(B/b)\to\mathbf{Spc}$, and the space $CF(b):=\mathrm{hocolim}_{\mathrm{Fac}(B/b)}F/b$.

For a morphism $g : b \to b'$ in B we have the 2-functor $g_* : \mathrm{Fac}(B/b) \to \mathrm{Fac}(B/b')$ and the natural transformation $\mathrm{id} : F/b \to F/b' \circ g_*$. Since $(gg')_* = g_* g'_*$, we thus have the functor

$$CF : B \to \mathbf{Spc}$$

The identity on b gives the inclusion of the trivial 2-category $*$ to $\mathrm{Fac}(B/b)$, and thus the natural map $\iota_b : F(b) \to CF(b)$. By Lemma 3.6.1, we have

Lemma 3.6.2. *$\iota_b : F(b) \to CF(b)$ is a homotopy retract.*

Thus, the hocolim construction transforms a "functor up to homotopy and all higher homotopies" on B to a strict functor on B.

Let $F : \mathcal{A} \to \mathbf{Spc}$ be a functor from a small category $\mathcal{A}$ to simplicial sets. We have the homotopy limit $\mathrm{holim}_{\mathcal{A}} F$, which is natural in F, i.e., if $\theta : F \to G$ is a natural transformation of functors, we have the map of cosimplicial spaces $\theta_* : \mathrm{holim}_{\mathcal{A}} F \to \mathrm{holim}_{\mathcal{A}} G$, and $\theta_* \circ \theta'_* = (\theta \circ \theta')_*$. If $i : \mathcal{B} \to \mathcal{A}$ is a functor of small categories, we have the map of spaces $i^* : \mathrm{holim}_{\mathcal{A}} F \to \mathrm{holim}_{\mathcal{B}}(F \circ i)$, and $(i \circ i')^* = i'^* \circ i^*$.

Let $\underline{n}$ be the category associated to the ordered set $\{0 < 1 < \ldots < n\}$. If $F : \mathcal{A} \to \mathbf{Spc}$ is a functor, we have the functor $F \circ p_2 : \underline{n} \times \mathcal{A} \to \mathbf{Spc}$ and the isomorphism

$$\mathcal{H}om(\Delta^n, \operatorname*{holim}_{\mathcal{A}} F) \cong \operatorname*{holim}_{\underline{n} \times \mathcal{A}} F \circ p_2.$$

Now suppose we have functors $i_j : \mathcal{B} \to \mathcal{A}$, $j = 0, \ldots, n$, and natural transformations $\theta^j : i_{j-1} \to i_j$, $j = 1, \ldots, n$, i.e., an n-simplex σ in the nerve of the functor category $\mathbf{Func}(\mathcal{B}, \mathcal{A})$. This yields in the evident manner a functor $(i_*, \theta_*) : \underline{n} \times \mathcal{B} \to \mathcal{A}$ and a natural transformation $\theta : (i_*, \theta_*) :\to i_n \circ p_2$. Thus, we have the map

$$\theta_* \circ (i_*, \theta_*)^* : \operatorname*{holim}_{\mathcal{A}} F \to \mathcal{H}om(\Delta^n, \operatorname*{holim}_{\mathcal{B}} F \circ i_n);$$

taking the adjoint yields the map

$$F(\sigma) : \Delta^n \times \operatorname*{holim}_{\mathcal{A}} F \to \operatorname*{holim}_{\mathcal{B}} F \circ i_n.$$

For $g : \underline{m} \to \underline{n}$ a functor we have the relation

$$\theta_*^{n,g(m)} \circ F(\sigma \circ g) = F(\sigma) \circ \Delta(g)$$

as maps $\Delta^m \times \mathrm{holim}_{\mathcal{A}} F \to \mathrm{holim}_{\mathcal{B}} F \circ i_n$, where $\theta^{n,g(m)} : i_{g(m)} \to i_n$ is the composition of the natural transformations θ^j, $j = g(m)+1, \ldots, n$, and $\Delta(g) : \Delta^m \to \Delta^n$ is the canonical map.

We apply these constructions to the following situation: Let $\pi : \mathcal{A} \to B$ be a functor of small categories. For each $b \in B$, we have the subcategory $\pi^{-1}(b)$ of $\mathcal{A}$ consisting of objects over b and morphisms over id_b; let $i_b : \pi^{-1}(b) \to \mathcal{A}$ be the inclusion. A *lax fibering* of π consists of the assignment:

(1) for each morphism $h : b \to b'$ in B, a functor $h^* : \pi^{-1}(b') \to \pi^{-1}(b)$, and a natural transformation $\theta_h : i_b \circ g^* \to i_{b'}$
(2) for each pair of composable morphisms $b \xrightarrow{h} b' \xrightarrow{g} b''$ in B, a natural transformation $\vartheta_{g,h} : h^* g^* \to (gh)^*$

which satisfies

a) $\pi(\theta_h(\alpha)) = h$ for $h : b \to b'$ in B, and $\alpha \in \pi^{-1}(b')$.
b) for composable morphisms $h : b \to b'$, $g : b' \to b''$, and for $\alpha \in \pi^{-1}(b'')$,

$$\theta_g(\alpha) \circ \theta_h(g^*\alpha) = \theta_{gh}(\alpha) \circ \vartheta_{g,h}(\alpha).$$

c) for composable morphisms $b \xrightarrow{h} b' \xrightarrow{g} b'' \xrightarrow{f} b'''$

$$\vartheta_{h,gf} \circ (\vartheta_{g,f} \circ h^*) = \vartheta_{hg,f} \circ (f^* \circ \vartheta_{h,g}).$$

Given a lax fibering $(\pi : \mathcal{A} \to B, {}^*, \theta, \vartheta)$, we have the 2-functor s from $\mathrm{Fac}B^{\mathrm{op}}$ to subcategories of $\mathcal{A}$ by sending b to $\pi^{-1}(b)$, $h : b \to b'$ in B to h^*, and the 2-morphism $h * g \to hg$ to $\vartheta_{g,h}$. If in addition we have a functor $F : \mathcal{A} \to \mathbf{Spc}$, sending b to $\mathrm{holim}_{\pi^1(b)} F \circ i_b$ extends to a continuous functor

$$\mathrm{holim}\, F : \mathrm{Fac}B \to \mathbf{Spc}.$$

Indeed, for each $h : b \to b'$ in B, we have the functor $h^* : \pi^{-1}(b') \to \pi^{-1}(b)$ and the natural transformation $\theta_h : i_b \circ h^* \to i_{b'}$. This induces the natural transformation $F(\theta_h) : (F \circ i_b) \circ h^* \to F \circ i_{b'}$ and thus the map

$$\mathrm{holim}\, F(h) := F(\theta_h)_* \circ (h^*)^* : \mathrm{holim}\, F(b) \to \mathrm{holim}\, F(b').$$

For a formal composition $h_n * \ldots * h_1$, we define

$$\mathrm{holim}\, F(h_n * \ldots * h_1) = \mathrm{holim}\, F(h_n) \circ \ldots \circ \mathrm{holim}\, F(h_1).$$

Using the 2-functor s and the properties of holim discussed above, we see that $\mathrm{holim}\, F$ extends to a continuous functor

$$\mathrm{holim}\, F : \mathrm{Fac}B \to \mathbf{Spc}.$$

Thus, we have the functor

$$C\mathrm{holim}\, F : B \to \mathbf{Spc},$$

the trivial cofibration $i_b : (\mathrm{holim}\, F)(b) \to (C\mathrm{holim}\, F)(b)$ and retraction $r_b : (C\mathrm{holim}\, F)(b) \to (\mathrm{holim}\, F)(b)$ for each $b \in B$.

Remark 3.6.3. Suppose we have a subcategory $i_0 : B_0 \to B$ of B such that the restriction of ϑ to pairs of maps in B_0 is the identity transformation, i.e., the restriction of π to $\pi^{-1}(B_0) \to B_0$ is a strict fibering. Then $b \mapsto (\mathrm{holim}\, F)(b)$ extends to a functor $\mathrm{holim}\, F_{|B_0} : B_0 \to \mathbf{Spc}$. In general, the cofibrations i_b will not extend to a natural weak equivalence $\mathrm{holim}\, F_{|B_0} \to C\mathrm{holim}\, F \circ i_0$, but the retractions r_b do: We have the natural transformation

$$r_{|B_0} : C\mathrm{holim}\, F \circ i_0 \to \mathrm{holim}\, F_{|B_0}$$

with $r_{|B_0}(b) = r_b$ for $b \in B_0$.

3.7. **Functoriality.** Using the results of the previous sections, we can transform the operation $X \mapsto E^{(p)}(X, -)$ into a functor on $\mathbf{Sm}_k$, along the lines worked out by Kahn [11].

For each $X \in \mathbf{Sm}_k$, the operation $Y \mapsto E^{(p)}(Y, -)$ defines a functor $E^{(p)}/\!\!/X$ on $\mathbf{Sm}/\!\!/X$. The tower

$$\ldots \to E^{(p)}(Y, -) \to E^{(p-1)}(Y, -) \to \ldots \to E^{(0)}(Y, -) \tag{3.7.1}$$

defines the functor $(Y, p) \mapsto E^{(p)}(Y, -)$ on $\mathbf{Sm}/\!\!/X \times \mathbb{N}$, which we denote by $E^{(*)}/\!\!/X$.

Let $\rho : \mathbf{Sm}/\!\!/k \to \mathbf{Sm}_k$ be the inclusion.

Theorem 3.7.1. *Suppose that $E : \mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{Spt}$ satisfies axiom 3; if k is finite, assume that E is the 0-spectrum of a $\mathbb{P}^1$-Ω-spectrum $\mathcal{E} \in \mathcal{SH}(k)$. Then,*

(1) *For each $p \geq 0$ there is a functor $E^{(p)} : \mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{Spt}$, together with an isomorphism*

$$\phi_p : E^{(p)} \circ \rho \to E^{(p)}/\!\!/k$$

in $\mathcal{H}\mathbf{Spt}(\mathbf{Sm}/\!\!/k)$.

(2) *There are natural transformations $\xi_p : E^{(p)} \to E^{(p-1)}$, $p \geq 0$, making the diagram*

$$\begin{array}{ccc} E^{(p)} \circ \rho & \xrightarrow{\phi_p} & E^{(p)}/\!\!/k \\ \downarrow & & \downarrow \\ E^{(p-1)} \circ \rho & \xrightarrow[\phi_{p-1}]{} & E^{(p-1)}/\!\!/k \end{array}$$

commute in $\mathcal{H}\mathbf{Spt}(\mathbf{Sm}/\!\!/k)$.

(3) *There are isomorphisms in $\mathcal{H}\mathbf{Spt}(\mathbf{Sm}_k)$*

$$\psi_p : (\Omega_T E)^{(p-1)} \to \Omega_T(E^{(p)}),\ p \geq 0,$$

intertwining the transformations ξ_{p-1} *and* $\Omega_T(\xi_p)$ *(here we set* $(\Omega_T E)^{(-1)} := (\Omega_T E)^{(0)}$*,* $\xi_{-1} = \xi_0$*). Similarly, there are isomorphisms in* $\mathcal{H}\mathbf{Spt}(\mathbf{Sm}_k)$

$$\psi_p : (\Omega_{\mathbb{P}^1} E)^{(p-1)} \to \Omega_{\mathbb{P}^1}(E^{(p)}),\ p \geq 0,$$

intertwining the transformations ξ_{p-1} *and* $\Omega_{\mathbb{P}^1}(\xi_p)$*. The diagram*

$$\begin{array}{ccc} (\Omega_T E)^{(p-1)} & \xrightarrow{\psi_p} & \Omega_T(E^{(p)}) \\ \downarrow & & \downarrow \\ (\Omega_{\mathbb{P}^1} E)^{(p-1)} & \xrightarrow[\psi_p]{} & \Omega_{\mathbb{P}^1}(E^{(p)}) \end{array}$$

commutes.

Additionally, the operation $E \mapsto (E^{(p)}, \phi_p, \xi_p, \psi_p)$ *are natural in* E*, and* $E^{(p)}$ *is a fibrant object in* $\mathbf{Spt}_{\mathrm{Nis}}(\mathbf{Sm}_k)$*.*

Proof. The construction is based on the similar construction due to Kahn [11]. We give the details of the construction for the reader's convenience.

We have the category $\mathcal{L}(\mathbf{Sm}_k)$ (see [**?**]). Objects are morphisms $f : X' \to X$ in $\mathbf{Sch}_k$ such that X' can be written as a disjoint union $X' = X'_0 \coprod X'_1$ with $f : X'_0 \to X$ an isomorphism. The choice of this decomposition is not part of the data. We identify $f : X' \to X$ and $f \coprod p : X' \coprod X'' \to X$ if $p : X'' \to X$ is smooth, and we identify $f_1 : X_1 \to X$ and $f_2 : X_2 \to X$ if there is an X-isomorphism $X_1 \cong X_2$. We note that a choice of component X'_0 of X' isomorphic to X via f determines a smooth section $s : X \to X'$.

A morphism $g : (f_X : X' \to X) \to (f_Y : Y' \to Y)$ is a morphism $g : X \to Y$ in $\mathbf{Sm}_k$ such that there exists a smooth morphism $g' : X' \to Y'$ with $f_Y \circ g' = g \circ f_X$. Using the smooth sections $s_X : X \to X'$, $s_Y : Y \to Y'$, one easily sees that this condition also respects the identifications $f \sim f \coprod p$ for p smooth; obviously, this condition respects the identification of isomorphisms over X or over Y. The choice of g' is not part of the data, so

$$\mathrm{Hom}_{\mathcal{L}(\mathbf{Sm}_k)}((f_X : X' \to X), (f_Y : Y' \to Y)) \subset \mathrm{Hom}_{\mathbf{Sm}_k}(X, Y).$$

Composition is induced by the composition in $\mathbf{Sm}_k$; this is well-defined since smooth morphisms are closed under composition.

Lemma 3.7.2. *Let* $g : (f_X : X' \to X) \to (f_Y : Y' \to Y)$ *be a morphism in* $\mathcal{L}(\mathbf{Sm}_k)$*. Then, for* $W \in \mathcal{S}^{(p)}_Y(n)_{f_Y}$*,* $(g \times \mathrm{id}_{\Delta^n})^{-1}(W)$ *is in* $\mathcal{S}^{(p)}_X(n)_{f_X}$*.*

Proof. Choose a component X'_0 of X' for which $f_X : X'_0 \to X$ is an isomorphism, and let $s : X \to X'$ be the composition

$$X \xrightarrow{f_X^{-1}} X'_0 \subset X'.$$

Then s is a smooth section to $f_X : X' \to X$, and we can factor g as $g = f_Y \circ g' \circ s$, with $g' : X' \to Y'$ smooth. Since $(f_Y \times \mathrm{id})^{-1}$ maps $\mathcal{S}_Y^{(p)}(n)_{f_Y}$ to $\mathcal{S}_{Y'}^{(p)}(n)$, by definition, and $g' \circ s$ is smooth, it follows that $W' := (g \times \mathrm{id}_{\Delta^n})^{-1}(W)$ is in $\mathcal{S}_X^{(p)}(n)$. Similarly, since $g \circ f_X = f_Y \circ g'$ and

$$\begin{aligned}(f_X \times \mathrm{id})^{-1}(W') &= ((f_X \circ g) \times \mathrm{id})^{-1}(W) \\ &= ((g' \circ f_Y) \times \mathrm{id})^{-1}(W) = (g' \times \mathrm{id})^{-1}((f_Y \times \mathrm{id})^{-1}(W)),\end{aligned}$$

we see that W' is in $\mathcal{S}_X^{(p)}(n)_{f_X}$. □

Sending $f_X : X' \to X$ to X defines the faithful functor

$$\pi : \mathcal{L}(\mathbf{Sm}_k)^{\mathrm{op}} \to \mathbf{Sm}_k^{\mathrm{op}}.$$

We make $\mathcal{L}(\mathbf{Sm}_k)^{\mathrm{op}}$ a lax fibered category over $\mathbf{Sm}_k^{\mathrm{op}}$ as follows: For each morphism $f : X \to Y$ in $\mathbf{Sm}_k^{\mathrm{op}}$(i.e. $f : Y \to X$ in $\mathbf{Sm}_k$), let $f^* : \pi^{-1}(Y) \to \pi^{-1}(X)$ be the functor

$$f^*(g : Y' \to Y) := (fg \coprod \mathrm{id}_X : Y' \coprod X \to X).$$

The map $\theta_f(g) : f^*(g : Y' \to Y) \to (g : Y' \to Y)$ in $\mathcal{L}(\mathbf{Sm}_k)^{\mathrm{op}}$ is the opposite of the map $f : (g : Y' \to Y) \to (fg \coprod \mathrm{id}_X : Y' \coprod X \to X)$; we see that this latter is a map in $\mathcal{L}(\mathbf{Sm}_k)$ by using the inclusion $Y' \to Y' \coprod X$ as the required smooth morphism. Given maps $f : X \to Y$ and $f' : Y \to Z$ in $\mathbf{Sm}_k^{\mathrm{op}}$, and $h : Z' \to Z$ in $\mathcal{L}(\mathbf{Sm}_k)$, set $\vartheta_{f',f}$ to be the map

$$f^*(f'^*(h : Z' \to Z)) \to (f'f)^*(h : Z' \to Z)$$

being the opposite of the map

$$\begin{aligned}&(ff'g \coprod \mathrm{id}_X : Z' \coprod X \to X) \\ &\qquad \to (ff'g \coprod f \coprod \mathrm{id}_X : Z' \coprod Y \coprod X \to X)\end{aligned}$$

induced by the identity on X. One easily checks that this data does indeed define a lax fibering.

Remark 3.7.3. It is easy to see that the restriction of $\mathcal{L}(\mathbf{Sm}_k)^{\mathrm{op}} \to \mathbf{Sm}_k^{\mathrm{op}}$ to $(\mathbf{Sm}/\!/k)^{\mathrm{op}}$ is a strict fibering.

We define the functor

$$E^{(p)}(?,-)_? : \mathcal{L}(\mathbf{Sm}_k)^{\mathrm{op}} \to \mathbf{Spt}$$

by sending $(f : X' \to X)$ to the functorial fibrant model (in $\mathbf{Spt}$) $\widetilde{E^{(p)}(X,-)}_f$ of $E^{(p)}(X,-)_f$. It follows from Lemma 3.7.2 that this really does define a functor. Let $hE^{(p)}(X)$ be the homotopy limit

$$hE^{(p)}(X) := \operatorname*{holim}_{\pi^{-1}(X)} E^{(p)}(?,-)_?.$$

By the construction of the previous section, we have the functor

$$\begin{aligned} ChE^{(p)} &: \mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{Spt}, \\ X &\mapsto ChE^{(p)}(X), \end{aligned}$$

and for each X a trivial cofibration $i_p : hE^{(p)}(X) \to ChE^{(p)}(X)$ with homotopy inverse the retraction $r_p : ChE^{(p)}(X) \to hE^{(p)}(X)$.

For fixed $X \in \mathbf{Sm}_k$, we have the diagram

(3.7.2)
$$\begin{array}{ccccc} & & hE^{(p)}(X) & \underset{r_p}{\overset{i_p}{\rightleftarrows}} & ChE^{(p)}(X) \\ & & \downarrow{\scriptstyle q_p} & & \\ E^{(p)}(X,-) & \xrightarrow[j_p]{} & \widetilde{E^{(p)}(X,-)}_{\mathrm{id}} & & \end{array}$$

Using Remarks 3.6.3 and 3.7.3 one sees that this diagram defines a diagram of functors on $\mathbf{Sm}/\!/k$:

(3.7.3)
$$\begin{array}{ccccc} & & hE^{(p)}/\!/k & \xleftarrow[r_p]{} & ChE^{(p)} \circ \rho \\ & & \downarrow{\scriptstyle q_p} & & \\ E^{(p)}/\!/k & \xrightarrow[j_p]{} & \widetilde{E^{(p)}/\!/k} & & \end{array}$$

r_p and j_p are pointwise weak equivalences. It follows from Theorem 3.3.5 and Lemma 3.3.1 that q_p is a weak equivalence in $\mathbf{Spt}_{\mathrm{Nis}}(\mathbf{Sm}/\!/k)$. ∎

Thus, we have isomorphism in $\mathcal{H}\mathbf{Spt}_{\mathrm{Nis}}(\mathbf{Sm}/\!/k)$

$$\pi_p : ChE^{(p)} \circ \rho \to E^{(p)}/\!/k.$$

We may vary p yielding the functor $ChE^{(*)}$ from $\mathbf{Sm}_k^{\mathrm{op}}$ to towers of spectra, i.e., a functor

$$ChE^{(*)} : (\mathbf{Sm}_k \times \mathbb{N})^{\mathrm{op}} \to \mathbf{Spt}.$$

Similarly, we have the functors

$$E^{(*)}/\!/k : (\mathbf{Sm}/\!/k \times \mathbb{N})^{\mathrm{op}} \to \mathbf{Spt},$$

etc., and the diagram (3.7.3) extends to

$$\begin{array}{ccc} & hE^{(*)}/\!\!/k & \xleftarrow[r]{} ChE^{(*)}\circ\rho \\ & {\scriptstyle q}\downarrow & \\ E^{(*)}/\!\!/k & \xrightarrow[j]{} \widetilde{E^{(*)}/\!\!/k} & \end{array} \tag{3.7.4}$$

We have the isomorphism (in $\mathcal{H}\mathbf{Spt}_{\mathrm{Nis}}(\mathbf{Sm}/\!\!/k\times\mathbb{N})$) $\pi: ChE^{(*)}\circ\rho\to E^{(*)}/\!\!/k$.

Let $\alpha: ChE^{(*)}\to ChE^{(*)}_{\mathrm{fib}}$ be the functorial fibrant replacement (with respect to the Nisnevic model structure); set

$$E^{(*)} := ChE^{(*)}_{\mathrm{fib}}.$$

Let $a: E^{(*)}/\!\!/k\to E^{(*)}_{\mathrm{fib}}/\!\!/k$ be the functorial fibrant replacement (with respect to the Nisnevic model structure).

We claim that

(3.7.5)

$$a: E^{(*)}/\!\!/k\to E^{(*)}_{\mathrm{fib}}/\!\!/k \text{ is a weak equivalence in } \mathbf{Spt}(\mathbf{Sm}/\!\!/k\times\mathbb{N}).$$

Indeed, it follows from axiom 2 that $E^{(*)}/\!\!/k$ satisfies Nisnevic descent; the same follows for $E^{(*)}_{\mathrm{fib}}/\!\!/k$ since this object is fibrant. As each object in $\mathbf{Sm}/\!\!/k\times\mathbb{N}$ has finite Nisnevic cohomological dimension, it follows from the local-to-global spectral sequence that a weak equivalence in $\mathbf{Spt}_{\mathrm{Nis}}(\mathbf{Sm}/\!\!/k\times\mathbb{N})$ between objects which satisfy Nisnevic descent is a weak equivalence in $\mathbf{Spt}(\mathbf{Sm}/\!\!/k\times\mathbb{N})$, whence the claim.

If we apply the functorial fibrant replacement $_{\mathrm{fib}}$ (with respect to the Nisnevic model structure) to the diagram (3.7.4), we thus have the isomorphism in $\mathcal{H}\mathbf{Spt}(\mathbf{Sm}/\!\!/k\times\mathbb{N})$

$$\phi: E^{(*)}\circ\rho\to E^{(*)}/\!\!/k.$$

Let $j_X: X\times(\mathbb{P}^1\setminus 0)\to X\times\mathbb{P}^1$ be the inclusion. The sequences

$$\mathcal{S}^{(p)}_X(r)_f \xrightarrow{i_0} \mathcal{S}^{(p+1)}_{X\times\mathbb{P}^1}(r)_{f\times\mathrm{id}} \xrightarrow{j_X^{-1}} \mathcal{S}^{(p+1)}_{X\times(\mathbb{P}^1\setminus 0)}(r)_{f\times\mathrm{id}}$$

and Lemma 3.5.4 yield the sequence of functors

$$(\Omega_{\mathbb{P}^1}E)^{(*-1)}/\!\!/k \xleftarrow{\iota} \mathrm{fib}(j^*)\to (E^{(*)})^{\mathbb{P}^1}/\!\!/k \xrightarrow{j^*} (E^{(*)})^{\mathbb{P}^1\setminus 0}/\!\!/k \tag{3.7.6}$$

and

$$(\Omega_{\mathbb{P}^1}E)^{(*-1)} \xleftarrow{\hat\iota} \mathrm{fib}(j^*)\to (E^{(*)})^{\mathbb{P}^1} \xrightarrow{j^*} (E^{(*)})^{\mathbb{P}^1\setminus 0}, \tag{3.7.7}$$

with ι and $\hat\iota$ (Nisnevic) weak equivalences in $\mathbf{Spt}_{\mathrm{Nis}}(\mathbf{Sm}/\!\!/k\times\mathbb{N})$. The map ϕ gives a map of sequences $\phi: (3.7.7)\circ\rho\to(3.7.6)$ which is an isomorphism of sequences in $\mathcal{H}\mathbf{Spt}_{\mathrm{Nis}}(\mathbf{Sm}/\!\!/k\times\mathbb{N})$.

By definition, $\Omega_T E^{(*)}$ is the homotopy fiber of $j^* : (E^{(*)})^{\mathbb{P}^1} \to (E^{(*)})^{\mathbb{P}^1\setminus 0}$. Since $E^{(*)}$ is homotopy invariant, $\Omega_T E^{(*)} \to \Omega_{\mathbb{P}^1} E^{(*)}$ is an isomorphism in $\mathcal{H}\mathbf{Spt}_{\mathrm{Nis}}(\mathbf{Sm}_k \times \mathbb{N})$. Thus we thus have the isomorphism in $\mathcal{H}\mathbf{Spt}_{\mathrm{Nis}}(\mathbf{Sm}_k \times \mathbb{N})$

$$\psi : (\Omega_{\mathbb{P}^1} E)^{(*-1)} \to \Omega_{\mathbb{P}^1}(E^{(*)}).$$

Let $E^{(p)}$, $\phi_p : E^{(p)} \circ \rho \to E^{(p)}/\!\!/k$ and $\psi_p : (\Omega_{\mathbb{P}^1} E)^{(p-1)} \to \Omega_{\mathbb{P}^1} E^{(p)}$, etc., denote the restrictions of the evident objects and maps under the appropriate inclusion functor $i_p : \mathcal{C} \to \mathcal{C} \times \mathbb{N}$. By Lemma 3.5.1, $E^{(p)}$ is fibrant, ϕ_p is an isomorphism in $\mathcal{H}\mathbf{Spt}_{\mathrm{Nis}}(\mathbf{Sm}/\!\!/k)$ and ψ_p is an isomorphism in $\mathcal{H}\mathbf{Spt}_{\mathrm{Nis}}(\mathbf{Sm}_k)$. Since $E^{(p)}$ and $E^{(p)}/\!\!/k$ both satisfy Nisnevic descent, ϕ_p is an isomorphism in $\mathcal{H}\mathbf{Spt}(\mathbf{Sm}/\!\!/k)$; similarly ψ_p is an isomorphism in $\mathcal{H}\mathbf{Spt}(\mathbf{Sm}_k)$. The remaining assertions of (1)-(3) follow directly from our construction, as does the naturality of $E \mapsto (E^{(p)}, \phi_p, \xi_p, \psi_p)$. □

Remark 3.7.4. The isomorphism

$$\psi : (\Omega_{\mathbb{P}^1} E)^{(*-1)} \to \Omega_{\mathbb{P}^1}(E^{(*)})$$

in $\mathcal{H}\mathbf{Spt}_{\mathrm{Nis}}(\mathbf{Sm}_k \times \mathbb{N})$ is natural in E in the following sense: There is a diagram of weak equivalences in $\mathbf{Spt}_{\mathrm{Nis}}(\mathbf{Sm}_k \times \mathbb{N})$

$$\begin{array}{ccc} & \widetilde{(\Omega_{\mathbb{P}^1} E)}^{(*-1)} & \\ \swarrow{\scriptstyle\psi'} & & \searrow{\scriptstyle\psi''} \\ (\Omega_{\mathbb{P}^1} E)^{(*-1)} & & \Omega_{\mathbb{P}^1}(E^{(*)}) \end{array}$$

with ψ' and ψ'' natural in E, and with $\psi = \psi'' \circ \psi'^{-1}$.

For $E \in \mathbf{Spt}(\mathbf{Sm}_k)$ satisfying axiom 3, we let $E^{(p/p+r)}$ be the cofiber of the map $E^{(p+r)} \to E^{(p)}$.

Corollary 3.7.5. *Under the hypotheses of Theorem 3.7.1, for integers $p, r \geq 0$, there is a functor $E^{(p/p+r)} : \mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{Spt}$ whose restriction to $\mathbf{Sm}/\!\!/k$ is isomorphic to $E^{(p/p+r)}(?, -) : \mathbf{Sm}/\!\!/k^{\mathrm{op}} \to \mathbf{Spt}$ in $\mathcal{H}\mathbf{Spt}(\mathbf{Sm}/\!\!/k)$. In addition*

(1) *The functor $E^{(0/1)}$ is* birational*: The restriction map*
$$E^{(0/1)}(X) \to E^{(0/1)}(k(X))$$
is a weak equivalence.

(2) *The functor $E^{(0/1)}$ is* rationally invariant*: If $F \to F(t)$ is a pure transcendental extension of fields (finitely generated over k), then $E^{(0/1)}(F) \to E^{(0/1)}(F(t))$ is a weak equivalence.*

Proof. The main statement and part (1) follows from Theorem 3.7.1. For (2), fix an irreducible $X \in \mathbf{Sm}_k$, and let $Z \to X$ be a proper closed subset. We have the localization fiber sequence

$$E_Z^{(0/1)}(X,-) \to E^{(0/1)}(X,-) \to E^{(0/1)}(X \setminus Z,-).$$

with $E_Z^{(0/1)}(X,-)$ the cofiber of $E_Z^{(1)}(X,-) \to E_Z^{(0)}(X,-)$. Since each closed subset $W \subset Z \times \Delta^n$ has codimension at least one on $X \times \Delta^n$, the map $E_Z^{(1)}(X,n) \to E_Z^{(0)}(X,n)$ is an isomorphism for each n. Thus $E_Z^{(0/1)}(X,-) = 0$ in $\mathcal{SH}$ and $E^{(0/1)}(X,-) \to E^{(0/1)}(X \setminus Z,-)$ is a weak equivalence. (2) follows by taking limits.

For (3), the homotopy property implies that

$$E^{(0/1)}(F,-) \to E^{(0/1)}(\mathbb{A}^1_F,-)$$

is a weak equivalence. Since $E^{(0/1)}(\mathbb{A}^1_F,-) \to E^{(0/1)}(F(t),-)$ is a weak equivalence by (2), the result is proved. □

3.8. **Supports.** We now consider an extension to singular schemes by taking the theory on a smooth scheme with supports. Let $P\mathbf{Sm}_k$ be the category of pairs (X,W) with $X \in \mathbf{Sm}_k$ and W a closed subset of X; a morphism $f : (X,W) \to (Y,W')$ is a map $f : X \to Y$ in $\mathbf{Sm}_k$ with $W \supset f^{-1}(W')$. We let $P\mathbf{Sm}_k(d)$ be the full subcategory with $\mathrm{codim}_X(W) \geq d$.

For a functor $E : \mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{Spt}$ we extend E to $P\mathbf{Sm}_k^{\mathrm{op}}$ by sending (X,W) to $E^W(X)$. For (X,W) in $P\mathbf{Sm}_k(d)$, let W^0 be the smooth locus of W. Write W^0 as a disjoint union $W^0 = W^0(d) \coprod W^0(> d)$, where $W^0(d)$ has pure codimension d on X, and $W^0(> d)$ has codimension $> d$ on X.

Let $E : \mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{Spt}$ be a functor satisfying axioms 1-3; if k is finite, assume that E is the 0-spectrum of a $\mathbb{P}^1$-Ω-spectrum $\mathcal{E} \in \mathcal{SH}(k)$.

Lemma 3.8.1. *For (X,W) in $P\mathbf{Sm}_k(d)$, there is a canonical isomorphism*

$$\sigma_d : (E^{(d/d+1)})^W(X) \xrightarrow{\sim} (\Omega^d_T E)^{(0/1)}(W^0(d)).$$

in $\mathcal{SH}$.

Proof. Let $X^0 = X \setminus (W \setminus W^0(d))$. By the localization property for $E^{(d/d+1)}$, the restriction

$$(E^{(d/d+1)})^W(X) \to (E^{(d/d+1)})^{W^0(d)}(X^0)$$

is a weak equivalence, so we reduce to the case $X = X^0$, $W = W^0(d)$.

By considering the deformation to the normal bundle, we have a canonical isomorphism

$$(E^{(d/d+1)})^W(X) \cong (E^{(d/d+1)})^W(N)$$

in $\mathcal{SH}$, where N is the normal bundle of W in X and W is included in N by the zero section $i_0 : W \to N$.

Let $N^0 := N \setminus \{i_0(W)\}$ with projection $q : N^0 \to W$. Using Corollary 3.7.5 and the localization property for $E^{(d/d+1)}$ again, the pull-back by q induces weak equivalences

$$(E^{(d/d+1)})^W(N) \xrightarrow{q^*} (E^{(d/d+1)})^{N^0}(q^*N)$$
$$(\Omega^d_T E)^{(0/1)}(W) \xrightarrow{q^*} (\Omega^d_T E)^{(0/1)}(N^0).$$

Using the diagonal section $\delta : N^0 \to q^*N^0 \subset q^*N$ as 1, we have a canonical isomorphism

$$\phi : q^*N \cong N^0 \times \mathbb{A}^1.$$

This in turn gives a canonical trivialization of the normal bundle of $i_0(N^0)$ in q^*N, hence a canonical isomorphism in $\mathcal{SH}$

$$(E^{(d/d+1)})^{N^0}(q^*N) \cong (\Omega^d_T E)^{(0/1)}(N^0).$$

This completes the construction. □

4. Generalized cycles

We use the results of the previous sections to give an interpretation of the layers in the homotopy coniveau tower. For this, we will need to strengthen axiom 3 to

A4. Suppose the base-field k is infinite. Then there is a functor $E_4 : \mathbf{Sm}_S^{\mathrm{op}} \to \mathbf{Spt}$ satisfying the axioms 1 and 2 and a natural weak equivalence

$$\sigma : E \to \Omega^4_T(E_4).$$

If k is finite, we assume that E is the 0-spectrum of a $\mathbb{P}^1$-Ω-spectrum.

Remark 4.0.2. Suppose E satisfies axiom 4. Then E satisfies axioms 1-3, and the functors $E^{(p)}$, $E^{(p/p+r)}$ also satisfy axioms 1-3. Indeed, if k is infinite and $E = \Omega^4_T E_4$, then $E^{(p)} = \Omega^2_T((\Omega^2_T E_4)^{(p+2)})$. Similarly, if $E = E_0$ is the 0-spectrum of a $\mathbb{P}^1$-Ω-spectrum $\mathcal{E} = (E_0, E_1, \ldots)$, then it will be shown in §6 that the associated spectra $E^{(p)}$ and $E^{(p/p+r)}$ also have this property. Thus, by Theorem 3.7.1, we may iterate the operation ${}^{(p/p+r)}$, forming functors $(E^{(p/p+r)})^{(q/q+s)} : \mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{Spt}$ which satisfy axioms 1 and 2.

4.1. **The semi-local Δ^*.** We recall that Δ^n has the vertices $v_0, \ldots, v_n$, where v_i is the closed subscheme defined by $t_j = 0$ $j \neq i$. For a scheme X, we let $\Delta^n_0(X)$ be the intersection of all open subschemes $U \subset X \times \Delta^n$ with $X \times v_i \subset U$ for all i.

Remark 4.1.1. If X is a semi-local scheme with closed points $x_1, \ldots, x_m$, then $\Delta^n_0(X)$ is just the semi-local scheme $\operatorname{Spec} \mathcal{O}_{X\times\Delta^n,S}$, where S is the closed subset $\{x_i \times v_j \mid i = 1, \ldots, m,\ j = 0, \ldots, n\}$. In particular $\Delta^n_0(X)$ is an affine scheme if X is semi-local.

For a scheme T, we let $\Delta^*_0(T)$ denote the cosimplicial ind-scheme $n \mapsto \Delta^n_0(T)$; if T is semi-local, then $\Delta^*_0(T)$ is a cosimplicial semi-local scheme. For F a field, we write $\Delta^*_{0,F}$ for $\Delta^*_0(\operatorname{Spec} F)$

4.2. **Some vanishing theorems.** We fix a functor $E : \mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{Spt}$.

Lemma 4.2.1. *Suppose that E satisfies axiom 3; if k is finite, we also suppose that E is the 0-spectrum of a $\mathbb{P}^1$-Ω-spectrum. Let $F = E^{(p)}$: $\mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{Spt}$ with $p > 0$. Then for X in $\mathbf{Sm}_k$, $F^{(0/1)}(X, -) \sim *$.*

Proof. Noting that $F^{(0/1)}(X, -) \sim F^{(0/1)}(\operatorname{Spec} k(X), -)$, we reduce to the case of a field K. In this case, we have $F^{(0/1)}(K, -) = E^{(p)}(\Delta^*_{0,K})$. Since each $\Delta^n_{0,K}$ is semi-local, we have the natural weak equivalences

$$E^{(p)}(\Delta^n_{0,K}) \leftarrow \widehat{E^{(p)}}(\Delta^n_{K,0}, -) \xrightarrow{\pi_p} E^{(p)}_{\mathcal{C}/\Delta^n_{0,K}}(\Delta^n_{K,0}, -),$$

where $\mathcal{C}/\Delta^n_{0,K}$ is the disjoint union of the maps of faces $\Delta^m_{0,K} \to \Delta^n_{0,K}$.

Thus, $F^{(0/1)}(K, -)$ is weakly equivalent to the total space of the bisimplicial spectrum

$$(n, m) \mapsto E^{(p)}(n, m),$$

where $E^{(p)}(n, m)$ is the limit of the spectra with support

$$E^W(\Delta^n_{0,K} \times_K \Delta^m_K),$$

as W runs over all closed subsets of $\Delta^n_{0,K} \times_K \Delta^m_K$ satisfying:

$$\operatorname{codim}_{F\times F'}(W \cap F \times F') \geq p$$

for all faces $F' \subset \Delta^m_K$, $F \subset \Delta^n_{0,K}$.

For each m, we have the restriction to a face (say the face $t_{m+1} = 0$)

$$\delta^* : E^{(p)}(-, m + 1) \to E^{(p)}(-, m),$$

with right inverse given by pull-back by the corresponding codegeneracy map

$$\sigma^* : E^{(p)}(-, m) \to E^{(p)}(-, m + 1).$$

By the same argument as for the homotopy property for $E^{(p)}(X,-)$, one shows that $\sigma^*\circ\delta^*$ is homotopic to the identity, hence δ^* is a homotopy equivalence.

Thus, the inclusion $E^{(p)}(-,0)\to E^{(p)}(-,-)$ is a weak equivalence. However, if W is an irreducible closed subset of $\Delta^n_{0,K}$ which intersects all faces in codimension $\geq p>0$, then in particular, W contains no vertex of $\Delta^n_{0,K}$. Since $\Delta^n_{0,K}$ is semi-local with closed points the vertices, this implies that W is empty, that is, $E^{(p)}(-,0)\sim *$, proving the result. □

Proposition 4.2.2. *Suppose that E satisfies axiom 4, and let $F=E^{(p)}:\mathbf{Sm}_k^{\mathrm{op}}\to\mathbf{Spt}$ with $p>0$. Then $F^{(q/q+1)}\sim *$ for all $0\leq q<p$. Similarly, $(E^{(p/p+1)})^{(q/q+1)}\sim *$ for $0\leq q<p$.*

Proof. We note that F satisfies axiom 3, and if k is finite, F is the 0-spectrum of a $\mathbb{P}^1$-Ω-spectrum. Since the operation $F\mapsto F^{(q/q+1)}$ is compatible with taking homotopy cofibers, the second assertion follows from the first.

For the first assertion, the case $p=1$ is handled by Lemma 4.2.1; we prove the general case by induction on p.

Note that, by Theorem 3.7.1(3), we have the weak equivalence in $\mathbf{Spt}(\mathbf{Sm}/\!\!/k)$

$$\psi^d:\Omega^d_TF\to(\Omega^d_TE)^{(p-d)}.$$

Thus, by our inductive assumption, we have

$$(\Omega^d_TF)^{(q/q+1)}\sim *$$

for $0\leq q<p-d$. By the localization spectral sequence (Corollary 3.1.5), this implies that the restriction map

$$F^{(q/q+1)}(\Delta^n_K)\to F^{(q/q+1)}(\Delta^n_{0,K})$$

is a weak equivalence for $0\leq q<p$, for all fields K and for all n. Thus, we have the isomorphisms in $\mathcal{SH}$

$$F^{(q/q+1)}(K)\cong(F^{(q/q+1)})^{(0)}(K,-)\cong(F^{(q/q+1)})^{(0/1)}(K,-)$$

for $0\leq q<p$. Applying Lemma 4.2.1 to F, we have

$$(F^{(q/q+1)})^{(0/1)}(K,-)\sim *$$

for $q>0$, which completes the proof. □

Proposition 4.2.3. *Suppose that E satisfies axiom 4, and let $F=E^{(p/p+1)}:\mathbf{Sm}_k^{\mathrm{op}}\to\mathbf{Spt}$ with $p>0$. Then $F^{(p+r)}\sim *$ for all $r>0$.*

Proof. $F^{(p+r)}(X)$ is isomorphic in $\mathcal{SH}$ to the total space of the simplicial spectrum $F^{(p+r)}(X,-)$. $F^{(p+r)}(X,n)$ in turn is the limit of the spectra with support $F^W(X\times\Delta^n)$, where W is a closed subset of $X\times\Delta^n$ which, among other properties, has codimension $\geq p+r$. By Lemma 3.8.1, it follows that $F^{(p+r)}(X,n)$ is weakly contractible, whence the result. □

4.3. **The main result.**

Theorem 4.3.1. *Let $E:\mathbf{Sm}_k^{\mathrm{op}}\to\mathbf{Spt}$ be a functor satisfying axiom 4. We have a natural isomorphism in $\mathcal{H}\mathbf{Spt}(\mathbf{Sm}\_k)$*

$$(E^{(p/p+1)})^{(q/q+1)}\cong\begin{cases}0 & \text{for } q\neq p,\\ E^{(p/p+1)} & \text{for } q=p.\end{cases}$$

Proof. The case $q\neq p$ follows from Proposition 4.2.2 and Proposition 4.2.3.

For the case $p=q$, let $F=E^{(p/p+1)}$. We thus have the natural map

$$F^{(p)}\to F^{(0)}\cong F=E^{(p/p+1)}.$$

By Proposition 4.2.3, $F^{(p+1)}\cong 0$, so the above map descends to the natural map

$$F^{(p/p+1)}\to F^{(0)}\cong F=E^{(p/p+1)}.$$

We have the tower

$$F^{(p/p+1)}\to F^{(p-1/p+1)}\to\ldots\to F^{(0/p+1)}\cong F^{(0)}.$$

with layers $F^{(q/q+1)}$, $q=0,\ldots,p-1$. By Proposition 4.2.2, all these layers are weakly contractible, so the map $F^{(p/p+1)}\to F^{(0)}$ is an isomorphism, completing the proof. □

Corollary 4.3.2. *Let $E:\mathbf{Sm}_k^{\mathrm{op}}\to\mathbf{Spt}$ be a functor satisfying axiom 4. Let X be in $\mathbf{Sm}_k$. Then $E^{(p/p+1)}(X)$ is naturally isomorphic in $\mathcal{SH}$ to a the total spectrum of a simplicial spectrum $E^{(p/p+1)}_{\mathrm{s.l.}}(X,-)$, with*

$$E^{(p/p+1)}_{\mathrm{s.l.}}(X,n)\cong\coprod_{x\in X^{(p)}(n)}(\Omega^p_TE)^{(0/1)}(k(x))$$

in $\mathcal{SH}$.

Proof. By Theorem 4.3.1, $E^{(p/p+1)}(X)$ is isomorphic (in $\mathcal{SH}$) to the total spectrum of the simplicial spectrum $n\mapsto(E^{(p/p+1)})^{(p/p+1)}(X,n)$. By Lemma 3.8.1, we have the isomorphism

$$(E^{(p/p+1)})^{(p/p+1)}(X,n)\cong\coprod_{x\in X^{(p)}(n)}(\Omega^p_TE)^{(0/1)}(k(x)),$$

in $\mathcal{SH}$, as desired. □

5. Computations

In this section, we consider a special type of theory $E : \mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{Spt}$ for which the "cells" $(\Omega^p_{\mathbb{P}^1}E)^{(0/1)}$ are particularly simple, essentially, that for a field F, $(\Omega^p_{\mathbb{P}^1}E)^{(0/1)}(F)$ is a $K(\pi, 0)$ with $\pi = \pi_0((\Omega^p_{\mathbb{P}^1}E)(F))$. For such theories, we can define an associated cycle theory $\mathrm{CH}^p(-; E, n)$ which generalizes the higher Chow groups of Bloch. We show that K-theory is of this type, and thus recover the Bloch-Lichtenbaum/ Friedlander-Suslin spectral sequence as our homotopy coniveau spectral sequence. This gives a new proof that this spectral sequence has the expected E_2-terms consisting of motivic cohomology. Motivic cohomology itself is also of this form, and, being the associated cycle theory of another theory, has a particularly simple spectral sequence.

5.1. **Well-connected theories.**

Definition 5.1.1. Let $E : \mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{Spt}$ be a functor satisfying axiom 4. We call E *well-connected* if

(1) for $X \in \mathbf{Sm}_k$ and $W \subset X$ a closed subset, $E^W(X)$ is -1-connected.
(2) for F a field finitely generated over k, $\pi_n((\Omega^d_T E)^{(0/1)}(F)) = 0$ for $n \neq 0$ and all $d \geq 0$.

Remark 5.1.2. Suppose E satisfies part (1) of Definition 5.1.1. Since $\Omega_T E(X) = E^{X\times 0}(X \times \mathbb{A}^1)$, it follows that $\Omega^d_T E$ also satisfies (1) for all $d \geq 0$.

Lemma 5.1.3. *Suppose E is well-connected. Let F be a field finitely generated over k. Then the natural map*

$$\pi_0((\Omega^p_T E)(F)) \to \pi_0((\Omega^p_T E)^{(0/1)}(F))$$

is an isomorphism for all $p \geq 0$.

Proof. We give the proof for $p = 0$ to simplify the notation. Since $E(\Delta^n_{0,F})$ is -1-connected, we have the exact sequence

$$\pi_0(E(\Delta^1_{0,F})) \xrightarrow{\delta_1^* - \delta_0^*} \pi_0(E(\Delta^0_{0,F})) \to \pi_0(E^{(0/1)}(F)) \to 0.$$

Similarly, we have the surjections $\pi_0(E(\Delta^n_F)) \to \pi_0(E(\Delta^n_{0,F}))$. Using the homotopy property, we find that the natural map

$$p^* : \pi_0(E(F)) \to \pi_0(E(\Delta^n_{0,F}))$$

is an isomorphism for all n, so the above exact sequence becomes

$$\pi_0(E(F)) \xrightarrow{0} \pi_0(E(F)) \cong \pi_0(E^{(0/1)}(F)).$$

□

5.2. **Cycles.** Let $E : \mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{Spt}$ be a well-connected theory. For $X \in \mathbf{Sm}_k$ and $W \subset X$ a closed subset, set

$$z^p_W(X;E) := \oplus_{\substack{x\in X^{(p)}\\ \bar{x}\subset W}} \pi_0(E_{-p}(k(x))).$$

We write $z^p(X;E)$ for $z^p_X(X;E)$.

Let $f : Y \to X$ be a morphism in $\mathbf{Sm}_k$, and let $W \subset X$ be a codimension p closed subset such that $f^{-1}(W)$ has codimension p on Y. Take $x \in X^{(p)}$ with $\bar{x} \subset W$, and let $y \in Y^{(p)}$ be a point in $f^{-1}(\bar{x})$. We have the pull-back homomorphism

$$f^*_{x/y} : \pi_0((\Omega^p_T E)(k(x))) \to \pi_0((\Omega^p_T E)(k(y)))$$

defined as the sequence

$$\begin{aligned}\pi_0((\Omega^p_T E)(k(x))) &\cong \pi_0((\Omega^p_T E)^{(0/1)}(k(x))) \cong \pi_0((E^{(p/p+1)})^{\bar{x}}(X))\\ &\xrightarrow{f^*} \pi_0((E^{(p/p+1)})^{f^{-1}\bar{x}}(Y)) \xrightarrow{\text{res}} \pi_0((E^{(p/p+1)})^{y}(\operatorname{Spec}\mathcal{O}_{Y,y}))\\ &\cong \pi_0((\Omega^p_T E)^{(0/1)}(k(y))) \cong \pi_0((\Omega^p_T E)(k(y))).\end{aligned}$$

Taking the sum of the $f^*_{x/y}$ defines the pull-back

$$f^* : z^p_W(X;E) \to z^p_{f^{-1}W}(Y;E)$$

which is easily seen to be functorial.

For $X \in \mathbf{Sm}_k$, we define the *higher cycles with E-coefficients* as

$$z^p(X;E,n) := \varinjlim_{W\in\mathcal{S}^{(p)}_X(n)} z^p_W(X\times\Delta^n;E),$$

forming the simplicial abelian group $n \mapsto z^p(X;E,n)$ and the associated complex $z^p(X;E,*)$.

Definition 5.2.1. Let X be in $\mathbf{Sm}_k$. The higher Chow groups of X with E-coefficients are the groups

$$\mathrm{CH}^p(X;E,n) := H_n(z^p(X;E,*)).$$

5.3. **Well-connectedness.** In this section, we give an alternative description of this property.

We let $\mathbf{Ord}$ be the category with objects the finite ordered sets $[n] := \{0 < 1 < \ldots < n\}$ and morphisms the order-preserving maps of sets. Let $\mathbf{Ord}^{inj}$ be the subcategory of injective maps. For a functor $E : \mathbf{Ord}^{\mathrm{op}} \to \mathbf{Ab}$, we have the complex $E([n],\partial)$, with

$$E([n],\partial)_{-m} := \oplus_{g:[n-m]\to[n]} E([n-m]),$$

and differential $d_{-m} : E([n], \partial)_{-m+1} \to E([n], \partial)_{-m}$ the signed sum of the maps

$$E(f)_g := E(f) : (E([n-m+1]), g) \to (E([n-m]), g \circ f);$$
$$f : [n-m] \to [n-m+1] \in \mathbf{Ord}^{inj},$$

where the sign is determined by the rule: Let $\{i_1 < \ldots < i_{m-1}\} = [n] \setminus g([n-m+1])$ and let $i \in [n]$ be the element with

$$\{i_1 < \ldots < i_{m-1}\} \cup \{i\} = [n] \setminus g \circ f([n-m]).$$

Then the map $E(f)$ above appears with sign $(-1)^j$, where $i_{j-1} < i < i_j$ (we set $i_0 = -1$).

We let $E([n], \partial^+)_{-m}$ be the quotient of $E([n], \partial)_{-m}$ by the subgroup

$$\oplus_{g,\ g(0)\neq 0} E([n-m]),$$

forming the quotient complex $E([n], \partial^+)$. One can identify $E([n], \partial^+)$ with the complex with

$$E([n], \partial^+)_{-m} = \oplus_{\substack{g:[n-m]\to[n] \\ g(0)=0}} E([n-m])$$

and differential $E([n], \partial^+)_{-m+1} \to E([n], \partial^+)_{-m}$ the signed sum of the maps $E(f)_g$ over $f : [n-m] \to [n-m+1]$ in $\mathbf{Ord}^{inj}$ with $f(0) = 0$.

Finally, we let E_* be the complex associated to the simplicial abelian group E, i.e., $E_m = E([m])$ and $d_m : E_{m+1} \to E_m$ is the usual sum of the maps $(-1)^i E(\delta_i) : E([m+1]) \to E([m])$, $\delta_i : [m] \to [m+1] \in \mathbf{Ord}^{inj}$ is the map which omits i.

We extend these notions to functors $E : \mathbf{Ord}^{\mathrm{op}} \to C(\mathbf{Ab})$ be taking the extended total complex of the evident double complexes.

Lemma 5.3.1. *Let $E : \mathbf{Ord}^{\mathrm{op}} \to C(\mathbf{Ab})$ be a functor with $E([n])$ -1-connected for all n. Then there is an exact sequence*

$$H_0(E([n+1], \partial^+)) \xrightarrow{E(\delta_0)} H_0(E([n], \partial)) \to H_n(NE) \to 0$$

for all $n \geq 0$. If $m \mapsto H_0(E([m]))$ is the constant functor, then

$$H_0(E([n], \partial)) \to H_n(NE)$$

is an isomorphism.

Proof. The second assertion follows from the first. Indeed, the degeneracy maps $\sigma_i^n : [n] \to [n-1]$ (where σ_i^n is the unique surjective order-preserving map $f : [n] \to [n-1]$ with $f(i) = f(i-1)$) define a splitting of the complexes $E([m], \partial^+)$, giving the exact sequence

$$0 \to H_m(E([n], \partial^+)) \to H_m(E([n])) \xrightarrow{\sum_i E(\delta_i)} \oplus_{i=1}^n H_m(E([n-1])).$$

Thus, the assumption that $m \mapsto H_0(E([m]))$ is constant implies that $H_0(E([n], \partial^+)) = 0$.

The first assertion is an easy consequence of the Dold-Kan correspondence, and is proved in, e.g., [12, ***]. □

Remark 5.3.2. Let $E : \mathbf{Sm}_k^{\mathrm{op}} \to C(\mathbf{Ab})$ be a functor, take Y in $\mathbf{Sm}_k$, and let $D = \sum_{i=1}^m D_i$ be a reduced strict normal crossing divisor on Y. For $I \subset \{1, \ldots, m\}$, let $D_I := \cap_{i \in I} D_i$, forming the m-cube of spectra

$$I \mapsto E(D_I).$$

Taking the iterated homotopy fiber over this m-cube defines the relative spectrum $E(Y; D)$.

For $Y = \Delta^n_F$, we let $\partial := \partial\Delta^n_F$ be the divisor $\sum_{i=0}^n (t_i = 0)$ and let $\partial^+ := \partial^+\Delta^n_F := \sum_{i=1}^n (t_i = 0)$. We have as well the restrictions to $\partial\Delta^n_{0,F}$, giving the spectra $E(\Delta^n_{0,F}, \partial)$ and $E(\Delta^n_{0,F}, \partial^+)$.

If $Y : \mathbf{Ord} \to \mathbf{Sm}_k$ is a cosimplicial scheme, we may apply the construction of this section to the composition $E \circ Y^{\mathrm{op}} : \mathbf{Ord}^{\mathrm{op}} \to C(\mathbf{Ab})$. If $Y = \Delta^*_{0,F}$, then we have the identities

$$(E \circ Y)([n], \partial) = E(\Delta^n_{F,0}, \partial)$$
$$(E \circ Y)([n], \partial^+) = E(\Delta^n_{F,0}, \partial^+),$$

and $(E \circ Y)_*$ is the complex associated to the simplicial complex $n \mapsto E(\Delta^n_{0,F})$.

Proposition 5.3.3. *Let $E : \mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{Spt}$ be a functor satisfying axiom 4. Then E is well-connected if and only if*

$$\pi_0((\Omega^d_T E)(\Delta^n_{0,F}, \partial)) = 0$$

for all $n \geq 1$, all $d \geq 0$ and all finitely generated field extensions F of k.

Proof. We have the spectral sequence

$$E^1_{p,q} = \pi_{p+q}(\Omega^d_T E)(\Delta^p_{0,F}) \Longrightarrow \pi_{p+q}(\Omega^d_T E)^{(0)}(F, -).$$

Since $\pi_m((\Omega^d_T E)(\Delta^p_{0,F})) = 0$ for $m < 0$, we have the exact sequence

$$\pi_0((\Omega^d_T E)(\Delta^1_{0,F})) \xrightarrow{d_1^{1,0}} \pi_0((\Omega^d_T E)(F)) \to \pi_0((\Omega^d_T E)^{(0)}(F, -)) \to 0.$$

As in the proof of Lemma 5.1.3, $\pi_0((\Omega^d_T E)(\Delta^1_{0,F})) = \pi_0((\Omega^d_T E)(F))$ and $d_1^{1,0}$ is the zero map, hence the map $\pi_0((\Omega^d_T E)(F)) \to \pi_0((\Omega^d_T E)^{(0)}(F, -))$ is an isomorphism.

Thus, to prove the proposition, it suffices to show that, if E satisfies axioms 1 and 2, then $\pi_n(E^{(0/1)}(F, -)) = 0$ for all $n \geq 1$ if and only if $\pi_0(E(\Delta^n_{0,F}, \partial)) = 0$ for all $n \geq 1$.

First, suppose that $\pi_0(E(\Delta^n_{F,0},\partial))=0$ for $1\le n\le N$. We will show that $\pi_n(E^{(0/1)}(F,-))=0$ for $1\le n\le N$.

We may assume by induction that $\pi_n(E^{(0/1)}(F,-))=0$ for $1\le n\le N-1$. Using the Hurewicz theorem, it suffices to show that $H_N(E^{(0/1)}(F,-))=0$. We have the fiber sequence

$$E(\Delta^n_{0,F},\partial))\to E(\Delta^n_{0,F},\partial))\to E(\Delta^{n-1}_{0,F},\partial)).$$

Thus, if $\pi_0(E(\Delta^n_{0,F},\partial))=0$ for $1\le n\le N$, it follows that

$$\pi_m(E(\Delta^N_{F,0},\partial))=0$$

for $m<0$. By the Hurewicz theorem again, it follows that

$$H_0(E(\Delta^N_{F,0},\partial))=0.$$

Therefore, we may replace E with its homology localization $\mathbb{Z}E$, and it suffices to prove the analog of the proposition for a functor $E:\mathbf{Sm}_k\to C(\mathbf{Ab})$.

In this case, (under our assumption that E is -1 connected), the Dold-Kan correspondence gives us the isomorphism

$$H_n(NE^{(0/1)}(F))\cong H_n(E^{(0/1)}(F,-))$$

for all n. Also, by Lemma 5.3.1 and axiom 1, we have the isomorphism

$$H_0(E(\Delta^N_{0,F},\partial))\to H_N(NE^{(0/1)}(F)).$$

Thus $H_N(E^{(0/1)}(F,-))=0$, as desired.

If $\pi_n(E^{(0/1)}(F,-))=0$ for $1\le n\le N$, then we may assume by induction that $\pi_0(E(\Delta^n_{0,F},\partial))=0$ for $1\le n\le N-1$. Reversing the above argument shows that $\pi_0(E(\Delta^N_{0,F},\partial))=0$, which completes the proof. □

Remark 5.3.4. The higher Chow groups of Bloch, $\mathrm{CH}^p(X,n)$, are defined without reference to an underlying cohomology theory E. Instead, one uses the usual cycle groups

$$z^p(X):=\oplus_{x\in X^{(p)}}\mathbb{Z}$$

as the building blocks for the cycle complex $z^p(X,*)$, where the pull-back map f^* is defined via Serre's intersection multiplicity formula.

The properties we have established for the spectra $E^{(p/p+1)}(X,-)$, namely: homotopy invariance, localization, extension to a functor, all are based on the analogous properties for the complexes $z^p(X,-)$ (*cf.* [1, 13, 12]). In the sequel, we will often identify $z^p(X,-)$ with the associated simplicial Eilenberg-Maclane spectrum, so as to enable a comparision with other simplicial spectra.

5.4. **The case of K-theory.** We show that the K-theory functor $K : \mathbf{Sm}_k \to \mathbf{Spt}$ satisfies axioms 1-3, and is well-connected. In fact, we will show that K is the 0-spectrum of a $\mathbb{P}^1$-Ω-spectrum $\mathcal{K} \in \mathcal{SH}(k)$, which covers the case of finite fields as well.

Proof. By Quillen's theorem, we have the weak equivalence of K-theory and G-theory for regular schemes. Thus, the homotopy invariance for G-theory gives the homotopy invariance property for K-theory on $\mathbf{Sm}_k$. Quillen's localization theorem yields the weak equivalence

$$K^W(X) \sim G(W)$$

for $W \subset X$ a closed subset, $X \in \mathbf{Sm}_k$, hence K-theory satisfies axioms 1 and 2.

Similarly, we have the weak equivalence

$$K^{X\times 0}(X \times \mathbb{A}^d) \sim G(X \times 0) \sim K(X)$$

for $X \in \mathbf{Sm}_k$. Thus $K_{-d} \sim K$, and we can take $K_2 = K$. Thus K satisfies axiom 3.

In the case of a finite base-field k, the same argument shows that the sequence $\mathcal{K} := (K, K, \ldots)$ with the above weak equivalence $K \sim K_{-1}$ defines a $\mathbb{P}^1$-Ω-spectrum with 0-spectrum K.

For well-connectedness, property (1) of Definition 5.1.1 follows by the localization theorem, since G-theory is -1-connected (for $U \subset W$, the map $G_0(W) \to G_0(U)$ is surjective).

For part (2), we use Weibel's homotopy K-theory, KH. By Vorst [18], the normal crossing divisor $\partial\Delta^n_{0,F} \subset \Delta^n_{0,F}$ is K_1-regular, hence we have the isomorphism

$$K_n(\partial\Delta^n_{0,F}) \cong KH_n(\partial\Delta^n_{0,F}).$$

for $n \leq 1$. Since KH satisfies Mayer-Vietoris for unions of closed subschemes, we have the weak equivalence of $KH(\Delta^n_{0,F}, \partial)$ with the homotopy fiber of the restriction

$$KH(\Delta^n_{0,F}) \to KH(\partial\Delta^n_{0,F})$$

By the K_1-regularity, we thus have the exact sequence

$$K_1(\Delta^n_{0,F}) \to K_1(\partial\Delta^n_{0,F}) \to K_0(\Delta^n_{0,F}, \partial) \to K_0(\Delta^n_{0,F}) \to K_0(\partial\Delta^n_{0,F}).$$

Since $\Delta^n_{0,F}$ is semi-local and affine, we have the surjection

$$\mathrm{GL}(R) \to \mathrm{GL}(R/I)$$

where R is the ring of functions on $\Delta^n_{0,F}$ and I is the ideal defining $\partial\Delta^n_{0,F}$. Since $\Delta^n_{0,F}$ is affine, we have surjections

$$\mathrm{GL}(R) \to K_1(\Delta^n_{0,F}); \quad \mathrm{GL}(R/I) \to K_1(\partial\Delta^n_{0,F}).$$

Also, $K_0(R)=\mathbb{Z}=K_0(R/I)$, so

$$K_0(\Delta^n_{0,F},\partial)=0.$$

Thus K-theory is well-connected. □

Theorem 5.4.1. *There is a natural isomorphism in $\mathcal{SH}$*

$$z^p(X,-)\cong K^{(p/p+1)}(X,-).$$

Proof. By Theorem 4.3.1, we have the isomorphism in $\mathcal{SH}$

$$K^{(p/p+1)}(X,-)\cong (K^{(p/p+1)})^{(p/p+1)}(X,-).$$

By Corollary 4.3.2, we have the isomorphism in $\mathcal{SH}$

$$(K^{(p/p+1)})^{(p/p+1)}(X,n)\cong\coprod_{x\in X^{(p)}(n)}(\Omega^p_TK)^{(0/1)}(k(x)).$$

Since K is well-connected and $K_0(k(x))=\mathbb{Z}$, $(\Omega^p_TK)^{(0/1)}(k(x))$ is the Eilenberg-Maclane spectrum $K(\mathbb{Z},0)$. Thus, we have the weak equivalence

$$(K^{(p/p+1)})^{(p/p+1)}(X,n)\xrightarrow{\sim}K(z^p(X,n),0). \tag{5.4.1}$$

It remains to see that the two sides agree as simplicial spectra.

The map(5.4.1) is just the weak equivalence of $(K^{(p/p+1)})^{(p/p+1)}(X,n)$ with its 0th Postnikov layer. Thus, we need only see that $z^p(X,-)$ and $\pi_0(K^{(p/p+1)})^{(p/p+1)}(X,-)$ agree as simplicial abelian groups.

For this, take $x\in X^{(p)}(n)$. We have the natural map

$$G(\bar{x})\sim K^{\bar{x}}(X\times\Delta^n)\to(K^{(p/p+1)})^{(p/p+1)}(X,n)$$

and, for each face map $g:\Delta^m\to\Delta^n$, the commutative diagram

$$\begin{array}{ccc}\pi_0G(\bar{x}) & \longrightarrow & \pi_0(K^{(p/p+1)})^{(p/p+1)}(X,n)\\ \big\downarrow{\scriptstyle g^*} & & \big\downarrow{\scriptstyle g^*}\\ \pi_0G(g^{-1}(\bar{x})) & \longrightarrow & \pi_0(K^{(p/p+1)})^{(p/p+1)}(X,m)\end{array}$$

Similarly, we have the surjection $\pi_0G(\bar{x})\to\pi_0G(k(x))$, $\pi_0G(g^{-1}(\bar{x}))\to\pi_0G(k(g^{-1}(\bar{x})))$ and the identifications

$$\pi_0G(k(x))=z^p_{\bar{x}}(X\times\Delta^n)$$
$$\pi_0G(k(g^{-1}(\bar{x})))=z^p_{g^{-1}(\bar{x})}(X\times\Delta^m)$$

Since the pull-back on cycles is defined via Serre's intersection multiplicity formula and Serre's vanishing theorem, we have the commutative diagram

$$\begin{array}{ccc} \pi_0 G(\bar{x}) & \longrightarrow & z^p_{\bar{x}}(X\times\Delta^n) \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle g^*} \\ \pi_0 G(g^{-1}(\bar{x})) & \longrightarrow & z^p_{g^{-1}(\bar{x})}(X\times\Delta^m) \end{array}$$

with surjective rows.

Putting these two commutative diagrams together with the weak equivalence from Corollary 4.3.2 gives the functoriality of the weak equivalence (5.4.1) with respect to the simplicial structure. □

5.5. **Bloch motivic cohomology.** As in Theorem 3.7.1 the method of Kahn [11], one can make the cycle complexes $z^p(X,*)$ functorial in $X\in\mathbf{Sm}_k$. Specifically, there are fibrant complexes of Nisnevic sheaves on $\mathbf{Sm}_k$, $\mathcal{Z}^p$, whose image in the derived category of Nisnevic sheaves on $\mathbf{Sm}/\!/k$ is isomorphic to the functor

$$X\mapsto z^p(X,2p-*).$$

Thus, for each $X\in\mathbf{Sm}_k$, we have the complex $\mathcal{Z}^p(X)^*$, with a natural isomorphism $\mathcal{Z}^p(X)^*\cong z^p(X,2p-*)$ in $\mathbf{D}^-_{\mathrm{Nis}}(X)$. The cohomology of $\mathcal{Z}^p(X)$ is the *Bloch motivic cohomology* of X:

$$H^n(X,\mathbb{Z}(p)) := H^n(\mathcal{Z}^p(X)) = H_{2p-n}(z^p(X,*)).$$

We consider $X\mapsto\mathcal{Z}^p(X)$ as a functor $\mathcal{Z}^p:\mathbf{Sm}_k^{\mathrm{op}}\to\mathbf{Spt}$ by taking the associated Eilenberg-Maclane spectrum. As in Theorem 3.7.1, the localization theorem for the complexes $z^p(X,*)$ yields the natural weak equivalences

$$\mathcal{Z}^p\to\Omega_T(\Sigma^2\mathcal{Z}^{p+1}).$$

Theorem 5.5.1. *For each $p\geq 0$ and each $q\geq 0$ we have the isomorphism in $\mathcal{H}\mathbf{Spt}(\mathbf{Sm}_k)$*

$$(\mathcal{Z}^p)^{(q/q+1)}\cong\begin{cases}0 & \text{for } q\neq p\\ \mathcal{Z}^p & \text{for } q=p.\end{cases}$$

Proof. From Theorem 5.4.1, we have the isomorphism in $\mathcal{H}\mathbf{Spt}(\mathbf{Sm}_k)$

$$\mathcal{Z}^p\cong\Sigma^{-2p}K^{(p/p+1)},$$

giving the isomorphism in $\mathcal{H}\mathbf{Spt}(\mathbf{Sm}_k)$

$$(\mathcal{Z}^p)^{(q/q+1)}\cong\Sigma^{-2p}(K^{(p/p+1)})^{(q/q+1)}.$$

The result then follows from Theorem 4.3.1. □

6. The $\mathbb{P}^1$-stable theory

6.1. **$\mathbb{P}^1$-spectra.** We now pass to the setting of $\mathbb{P}^1$-spectra. For $i : Y \to X$ a closed embedding or open immmersion, and $E : \mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{Spt}$ a functor, we write $E(X/Y)$ for the homotopy fiber of $i^* : E(X) \to E(Y)$.This notation extends in the evident way to define $E((X,x) \wedge (Y,y))$ for pointed smooth k-scheme (X,x) and (Y,y), where $E(X \times y \vee x \times Y)$ is the fiber of

$$E(X \times y \coprod x \times Y) \to E(x \times y)$$

In the same way, we have the spectrum $E(\Lambda_{i=1}^n(X_i, x_i))$.

Given $E : \mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{Spt}$, define $\Omega^1_{\mathbb{P}^1} E : \mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{Spt}$ by

$$\Omega^1_{\mathbb{P}^1} E(X) = E(\mathbb{P}^1 \wedge X_+).$$

Note that, if E satisfies axioms 1 and 2, then the inclusion $\infty \to \mathbb{P}^1 \setminus \{0\} \cong \mathbb{A}^1$ defines weak equivalences

$$(\Omega_T E)(X) = E((\mathbb{P}^1/\mathbb{P}^1 \setminus \{0\}) \wedge X_+) \to \Omega^1_{\mathbb{P}^1} E(X$$

We give two definitions to fix ideas:

Definition 6.1.1. A $\mathbb{P}^1$-Ω-spectrum $\mathcal{E}$ on $\mathbf{Sm}_k$ is given by

(1) A sequence $(E_0, E_1, \ldots)$, where $E_j : \mathbf{Sm}_k \to \mathbf{Spt}$ is a functor satisfying axioms 1 and 2.
(2) Weak equivalences in $\mathbf{Spt}(\mathbf{Sm}_k)$, $\epsilon_j : E_j \to \Omega^1_{\mathbb{P}^1} E_{j+1}$, $j = 0, 1 \ldots$

Maps are maps of sequences respecting the maps in (2). We denote the category of $\mathbb{P}^1$-Ω-spectra on $\mathbf{Sm}_k$ by $\mathbf{Spt}^{\Omega}_{\mathbb{P}^1}(\mathbf{Sm}_k)$.

In particular, if $\mathcal{E}$ is a $\mathbb{P}^1$-Ω-spectrum on $\mathbf{Sm}_k$, then each spectrum E_j satisfies axioms 1, 2 and 3.

For the next definition, we use the notion of a *space over k* in the sense of Morel-Voevodsky [16]: recall that a space over k is a Nisnevic sheaf of simplicial sets on $\mathbf{Sm}_k$. Unless specifically mentioned, we will consider $\mathbb{P}^1$ as a pointed space over k using ∞ as the base-point. For a pointed space Z, we write $\Sigma_{\mathbb{P}^1} T$ for $\mathbb{P}^1 \wedge Z$.

Definition 6.1.2. A $\mathbb{P}^1$-spectrum $\mathcal{E}$ on $\mathbf{Sm}_k$ is given by

(1) A sequence $(E_0, E_1, \ldots)$, where each E_j is a pointed space.
(2) Maps of spaces over k, $\epsilon_j : \mathbb{P}^1 \wedge E_j \to E_{j+1}$, $j = 0, 1 \ldots$

Maps are maps of sequences respecting maps in (2). We denote the category of $\mathbb{P}^1$-spectra on $\mathbf{Sm}_k$ by $\mathbf{Spt}_{\mathbb{P}^1}(\mathbf{Sm}_k)$.

If $\mathcal{E} = (E_0, E_1, \ldots)$ is a $\mathbb{P}^1$-spectrum or a $\mathbb{P}^1$-Ω-spectrum, we have the suspensions

$$\Sigma^1_{\mathbb{P}^1} \mathcal{E} := (E_1, E_2, \ldots) \qquad \Sigma^{-1}_{\mathbb{P}^1} \mathcal{E} := (\Omega^1_{\mathbb{P}^1} E_0, \Omega^1_{\mathbb{P}^1} E_1, \ldots).$$

6.2. **Model structure and homotopy categories.** We recall the category $\mathcal{SH}(k)$ and its relation to Definitions 6.1.1 and 6.1.2. For details, we refer the reader to [14, 15].

Denote the category $\mathcal{H}\mathbf{Spt}_{\mathrm{Nis}}(\mathbf{Sm}_k)$ by $\mathcal{SH}_s(k)$.

Consider the model category $\mathbf{Spt}_{\mathrm{Nis}}(\mathbf{Sm}_k)$. Call an object E of $\mathbf{Spt}_{\mathrm{Nis}}(\mathbf{Sm}_k)$ $\mathbb{A}^1$-*local* if the projections $X \times \mathbb{A}^1 \to X$ induce a weak equivalence (in $\mathbf{Spt}_{\mathrm{Nis}}(\mathbf{Sm}_k)$) $E \to E^{\mathbb{A}^1}$. Call a map $f : E \to F$ an $\mathbb{A}^1$-*weak equivalence* if, for each $\mathbb{A}^1$-local Z, the map

$$f^* : \mathrm{Hom}_{\mathcal{SH}_s(k)}(F, Z) \to \mathrm{Hom}_{\mathcal{SH}_s(k)}(E, Z)$$

is an isomorphism. Note that a weak equivalence in $\mathbf{Spt}_{\mathrm{Nis}}(\mathbf{Sm}_k)$ is automatically an $\mathbb{A}^1$-weak equivalence. We have the Bousfield localization of $\mathbf{Spt}_{\mathrm{Nis}}(\mathbf{Sm}_k)$ with respect to the $\mathbb{A}^1$-weak equivalences, i.e., the model structure on $\mathbf{Spt}_{\mathrm{Nis}}(\mathbf{Sm}_k)$ with the same cofibrations as $\mathbf{Spt}_{\mathrm{Nis}}(\mathbf{Sm}_k)$, the weak equivalences the $\mathbb{A}^1$-weak equivalences, and the fibrations determined by the RLP with respect to trivial cofibrations. We denote this model category by $\mathbf{Spt}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathbf{Sm}_k)$ and the resulting homotopy category by $\mathcal{SH}^{\mathbb{A}^1}_s(k)$.

We next consider the category of (s,p)-*spectra on* $\mathbf{Sm}_k$. The objects are sequences $\mathcal{E} := (E_0, E_1, \ldots)$ of functors $E_n : \mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{Spt}$, together with connecting morphisms $\epsilon_n : \mathbb{P}^1 \wedge E_n \to E_{n+1}$. Maps are sequences of maps in $\mathbf{Spt}(\mathbf{Sm}_k)$ respecting the connecting morphisms.

For an (s,p)-spectrum $(\mathcal{E}, \epsilon_n)$, the ϵ_n induce, for each $X \in \mathbf{Sm}_k$, the map

$$\epsilon_n(X) : E_n(X) \to E_{n+1}(\mathbb{P}^1 \wedge X_+).$$

For X is in $\mathbf{Sm}_k$, we have the bi-graded stable homotopy groups

$$\pi^s_{a,b}(\mathcal{E}(X)) = \lim_{n\to\infty} \pi^s_{a+2n}(E_n((\mathbb{P}^1)^{n+b} \wedge X_+)),$$

using the maps $\epsilon_n(-)$ for the transition maps in the inductive system of homotopy groups. The $\pi^s_{a,b}(\mathcal{E}(X))$ form a presheaf of abelian groups on $\mathbf{Sm}_k$; we let $\pi^s_{a,b}(\mathcal{E})$ denote the associated Nisnevic sheaf.

A map $f : \mathcal{E} \to \mathcal{F}$ of (s,p)-spectra is called a weak equivalence if f induces an isomorphism $f_* : \pi^s_{*,*}(\mathcal{E}) \to \pi^s_{*,*}(\mathcal{F})$ on the homotopy sheaves. f is a cofibration if f_0 is a cofibration in $\mathbf{Spt}_{\mathrm{Nis}}(\mathbf{Sm}_k)$, and for each $n \geq 0$, the map

$$E_{n+1} \coprod_{\mathbb{P}^1 \wedge E_n} \mathbb{P}^1 \wedge F_n \to F_{n+1}$$

is a cofibration in $\mathbf{Spt}_{\mathrm{Nis}}(\mathbf{Sm}_k)$. The fibrations are characterized by having the RLP with respect to trivial cofibrations. This gives us the model category of (s,p)-spectra on $\mathbf{Sm}_k$, denoted $\mathbf{Spt}_{(s,p)}(\mathbf{Sm}_k)$. The homotopy category is denoted $\mathcal{SH}(k)$.

If $\mathcal{E} = (E_0, E_1, \ldots)$ is a $\mathbb{P}^1$-spectrum, we can form the associated (s,p)-spectrum by taking the term-wise (simplicial) suspension spectra $(\Sigma^*_s E_0, \Sigma^*_s E_1, \ldots)$. If $\mathcal{E} = (E_0, E_1, \ldots)$ is a $\mathbb{P}^1$-Ω-spectrum, the maps $\epsilon_n : E_n \to \Omega_{\mathbb{P}^1} E_{n+1}$ induce by adjointness the maps $\epsilon'_n : \mathbb{P}^1 \wedge E_n \to E_{n+1}$, forming the associated (s,p)-spectrum. Similarly, if $\mathcal{E} = ((E_0, E_1, \ldots), \epsilon_n)$ is an (s,p)-spectrum, we have by adjointness the maps $\epsilon'_n : E_n \to \Omega_{\mathbb{P}^1} E_{n+1}$; if $\mathcal{E}$ is fibrant, this forms a $\mathbb{P}^1$-Ω-spectrum. Thus, we may pass from $\mathbb{P}^1$-spectra to $\mathbb{P}^1$-Ω-spectra by first forming the associated (s,p)-spectrum, taking a fibrant model, and then using adjointness. We denote this functor by $\mathcal{E} \mapsto \Omega^\infty_{\mathbb{P}^1}\mathcal{E}$.

We can also take the $\mathbb{P}^1$ infinite loop spaces of an (s,p)-spectrum or a $\mathbb{P}^1$-Ω-spectrum, forming a $\mathbb{P}^1$-spectrum:

$$\mathcal{E} = (E_0, E_1, \ldots) \mapsto (\lim_m \Omega^m_s E_{0m}, \lim_m \Omega^m_s E_{1m}, \ldots).$$

Here Ω^m_s is the loop-space functor with respect to the simplicial structure.

Via these functors, the model structure on $\mathbf{Spt}_{(s,p)}(\mathbf{Sm}_k)$ induces model structures on the categories of $\mathbb{P}^1$-spectra and on $\mathbb{P}^1$-Ω-spectra, and gives a Quillen equivalence of these three model categories. In particular, we can consider $\mathbb{P}^1$-spectra and $\mathbb{P}^1$-Ω-spectra as objects in $\mathbf{Spt}_{(s,p)}(k)$ or in $\mathcal{SH}(k)$.

Remarks 6.2.1. (1) The above definition of $\mathcal{SH}(k)$ is slightly different than the one given by [15]. First of all, Morel uses the suspension functor with respect to $(\mathbb{A}^1 \setminus \{0\}, 1)$, rather than $(\mathbb{P}^1, \infty)$. Secondly, the individual spaces occurring in the spectra E_n are required to be sheaves of pointed simplicial sets rather than presheaves. Since $(\mathbb{P}^1, \infty)$ is homotopy equivalent to $S^1 \wedge (\mathbb{A}^1 \setminus \{0\}, 1)$, the two different choices of suspensions lead to Quillen equivalent model categories, and as the cofibrations and weak equivalences in the presheaf category are defined stalk-wise, using presheaves or sheaves also yield Quillen equivalent model categories. Thus, we may use the same notation $\mathcal{SH}(k)$ for the homotopy category.

(2) We have the functor $\Sigma^\infty_{\mathbb{P}^1} : \mathbf{Spt}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathbf{Sm}_k) \to \mathbf{Spt}_{(s,p)}(\mathbf{Sm}_k)$, defined by sending E to the sequence $\Sigma^\infty_{\mathbb{P}^1} E := (E, \mathbb{P}^1 \wedge E, \ldots)$, with the evident connecting maps. $\Sigma^\infty_{\mathbb{P}^1}$ is a left Quillen functor, with right adjoint the $\mathbb{P}^1$-infinite loop space functor $\mathcal{E} \mapsto \Omega^\infty_{\mathbb{P}^1}\mathcal{E}$, where

$$\Omega^\infty_{\mathbb{P}^1}\mathcal{E} := \lim_{n\to\infty} \Omega^n_{\mathbb{P}^1} E_n$$

if $\mathcal{E} = (E_0, E_1, \ldots)$. In particular, this shows that a weak equivalence $\mathcal{E} \to \mathcal{F}$ between fibrant objects in $\mathbf{Spt}_{(s,p)}(\mathbf{Sm}_k)$ induces a point-wise weak equivalence $f_n : E_n \to F_n$ on the various S^1-spectra.

Examples 6.2.2. (1) Each $X \in \mathbf{Sm}_k$ determines the $\mathbb{P}^1$-*suspension spectrum*

$$\Sigma^\infty_{\mathbb{P}^1} X_+ := (X_+, \Sigma^1_{\mathbb{P}^1} X_+, \Sigma^2_{\mathbb{P}^1} X_+, \ldots)$$

and the corresponding $\mathbb{P}^1$-Ω-spectrum $\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1} X_+$. For $X = \operatorname{Spec} k$, we write S^0_k for $\operatorname{Spec} k_+$. We have the $\mathbb{P}^1$-*sphere spectrum*

$$\Sigma^\infty_{\mathbb{P}^1} S^0_k =: (S^0_k, \mathbb{P}^1_k, \ldots, \Lambda^d \mathbb{P}^1_k, \ldots)$$

and the $\mathbb{P}^1$-Ω-spectrum $\mathbb{S} := \Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1} S^0_k$.

(2) For $X \in \mathbf{Sm}_k$, let $\mathrm{Cyc}^d(X)$ denote the set of effective cycles $W = \sum_i n_i W_i$, $n_i > 0$, in $X \times (\mathbb{P}^1)^d$, with each irreducible component W_i finite over X, and dominating some component of X. This defines the Nisnevic sheaf $X \mapsto \mathrm{Cyc}^d(X)$. The reduced version is the quotient

$$\widetilde{\mathrm{Cyc}}^d(X) := \mathrm{Cyc}^d(X) / \sum_{i=1}^d p_{i*}(\mathrm{Cyc}^{d-1}(X)),$$

where p_{i*} is the map induced by the inclusion $p_i : (\mathbb{P}^1)^{d-1} \to (\mathbb{P}^1)^d$ defined by inserting ∞ in the ith spot.

We have the map $\mathbb{P}^1 \to \mathrm{Cyc}^1$ defined by taking the graph of a map $f : X \to \mathbb{P}^1$. Taking the product over X gives the map

$$\widetilde{\mathrm{Cyc}}^d(X) \wedge \widetilde{\mathrm{Cyc}}^{d'}(X) \to \widetilde{\mathrm{Cyc}}^{d+d'}(X),$$

which thus gives us the structure morphisms

$$\mathbb{P}^1 \wedge \widetilde{\mathrm{Cyc}}^d(X) \to \widetilde{\mathrm{Cyc}}^{d+1}(X)$$

This structure defines the $\mathbb{P}^1$-spectrum $\mathcal{H}\mathbb{Z}$; in characteristic zero, $\mathcal{H}\mathbb{Z}$ is represented by the symmetric powers of $(\mathbb{P}^1)^{\wedge d}$ in the evident way.

The associated $\mathbb{P}^1$-Ω-spectrum, $\Omega^\infty_{\mathbb{P}^1}\mathcal{H}\mathbb{Z}$, is equivalent to the sequence

$$(\mathcal{Z}^0, \Sigma^2\mathcal{Z}^1, \ldots, \Sigma^{2d}\mathcal{Z}^d)$$

with connecting maps the localization weak equivalences

$$\mathcal{Z}^p \to \Omega_{\mathbb{P}^1}\Sigma^2(\mathcal{Z}^{p+1}).$$

A direct map relating the two constructions is given as follows: Send $W \in \mathrm{Cyc}^d(X \times \Delta^n)$ to the cycle $W \in z^d(X \times (\mathbb{P}^1)^d, n)$, which we then restrict to $W^0 \in z^d(X \times \mathbb{A}^d, n)$. This gives a natural transformation

$$\widetilde{\mathrm{Cyc}}^d(X \times \Delta^*) \to z^d(X \times \mathbb{A}^d, *) \sim z^d(X, *),$$

which gives the direct relation. That this map gives a weak equivalence is proved by Voevodsky-Suslin [7] assuming resolution of singularities, and in general by Voevodsky [19].

6.3. **The connecting map.** Before defining the homotopy coniveau tower on the level of $\mathbb{P}^1$-spectra, we need a technical modification of the functor $E^{(p)}$.

Let $\mathcal{E} = ((E_0, E_1, \ldots), \epsilon_*)$ be a $\mathbb{P}^1$-Ω-spectrum. Composing the connecting maps ϵ_* give us weak equivalences

$$\epsilon_{n,m} : E_n \to \Omega^m_{\mathbb{P}^1} E_{n+m}.$$

Using the diagram of Remark 3.7.4, these maps give a natural tower of weak equivalences

$$\ldots \to \Omega^m_{\mathbb{P}^1} \widetilde{E^{(p+m)}_{n+m}} \to \Omega^{m-1}_{\mathbb{P}^1} \widetilde{E^{(p+m-1)}_{n+m-1}} \to \ldots \to E^{(p)}_n$$

If we then take the homotopy limit over this tower, with projection

$$\widehat{E^{(p)}_n} \to E^{(p)}_n,$$

then the map $\psi_{p,n} : E^{(p)}_n \to \Omega_{\mathbb{P}^1} E^{(p+1)}_{n+1}$ in $\mathcal{H}\mathbf{Spt}_{\mathrm{Nis}}(\mathbf{Sm}_k)$ lifts to the map

$$\hat{\psi_{p,n}} \widehat{E^{(p)}_n} \to \Omega_{\mathbb{P}^1} \widehat{E^{(p+1)}_{n+1}}$$

in $\mathbf{Spt}_{\mathrm{Nis}}(\mathbf{Sm}_k)$.

We will henceforth replace $E^{(p)}_n$ with a functorial bifibrant model of $\widehat{E^{(p)}_n}$, so that we may assume that the maps $\psi_{p,n} : E^{(p)}_n \to \Omega_{\mathbb{P}^1} E^{(p+1)}_{n+1}$ exist in $\mathbf{Spt}_{\mathrm{Nis}}(\mathbf{Sm}_k)$, are natural in $\mathcal{E}$, and are compatible with change in p.

6.4. **The stable homotopy coniveau filtration.** For a $\mathbb{P}^1$-Ω-spectrum $\mathcal{E} = ((E_0, E_1, \ldots), \epsilon_*)$, and integer p, set $\phi_p\mathcal{E} := (E^{(p)}_0, E^{(p+1)}_1, \ldots)$, where the maps ϵ_d are given by the localization weak equivalence of Theorem 3.7.1(3) (suitably refined as in §6.3)

$$(E_d)^{(d+p)} \xrightarrow{\epsilon_d^{(d+p)}} (\Omega_{\mathbb{P}^1} E_{d+1})^{(d+p)} \xrightarrow{\psi_{d+p+1}} \Omega_{\mathbb{P}^1}(E^{(d+p+1)}_{d+1}).$$

The natural maps $E^{(p+j)}_j \to E_j$ define the map of $\mathbb{P}^1$-Ω-spectra

$$\phi_p\mathcal{E} \to \mathcal{E}.$$

Recall that $E^{(n)} = E^{(0)}$ for $n < 0$.

We thus have the tower of $\mathbb{P}^1$-Ω-spectra

$$\ldots \to \phi_{p+1}\mathcal{E} \to \phi_p\mathcal{E} \to \ldots \to \phi_0\mathcal{E} \to \phi_{-1}\mathcal{E} \to \ldots \to \mathcal{E}. \tag{6.4.1}$$

We write $\phi_{p/p+r}\mathcal{E}$ for the cofiber $\phi_{p+r}\mathcal{E} \to \phi_p\mathcal{E}$ and $\sigma_p\mathcal{E}$ for $\phi_{p/p+1}\mathcal{E}$.

Remark 6.4.1. For $q \geq 0$ and $p \geq 0$, we have the identity

$$\Sigma^q_{\mathbb{P}^1}(\phi_p\mathcal{E}) = \phi_{p+q}\Sigma^q_{\mathbb{P}^1}\mathcal{E};$$

for $q \geq 0$ and $p < 0$, we have this identity in "sufficiently large degree"; in any case, a weak equivalence. Similarly, for $q < 0$, the localization weak equivalence

$$\Omega^{-q}_{\mathbb{P}^1}(E^{(m)}_n) \sim E^{(m+q)}_{n+q}$$

gives a natural isomorphism

$$\Sigma^q_{\mathbb{P}^1}(\phi_p\mathcal{E}) \sim \phi_{p+q}\Sigma^q_{\mathbb{P}^1}\mathcal{E}$$

in $\mathcal{SH}(k)$.

7. Suspension spectra and the slice filtration

In the $\mathbb{A}^1$-stable homotopy category $\mathcal{SH}(k)$, one has the smallest localizing subcategory closed under colimits and containing all suspension spectra $\Sigma^\infty_{\mathbb{P}^1}X_+$ with $X \in \mathbf{Sm}_k$, denoted $\mathcal{SH}^{\text{eff}}(k)$; we have as well the various suspensions $\Sigma^p_{\mathbb{P}^1}\mathcal{SH}^{\text{eff}}(k)$. Voevodsky [20] has defined a sequence of functors $f_p : \mathcal{SH}(k) \to \mathcal{SH}(k)$ by using the right adjoint r_p to the inclusion $i_p : \Sigma^p_{\mathbb{P}^1}\mathcal{SH}^{\text{eff}}(k) \to \mathcal{SH}(k)$: $f_p = i_p \circ r_p$. Concretely, for $\mathcal{E}$ in $\mathcal{SH}(k)$, one has a canonical map $f_p\mathcal{E} \to \mathcal{E}$ which is universal for maps $i_p\mathcal{F} \to \mathcal{E}$, $\mathcal{F} \in \Sigma^p_{\mathbb{P}^1}\mathcal{SH}^{\text{eff}}(k)$. Additionally, one has the tower

$$\ldots \to f_{d+1}\mathcal{E} \to f_d\mathcal{E} \to \ldots \to f_0\mathcal{E} \to f_{-1}\mathcal{E} \to \ldots \to \mathcal{E},$$

defining the *slice filtration* of $\mathcal{E}$. The cofiber of $f_{d+1}\mathcal{E} \to f_d\mathcal{E}$ is denoted $s_d\mathcal{E}$.

In this section, we analyze the filtration ϕ_p for suspension spectra, and show that $\phi_p = f_p$. Thus, the stable homotopy coniveau filtration agrees with the slice filtration.

7.1. **A modified tower.** For a $\mathbb{P}^1$-Ω-spectrum $\mathcal{E}$, we define a modified version $\phi^{\mathbb{A}^*}_p\mathcal{E}$ of $\phi_p\mathcal{E}$.

For a functor $E : \mathbf{Sm}^{\text{op}}_k \to \mathbf{Spt}$, we have the associated functor $E^{\mathbb{A}^n}$ with

$$E^{\mathbb{A}^n}(X) := E(X \times \mathbb{A}^n).$$

If $\mathcal{F} = (F_0, F_1, \ldots)$ is a $\mathbb{P}^1$-Ω-spectrum, define $\mathcal{F}^{\mathbb{A}^*}$ to be the $\mathbb{P}^1$-Ω-spectrum

$$(F_0, F_1^{\mathbb{A}^1}, \ldots, F_n^{\mathbb{A}^n}, \ldots),$$

with connecting maps given by

$$F_j(X \times \mathbb{A}^j) \xrightarrow{\epsilon_j(X\times\mathbb{A}^j)} \Omega^1_{\mathbb{P}^1}F_{j+1}(X \times \mathbb{A}^j)$$
$$\xrightarrow{p^*} \Omega^1_{\mathbb{P}^1}F_{j+1}(X \times \mathbb{A}^j \times \mathbb{A}^1) = \Omega^1_{\mathbb{P}^1}F_{j+1}(X \times \mathbb{A}^{j+1}).$$

The pull-back maps $p^* : E(X) \to E(X \times \mathbb{A}^n)$ define the natural transformation

$$p_n : E \to E^{\mathbb{A}^n};$$

if E satisfies axiom 1, then p_n is a weak equivalence. Similarly, we have the weak equivalence

$$p : \mathcal{F} \to \mathcal{F}^{\mathbb{A}^*}$$

Suppose that E satisfies axiom 3. We define a weak equivalence

$$\delta_n : ((\Omega_{\mathbb{P}^1} E)^{(p)})^{\mathbb{A}^n} \to \Omega_{\mathbb{P}^1}((E^{(p+1)})^{\mathbb{A}^{n+1}}).$$

Let $\Delta^0 \subset \mathbb{A}^1 \times \mathbb{P}^1$ be graph of the standard open immersion $\mathbb{A}^1 \to \mathbb{P}^1$ with image $\mathbb{P}^1 \setminus \infty$. We have the functors

$$\begin{aligned}
F_1 :=& X \mapsto (\Omega_{\mathbb{P}^1} E)^{(p)}(X \times \mathbb{A}^n \times \Delta^0),\\
F_2 :=& X \mapsto (E^{(p+1)})^{X \times \mathbb{A}^n \times \Delta^0}(X \times \mathbb{A}^n \times \mathbb{A}^1 \times \mathbb{P}^1)\\
F_3 :=& X \mapsto (\Omega_{\mathbb{P}^1} E^{(p+1)})(X \times \mathbb{A}^n \times \mathbb{A}^1)
\end{aligned}$$

The projection $\mathbb{A}^n \times \Delta^0 \to \mathbb{A}^n$ gives the isomorphism in $\mathcal{H}\mathbf{Spt}(\mathbf{Sm}_k)$

$$\psi_0^n : ((\Omega_{\mathbb{P}^1} E)^{(p)})^{\mathbb{A}^n} \to F_1.$$

Using the defining equation $X_1 - tX_0$ for Δ^0, and the localization theorem for $E^{(n+1)}$, we have the isomorphism in $\mathcal{H}\mathbf{Spt}(\mathbf{Sm}_k)$

$$\psi_1^n : F_1 \to F_2.$$

Similarly, as $\Delta^0 \cap \mathbb{A}^1 \times \infty = \emptyset$, we have the isomorphism in $\mathcal{H}\mathbf{Spt}(\mathbf{Sm}_k)$

$$\psi_2^n : F_2 \to F_3.$$

Thus, we have the isomorphism

$$\delta_n : ((\Omega_{\mathbb{P}^1} E)^{(p)})^{\mathbb{A}^n} \to \Omega_{\mathbb{P}^1}((E^{(p+1)})^{\mathbb{A}^{n+1}}),$$

in $\mathcal{H}\mathbf{Spt}(\mathbf{Sm}_k)$. Using the method of Theorem 3.7.1 and §6.3, we may assume that the maps δ_n are weak equivalences in $\mathbf{Spt}_{\mathrm{Nis}}(\mathbf{Sm}_k)$, natural in E, and compatible with change in n.

For $\mathcal{E} = (E_0, E_1, \ldots)$ a $\mathbb{P}^1$-Ω-spectrum, we may thus form the $\mathbb{P}^1$-Ω-spectrum

$$\phi_p^{\mathbb{A}^*}\mathcal{E} := (E_0^{(p)}, (E_1^{(p+1)})^{\mathbb{A}^1}, \ldots, (E_n^{(p+n)})^{\mathbb{A}^n}, \ldots)$$

with connecting maps given by the δ_n.

Lemma 7.1.1. *There are isomorphisms in $\mathcal{SH}(k)$*

$$\phi_p^{\mathbb{A}^*}\mathcal{E} \cong \phi_p\mathcal{E},$$

natural in $\mathcal{E}$ and compatible with change in p.

Proof. Let $E : \mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{Spt}$ be a functor satisfying axioms 1 and 2. The inclusion of pairs $i_0 : (\mathbb{P}^1, 0) \to (\mathbb{A}^n \times \mathbb{A}^1 \times \mathbb{P}^1, \mathbb{A}^n \times \Delta^0)$ defined by the 0-section $\mathbb{P}^1 \to \mathbb{A}^n \times \mathbb{A}^1 \times \mathbb{P}^1$ induces the map of functors

$$(X \mapsto E^{X \times \mathbb{A}^n \times \Delta^0}(X \times \mathbb{A}^n \times \mathbb{A}^1 \times \mathbb{P}^1)) \xrightarrow{i_0^*} (X \mapsto E^{X \times 0}(X \times \mathbb{P}^1))$$

which is easily seen to be a weak equivalence. One easily sees that this gives us the desired isomorphism

$$i_0^* : \phi_p^{\mathbb{A}^*}\mathcal{E} \cong \phi_p \mathcal{E}.$$

□

7.2. **Suspension spectra.** In this section, we define natural maps of $\mathbb{P}^1$-spectra

$$\bar{\iota}_X : \Sigma^\infty_{\mathbb{P}^1} X_+ \to \phi_0 \Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1} X_+$$
$$\iota_X : \Sigma^\infty_{\mathbb{P}^1} X_+ \to \phi_0^{\mathbb{A}^*} \Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1} X_+.$$

In the construction, we note that the $\mathbb{P}^1$-Ω suspension spectrum $\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1} X_+$ is a bifibrant object of $\mathbf{Spt}(\mathbf{Sm}_k)$, so that each spectrum $(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1} X_+)_n(Y)$ is bifibrant, for each $Y \in \mathbf{Sm}_k$. In particular, each map $(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1} X_+)_n(Y) \to (\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1} X_+)_m(W)$ in $\mathcal{SH}$ lifts (uniquely up to homotopy) to a map $(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1} X_+)_n(Y) \to (\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1} X_+)_m(W)$ in $\mathbf{Spt}$.

To define ι_X, we map $\Sigma^d_{\mathbb{P}^1} X_+$ to $((\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1} X_+)^{(d)}_d)^{\mathbb{A}^d}$ as follows: Let $j : \mathbb{A}^d \to (\mathbb{P}^1)^d$ be the standard open inclusion identifying $\mathbb{A}^d$ with the open subscheme $(\mathbb{P}^1 - \infty)^d$, and let $\Gamma_d \subset (\mathbb{P}^1)^d \times \mathbb{A}^d$ be the transpose of the graph of j. Note that the projection $\Gamma_d \to \mathbb{A}^d$ is an isomorphism, and the map

$$\begin{aligned}&(X_{0,1} : X_{1,1}, \ldots, X_{0,d}, X_{1,d}; T_1, \ldots, T_d)\\ &\qquad \mapsto (X_{0,1} : X_{1,1} - T_1 X_{0,1}, \ldots, X_{0,d}, X_{1,d} - T_d X_{0,d}; T_1, \ldots, T_d)\end{aligned}$$

sends $((\mathbb{P}^1)^d \times \mathbb{A}^d, \Gamma_d)$ to $((\mathbb{P}^1)^d, 0^d) \times \mathbb{A}^d$. Thus we have a weak equivalence

$$(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1} X_+)_0(X \times \Gamma_d) \xrightarrow{\sim} (\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1} X_+)_d^{X \times \Gamma_d}(X \times (\mathbb{P}^1)^d \times \mathbb{A}^d).$$

Since $X \times \Gamma_d$ is contained in $X \times (\mathbb{P}^1 \setminus \infty)^d \times \mathbb{A}^d$, this weak equivalence descends to a weak equivalence

$$(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1} X_+)_0(X \times \Gamma_d) \xrightarrow{\sim} (\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1} X_+)_d^{X \times \Gamma_d}(\Sigma^d_{\mathbb{P}^1}(X \times \mathbb{A}^d)_+).$$

The projection $X \times \Gamma_d \to X$ composed with the canonical map

$$X \to X_+ \to (\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1} X_+)_0$$

gives a section

$$\iota_\Gamma \in (\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1} X_+)_0(X \times \Gamma_d),$$

and thus gives the canonical section in $(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)^{X\times\Gamma_d}_d(\Sigma^d_{\mathbb{P}^1}(X\times\mathbb{A}^d)_+)$. Composing with the natural map

$$(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)^{X\times\Gamma_d}_d(\Sigma^d_{\mathbb{P}^1}(X\times\mathbb{A}^d)_+)\to(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)^{(d)}_d(\Sigma^d_{\mathbb{P}^1}(X\times\mathbb{A}^d)_+)$$

gives a canonical section

$$\iota_d\in(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)^{(d)}_d(\Sigma^d_{\mathbb{P}^1}(X\times\mathbb{A}^d)_+).$$

Replacing Γ_d with the point $0^d\in(\mathbb{P}^1)^d$, the same procedure yields the canonical section

$$\bar\iota_d\in(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)^{(d)}_d(\Sigma^d_{\mathbb{P}^1}X_+).$$

By evaluation, ι_d gives the natural transformation

$$Y\mapsto\iota_d(Y):(\Sigma^d_{\mathbb{P}^1}X_+)(Y)\to(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)^{(d)}_d(Y\times\mathbb{A}^d),$$

and $\bar\iota_d$ gives the natural transformation

$$Y\mapsto\bar\iota_d(Y):(\Sigma^d_{\mathbb{P}^1}X_+)(Y)\to(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)^{(d)}_d(Y).$$

This gives us the two maps of functors on $\mathbf{Sm}_k$

$$\iota_d:\Sigma^d_{\mathbb{P}^1}X_+\to((\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)^{(d)}_d)^{\mathbb{A}^d}$$
$$\bar\iota_d:\Sigma^d_{\mathbb{P}^1}X_+\to(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)^{(d)}_d.$$

Lemma 7.2.1. *(1) The composition*

$$\Sigma^d_{\mathbb{P}^1}X_+\xrightarrow{\bar\iota_d}(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)^{(d)}_d\to(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)_d$$

is naturally homotopic to the canonical map $\Sigma^d_{\mathbb{P}^1}X_+\to(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)_d$.

(2) The composition

$$\Sigma^d_{\mathbb{P}^1}X_+\xrightarrow{\bar\iota_d}(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)^{(d)}_d\xrightarrow{p_d}((\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)^{(d)}_d)^{\mathbb{A}^d}$$

is naturally homotopic to ι_d.

Proof. Via the defining equations for 0^d, we identify the normal bundle of 0^d in $(\mathbb{P}^1)^d$ with the open neighborhood $\mathbb{A}^d=(\mathbb{P}^1-\infty)^d$ of 0^d. This gives the canonical homotopy equivalence of $\Sigma^d_{\mathbb{P}^1}X_+$ with $((\mathbb{P}^1)^d/((\mathbb{P}^1)^d\setminus 0^d))\wedge X_+$, which leads to the sequence of weak equivalences

$$\begin{array}{ccc}(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)_0(X) & \xrightarrow{\sim} & (\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)^{X\times 0}_d(X\times(\mathbb{P}^1)^d)\\ & & \|\\ & & (\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)_d(((\mathbb{P}^1)^d/((\mathbb{P}^1)^d\setminus 0^d))\wedge X_+)\\ & & \uparrow\wr\\ & & (\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)_d(\Sigma^d_{\mathbb{P}^1}X_+)\end{array}$$

Since $\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+$ is the $\mathbb{P}^1$-Ω-spectrum associated to the $\mathbb{P}^1$-suspension spectrum $\Sigma^\infty_{\mathbb{P}^1}X_+$, it follows that the map

$$\begin{aligned}\mathrm{Hom}_{\mathcal{H}\mathbf{Spt}(\mathbf{Sm}_k)}&(\Sigma^\infty_s X_+, (\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)_0)\\ &\to \mathrm{Hom}_{\mathcal{H}\mathbf{Spt}(\mathbf{Sm}_k)}(\Sigma^\infty_s\Sigma^d_{\mathbb{P}^1}X_+, (\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)_d)\end{aligned}$$

induced by this diagram is just the standard map sending f to the map induced by $\Sigma^d_{\mathbb{P}^1}f$. Thus, the image of the canonical map $\Sigma^\infty_s X_+ \to (\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)_0$ is the canonical map $\Sigma^\infty_s\Sigma^d_{\mathbb{P}^1}X_+ \to (\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)_d$. Referring to the definition of $\bar\iota_d$, this proves (1).

For (2), we have the map

$$\sigma : \mathbb{A}^d \times \Delta^1 \to (\mathbb{P}^1)^d \times \mathbb{A}^d \times \Delta^1$$

defined by

$$\sigma(T_1, \ldots, T_d; t_0, t_1) = ((1 : t_0T_1), \ldots, (1 : t_0T_d); T_1, \ldots, T_d; t_0, t_1)).$$

This clearly defines the isomorphism $\mathbb{A}^d \cong \Gamma$ for $t_1 = 0$, and the isomorphism $\mathbb{A}^d \cong 0^d \times \mathbb{A}^d$ for $t_1 = 1$. Letting $\Gamma(*) \subset (\mathbb{P}^1)^d \times \mathbb{A}^d \times \Delta^1$ be the image of σ, and letting $\iota(*)$ be the map

$$(\Sigma^d_{\mathbb{P}^1}X_+) \times \mathbb{A}^d \times \Delta^1 \to (\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)^{(d)}_d$$

constructed as ι, we see that $\iota(*)$ gives the desired homotopy. □

Lemma 7.2.2. *The maps ι_d and $\bar\iota_d$ give rise to maps of $\mathbb{P}^1$-spectra*

$$\begin{aligned}\bar\iota_X &: \Sigma^\infty_{\mathbb{P}^1}X_+ \to \phi_0\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+.\\ \iota_X &: \Sigma^\infty_{\mathbb{P}^1}X\_+ \to \phi_0^{\mathbb{A}^*}\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+.\end{aligned}$$

Proof. We describe the case of ι_X; the argument for $\bar\iota_X$ is similar, but simpler, and is left to the reader.

Using the notation of the above section, and §7.1, we note that the exchange of factors $(\mathbb{P}^1)^d \times \mathbb{A}^d \times \mathbb{A}^1 \times \mathbb{P}^1 \to (\mathbb{P}^1)^{d+1} \times \mathbb{A}^{d+1}$ sends the subscheme $\Gamma_d \times \Delta^0$ isomorphically to Γ_{d+1}. From this, one easily traces through the definitions to find that the diagram

$$\begin{array}{ccc}\mathbb{P}^1 \wedge \Sigma^d_{\mathbb{P}^1}X_+ & \xrightarrow{\mathrm{id}\wedge\iota_d} & \mathbb{P}^1 \wedge (\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)^{(d)}_d\\ {\scriptstyle\mathrm{id}}\downarrow & & \downarrow{\scriptstyle\epsilon'_d}\\ \Sigma^{d+1}_{\mathbb{P}^1}X_+ & \xrightarrow[\iota_{d+1}]{} & (\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)^{(d+1)}_{d+1}\end{array}$$

commutes in $\mathcal{H}\mathbf{Spt}(\mathbf{Sm}_k)$.

We have the connecting maps

$$(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)^{(d)}_d(\Sigma^d_{\mathbb{P}^1}X\times\mathbb{A}^d_+)\xrightarrow{\epsilon_d}(\Omega_{\mathbb{P}^1}(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)^{(d+1)}_{d+1})(\Sigma^{d+1}_{\mathbb{P}^1}X\times\mathbb{A}^{d+1}_+)$$

defining the spectrum $\phi_0^{\mathbb{A}^*}\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+$ (after evaluation on $\Sigma^d_{\mathbb{P}^1}X_+$). We have the classes

$$\iota_d\in(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)^{(d)}_d(\Sigma^d_{\mathbb{P}^1}X\times\mathbb{A}^d_+)$$
$$\iota_{d+1}\in(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)^{(d+1)}_{d+1}(\Sigma^{d+1}_{\mathbb{P}^1}X\times\mathbb{A}^{d+1}_+)$$

defining the maps

$$\iota_d:\Sigma^d_{\mathbb{P}^1}X_+\to((\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)^{(d)}_d)^{\mathbb{A}^d}$$
$$\iota_{d+1}:\Sigma^{d+1}_{\mathbb{P}^1}X\to(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)^{(d+1)}_{d+1})^{\mathbb{A}^{d+1}}$$

To prove the lemma, it suffices to show that

$$\epsilon_d(\iota_d)=\iota_{d+1}. \tag{7.2.1}$$

As everything is natural in X, we reduce to the case $X=\operatorname{Spec}k$.

In fact, the equation (7.2.1) will not be satisfied on the nose, due to the fact that the definition of the ι_d involved inverting some natural weak equivalences. It suffices, however, to verify the identity (7.2.1) after tracing through the weak equivalences we used, as then (7.2.1) will be satisfied up to a map which factors through a functor $A:\mathbf{Sm}_k^{\mathrm{op}}\to\mathbf{Spt}$ with $A(Y)$ contractible for each Y. This is sufficient for constructing the desired map of $\mathbb{P}^1$-Ω-spectra.

The map ϵ_d is constructed by the localization theorem which gives the isomorphism in $\mathcal{SH}$

$$\mathbb{S}^{(d)}_d(\Sigma^d_{\mathbb{P}^1}\mathbb{A}^d_+)\to(\mathbb{S}^{(d+1)}_{d+1})^{\Sigma^d_{\mathbb{P}^1}\mathbb{A}^d_+\times\Delta^0}(\Sigma^d_{\mathbb{P}^1}\mathbb{A}^d_+\times\mathbb{A}^1\times\mathbb{P}^1)$$

Similarly, the section ι_d is given by the isomorphisms in $\mathcal{SH}$

$$\begin{aligned}\mathbb{S}^{\Gamma_d}_d(\Sigma^d_{\mathbb{P}^1}\mathbb{A}^d_+)&\cong\mathbb{S}_0(\Gamma_d)\\&\cong\mathbb{S}_0(k)\end{aligned}$$

via the image of the section $\mathrm{id}\in(S^0_k)(k)$ under the canonical map

$$(S^0_k)(k)\to\mathbb{S}_0(k).$$

The section ι_{d+1} is defined similarly. Let $p:\Gamma_{d+1}\to\Gamma_d$ be the projection induced by the projection $(\mathbb{P}^1)^{d+1}\times\mathbb{A}^{d+1}\to(\mathbb{P}^1)^d\times\mathbb{A}^d$ on the respective first d factors, and let $\pi_d:\Gamma_d\to\operatorname{Spec}k$ and $\pi_{d+1}:\Gamma_{d+1}\to\operatorname{Spec}k$ be the structure morphisms.

We note that, after the evident reordering of factors, we have $\Gamma_d \times \Delta^0 = \Gamma_{d+1}$. Thus, we have the commutative diagram of isomorphisms in $\mathcal{SH}$

$$\begin{array}{ccccccc}
\mathbb{S}_0(k) & \xrightarrow{\pi_d^*} & \mathbb{S}_0(\Gamma_d) & \longrightarrow & \mathbb{S}_d^{\Gamma_d}(\Sigma^d_{\mathbb{P}^1}\mathbb{A}^d_+) & \longrightarrow & \mathbb{S}_d^{(d)}(\Sigma^d_{\mathbb{P}^1}\mathbb{A}^d_+) \\
& {\scriptstyle \pi_{d+1}^*}\searrow & \downarrow{\scriptstyle p^*} & & \downarrow & & \downarrow{\scriptstyle \epsilon_d} \\
& & \mathbb{S}_0(\Gamma_{d+1}) & \longrightarrow & \mathbb{S}_d^{\Gamma_{d+1}}(\Sigma^{d+1}_{\mathbb{P}^1}\mathbb{A}^{d+1}_+) & \longrightarrow & \mathbb{S}_d^{(d+1)}(\Sigma^{d+1}_{\mathbb{P}^1}\mathbb{A}^{d+1}_+),
\end{array}$$

which yields the necessary compatibility.

□

Lemma 7.2.3. *The composition*

$$\Sigma^\infty_{\mathbb{P}^1}X_+ \xrightarrow{\iota_X} \phi_0^{\mathbb{A}^*}\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+ \xrightarrow{\kappa_X^{\mathbb{A}^*}} (\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)^{\mathbb{A}^*}$$

is equal to the composition

$$\Sigma^\infty_{\mathbb{P}^1}X_+ \xrightarrow{\omega_X} \Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+ \xrightarrow{p_X} (\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)^{\mathbb{A}^*}.$$

in $\mathcal{SH}(k)$.

Proof. Using the compatibility we showed in the proof of Lemma 7.2.2, one sees that the homotopies constructed in Lemma 7.2.1 give rise to the desired homotopy of maps of $\mathbb{P}^1$-spectra. □

7.3. The layers of a suspension spectrum.

Theorem 7.3.1. *Let X be in $\mathbf{Sm}_k$. For each $d \geq 0$, the canonical map*

$$(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)^{(q)}_d \xrightarrow{\phi_q} (\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)_d$$

is an isomorphism in $\mathcal{SH}_s(k)$ for $0 \leq q \leq d$.

Proof. The map $\bar\iota_X$ induces the map of $\mathbb{P}^1$-Ω-spectra

$$\bar\iota_X^\infty : \Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+ \to \phi_0(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)$$

such that the composition with the canonical map

$$\kappa_X : \phi_0(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+) \to \Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+$$

is the identity in $\mathcal{SH}(k)$.

Restricting to the dth spectra, we have the map in $\mathcal{SH}_s(k)$

$$\bar\iota_{X,d}^\infty : (\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)_d \to (\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)^{(d)}_d$$

such that the composition with the canonical map

$$\kappa_{X,d} : (\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)^{(d)}_d \to (\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)_d$$

is the identity.

Since the construction of $E^{(p)}$ is functorial in E, we have, for each $p \geq 0$, the sequence of maps in $\mathcal{SH}_s(k)$

$$(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)_d^{(p/p+1)} \xrightarrow{(\bar{\iota}_d^\infty)^{(p/p+1)}} ((\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)_d^{(d)})^{(p/p+1)} \xrightarrow{\kappa_{X,d}^{(p/p+1)}} (\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)_d^{(p/p+1)}$$

and the composition is the identity.

By Proposition 4.2.2, $((\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)_d^{(d)})^{(p/p+1)} \sim *$ for $0 \leq p < d$, hence

$$(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)_d^{(p/p+1)} \sim *$$

for $0 \leq p < d$, which implies that ϕ_q is an isomorphism for $0 \leq q \leq d$, as desired. □

Corollary 7.3.2. *Let X be in $\mathbf{Sm}_k$, and let $\mathcal{E} = \Sigma^d_{\mathbb{P}^1}\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+$. Then the maps in the tower*

$$\phi_d\mathcal{E} \to \phi_{d-1}\mathcal{E} \to \ldots \to \mathcal{E}$$

are all weak equivalences.

Proof. We write

$$\mathcal{E} = (E_0, E_1, \ldots); \quad E_j = (\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)_{j+d}.$$

Take $q \leq d$ and $j \geq 0$, $j \geq -q$. Then

$$(\phi_q\mathcal{E})_j = (\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+)_{d+j}^{(q+j)}.$$

By Theorem 7.3.1, the map

$$(\phi_q\mathcal{E})_j \to E_j$$

is a weak equivalence, whence the result. □

7.4. **Comparison with the slice filtration.** We can now show that the homotopy coniveau tower for $\mathbb{P}^1$-Ω-spectra agrees with Voevodsky's slice filtration. We assume that the base field k is perfect.

Let $\mathcal{E} = (E_0, E_1, \ldots)$ be a $\mathbb{P}^1$-Ω-spectrum and let $f_n\mathcal{E} \to \mathcal{E}$ be the canonical map. Applying ϕ_p gives the map $\phi_p(f_nE) \to \phi_pE$.

Write the $\mathbb{P}^1$-Ω-spectrum $f_n\mathcal{E}$ as

$$f_n\mathcal{E} = ((f_n\mathcal{E})_0, (f_n\mathcal{E})_1, \ldots)$$

The morphism $f_n\mathcal{E} \to \mathcal{E}$ gives us the sequence of maps $(f_n\mathcal{E})_m \to E_m$, compatible with the connecting weak equivalences.

Lemma 7.4.1. *Let X be in $\mathbf{Sm}_k$ and $W \subset X$ a closed subset with* $\mathrm{codim}_X W \geq n+m$. *Then the map*

$$(f_n\mathcal{E})^W_m(X) \to E^W_m(X)$$

is a weak equivalence.

Proof. We may assume that $\mathcal{E}$ and $f_n\mathcal{E}$ are fibrant. For a pointed simplicial Nisnevic sheaf $\mathcal{A}$ on $\mathbf{Sm}_k$, we let $\Sigma^\infty_{(s,p)}\mathcal{A}$ denote the (s,p)-spectrum with nth S^1-spectrum $\Sigma^n_{\mathbb{P}^1}(\Sigma^\infty_s\mathcal{A})$. $\Sigma^\infty_{(s,p)}\mathcal{A}$ is a cofibrant object in $\mathbf{Spt}_{(s,p)}(\mathbf{Sm}_k)$.

For a fibrant (s,p)-spectrum $\mathcal{F} := (F_0, F_1, \ldots)$, we have the identity

$$\mathcal{H}om_{\mathbf{Spt}_{(s,p)}(\mathbf{Sm}_k)}(\Sigma^{-m}_{\mathbb{P}^1}\Sigma^\infty_{(s,p)}\mathcal{A}, \mathcal{F}) = \mathcal{H}om_{\mathbf{Spt}_{\mathrm{Nis}}(\mathbf{Sm}_k)}(\Sigma^\infty_s\mathcal{A}, F_m).$$

Thus, for $Y \in \mathbf{Sm}_k$ with closed subset T, we have the identity

$$\mathcal{H}om_{\mathbf{Spt}_{(s,p)}(\mathbf{Sm}_k)}(\Sigma^{-m}_{\mathbb{P}^1}\Sigma^\infty_{(s,p)}(Y/Y\setminus T), \mathcal{F}) = F^T_m(Y).$$

Stratify W by closed subsets

$$\emptyset = W_{-1} \subset W_0 \subset \ldots \subset W_N = W$$

so that $W_i \setminus W_{i-1}$ is smooth over k. Using the fiber sequences

$$E^{W_{i-1}}_m(X) \to E^{W_i}_m(X) \to E^{W_i\setminus W_{i-1}}_m(X \setminus W_{i-1})$$
$$(f_n\mathcal{E})^{W_{i-1}}_m(X) \to (f_n\mathcal{E})^{W_i}_m(X) \to (f_n\mathcal{E})^{W_i\setminus W_{i-1}}_m(X \setminus W_{i-1})$$

we reduce to the case of smooth irreducible W.

Similarly, we may assume that W has trivial normal bundle N in X. Let $d \geq n+m$ be the codimension of W in X. Then

$$E^W_m(X) = \mathcal{H}om_{\mathbf{Spt}_{(s,p)}(\mathbf{Sm}_k)}(\Sigma^{-m}_{\mathbb{P}^1}\Sigma^\infty_{(s,p)}(X/(X\setminus W)), \mathcal{E}),$$

and we have a similar description of $(f_n\mathcal{E})^W_m(X)$.

We have the homotopy equivalences

$$\Sigma^{-m}_{\mathbb{P}^1}\Sigma^\infty_{(s,p)}(X/(X\setminus W)) \sim \Sigma^{-m}_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}(N/(N\setminus 0_W)) \sim \Sigma^{d-m}_{\mathbb{P}^1}\Sigma^\infty_{(s,p)}W_+.$$

Since $\Sigma^{-m}_{\mathbb{P}^1}\Sigma^\infty_{(s,p)}(X/(X\setminus W))$ is cofibrant, and $f_n\mathcal{E}$ and $\mathcal{E}$ are fibrant, the universal property of $f_n\mathcal{E} \to \mathcal{E}$ yields the weak equivalance of function spectra

$$\mathcal{H}om_{\mathbf{Spt}_{(s,p)}(\mathbf{Sm}_k)}(\Sigma^{-m}_{\mathbb{P}^1}\Sigma^\infty_{(s,p)}(X/(X\setminus W)), f_n\mathcal{E}) \to \mathcal{H}om_{\mathbf{Spt}_{(s,p)}(\mathbf{Sm}_k)}(\Sigma^{-m}_{\mathbb{P}^1}\Sigma^\infty_{(s,p)}(X/(X\setminus W)), \mathcal{E}).$$

This weak equivalence of function spectra thus gives the weak equivalence of spectra

$$(f_n\mathcal{E})^W_m(X) \to E^W_m(X),$$

completing the proof. □

Lemma 7.4.2. *The map $\phi_p(f_n\mathcal{E}) \to \phi_p\mathcal{E}$ is a weak equivalence for all $p \geq n$.*

Proof. Fix an X in $\mathbf{Sm}_k$. On the mth spectra (for $m \geq -p$), the map map $\phi_p(f_n\mathcal{E}) \to \phi_p\mathcal{E}$ yields the map

$$(f_n\mathcal{E})_m^{(p+m)}(X) \to E_m^{(p+m)}(X).$$

Recall that $E_m^{(p+m)}(X)$ is equivalent to the simplicial spectrum

$$q \mapsto E_m^{(p+m)}(X, q) := \operatorname*{hocolim}_{W \in \mathcal{S}_X^{(p+m)}(q)} E_m^W(X \times \Delta^q).$$

In particular, the W in $\mathcal{S}_X^{(p+m)}(q)$ have codimension $\geq p+m \geq n+m$. We have a similar description of $(f_n\mathcal{E})_m^{(p+m)}(X)$. Thus, by Lemma 7.4.1, the map $(f_n\mathcal{E})_m^{(p+m)}(X) \to E_m^{(p+m)}(X)$ is a weak equivalence for all m. □

Lemma 7.4.3. *For $\mathcal{E}$ in $\Sigma^p_{\mathbb{P}^1}\mathcal{SH}^{\mathrm{eff}}(k)$, $\phi_p\mathcal{E} \to \mathcal{E}$ is a weak equivalence.*

Proof. The suspension spectra $\Sigma^d_{\mathbb{P}^1}\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+$, $d \geq p$, are dense in $\Sigma^p_{\mathbb{P}^1}\mathcal{SH}^{\mathrm{eff}}(k)$, so it suffices to prove the result for such spectra. This is proven in Corollary 7.3.2. □

Lemma 7.4.4. *Let $\mathcal{E}$ be a $\mathbb{P}^1$-Ω-spectrum. Then the spectrum $\phi_p\mathcal{E}$ is in $\Sigma^p_{\mathbb{P}^1}\mathcal{SH}^{\mathrm{eff}}(k)$.*

Proof. It follows from Lemma 7.4.3 that $\phi_p(f_p\mathcal{E})$ is in $\Sigma^p_{\mathbb{P}^1}\mathcal{SH}^{\mathrm{eff}}(k)$. By Lemma 7.4.2, the map $\phi_p(f_p\mathcal{E}) \to \phi_p\mathcal{E}$ is a weak equivalence, which verifies our assertion □

Theorem 7.4.5. *Let $\mathcal{E}$ be a $\mathbb{P}^1$-Ω-spectrum, p an integer. Then the natural map $\phi_p\mathcal{E} \to \mathcal{E}$ factors canonically through the map $f_p\mathcal{E} \to \mathcal{E}$, and induces a weak equivalence*

$$\phi_p\mathcal{E} \sim f_p\mathcal{E}.$$

Proof. By Lemma 7.4.4, $\phi_p\mathcal{E}$ is in $\Sigma^p_{\mathbb{P}^1}\mathcal{SH}^{\mathrm{eff}}(k)$. In particular, the map $\phi_p\mathcal{E} \to \mathcal{E}$ factors uniquely through $f_p\mathcal{E} \to \mathcal{E}$, giving the morphism $h : \phi_p\mathcal{E} \to f_p\mathcal{E}$.

We apply ϕ_p to the diagram

$$\begin{array}{ccc} \phi_p\mathcal{E} & \longrightarrow & \mathcal{E} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ f_p\mathcal{E} & \longrightarrow & \mathcal{E}, \end{array}$$

giving

$$\begin{array}{ccc} \phi_p\phi_p\mathcal{E} & \xrightarrow{i} & \phi_p\mathcal{E} \\ {\scriptstyle \phi_p h}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} \\ \phi_p f_p\mathcal{E} & \xrightarrow[j]{} & \phi_p\mathcal{E} \\ {\scriptstyle r}\big\downarrow & & \\ f_pE, & & \end{array}$$

where $r : \phi_p f_p\mathcal{E} \to f_p\mathcal{E}$ is the canonical map. Since $f_p\mathcal{E}$ and $\phi_p\mathcal{E}$ are in $\Sigma^p_{\mathbb{P}^1}\mathcal{SH}^{\mathrm{eff}}(k)$, the maps i and r are weak equivalences, by Lemma 7.4.3. The map j is a weak equivalence by Lemma 7.4.2, so $\phi_p h$ is a weak equivalence. Since the diagram

$$\begin{array}{ccc} \phi_p\phi_p\mathcal{E} & \xrightarrow{i} & \phi_p\mathcal{E} \\ {\scriptstyle \phi_p h}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ \phi_p f_p\mathcal{E} & \xrightarrow[r]{} & f_p\mathcal{E} \end{array}$$

commutes, h is a weak equivalence.

□

Let X be in $\mathbf{Sm}_k$, and let $\mathcal{E}$ be in $\Sigma^p_{\mathbb{P}^1}\mathcal{SH}^{\mathrm{eff}}(k)$. By Lemma 7.4.3, we have the map in $\mathcal{SH}(k)$

$$\phi_{\mathcal{E}} : \mathcal{E} \to \sigma_p\mathcal{E}.$$

Applying the functor σ_p to $\phi_{\mathcal{E}}$ gives

$$\sigma_p(\phi_{\mathcal{E}}) : \sigma_p\mathcal{E} \to \sigma_p(\sigma_p\mathcal{E}).$$

Corollary 7.4.6. *The map $\sigma_p(\phi_{\mathcal{E}})$ is a weak equivalence.*

Proof. It is obvious that $s_p(s_p\mathcal{E}) \to s_p\mathcal{E}$ is an isomorphism; by Theorem 7.4.5, the same holds for σ_p. □

7.5. **Products.** Let $\mathcal{E}$ and $\mathcal{E}'$ be $\mathbb{P}^1$-Ω-spectra. The canonical maps $\phi_n\mathcal{E} \to \mathcal{E}$ and $\phi_m\mathcal{E}' \to \mathcal{E}'$ induce the map $\mu : \phi_n\mathcal{E} \wedge \phi_m\mathcal{E}' \to \mathcal{E} \wedge \mathcal{E}'$.

By Lemma 7.4.4, $\phi_n\mathcal{E}$ is in $\Sigma^n_{\mathbb{P}^1}\mathcal{SH}^{\mathrm{eff}}(k)$ and $\phi_m\mathcal{E}$ is in $\Sigma^m_{\mathbb{P}^1}\mathcal{SH}^{\mathrm{eff}}(k)$, hence $\phi_n\mathcal{E} \wedge \phi_m\mathcal{E}'$ is in $\Sigma^{n+m}_{\mathbb{P}^1}\mathcal{SH}^{\mathrm{eff}}(k)$.

Applying ϕ_{n+m} to μ and using Lemma 7.4.3, we have the diagram

$$\begin{array}{ccc} \phi_{n+m}(\phi_n\mathcal{E} \wedge \phi_m\mathcal{E}') & \xrightarrow{\phi_{n+m}(\mu)} & \phi_{n+m}(\mathcal{E} \wedge \mathcal{E}') \\ {\scriptstyle i_{n+m}}\big\downarrow & & \\ \phi_n\mathcal{E} \wedge \phi_m\mathcal{E}' & & \end{array}$$

with i_{n+m} a weak equivalence. Thus, we have the multiplication

$$\mu_{n,m} : \phi_n\mathcal{E} \wedge \phi_m\mathcal{E}' \to \phi_{n+m}(\mathcal{E} \wedge \mathcal{E}')$$

One checks that the $\mu_{*,*}$ are associative in $\mathcal{SH}(k)$ and are compatible with respect to increasing n and m. In particular, we have

Proposition 7.5.1. *Let $\mathcal{E}$ be a $\mathbb{P}^1$-Ω-spectrum. Then $\sigma_n\mathcal{E}$ is a $\sigma_0\mathbb{S}$-module:*

$$\mu_{\mathcal{E},n} : \sigma_0\mathbb{S} \wedge \sigma_n\mathcal{E} \to \sigma_n\mathcal{E}.$$

Proof. This follows from the above discussion applied to the canonical $\mathbb{S}$-module structure on $\mathcal{E}$. □

8. The sphere spectrum and the $\mathcal{H}\mathbb{Z}$-module structure

In this section, we analyze the layer $\sigma_0\mathbb{S}$, and show that this spectrum is weakly equivalent to the motivic cohomology spectrum $\mathcal{H}\mathbb{Z}$. By Proposition 7.5.1, this gives the σ_p an $\mathcal{H}\mathbb{Z}$-module structure, and shows that the E_1-terms in the basic spectral sequence (Proposition 1.4.2) may be interpreted as generalized motivic cohomology. Throughout this section, we assume that the base-field k is perfect.

8.1. **The reverse cycle map.** In this section, we show how to map $\mathcal{H}\mathbb{Z}$ back to the layer $\sigma_0\mathbb{S}$.

We construct a map $\mathrm{rev} : \mathcal{H}\mathbb{Z} \to \sigma_0\mathbb{S}$ by first constructing maps

$$\mathrm{rev}_d : \mathcal{H}\mathbb{Z}_d = \widetilde{\mathrm{Cyc}}^d \to (\sigma_0\mathbb{S})_d$$

in $\mathbf{Spt}(\mathbf{Sm}_k)$, which we then patch together to yield the map rev.

In the discussion below, if $E = ((E_0, E_1, \ldots), S^1 \wedge E_n \to E_{n+1})$ is a spectrum, we write $x \in E$ to indicate an element x in the 0-simplicies of the simplicial set E_0, equivalently, a morphism of spectra:

$$x : \Sigma^\infty S^0 \to E.$$

We first consider the case $d = 1$. Cyc^1 is represented by the union

$$\coprod_{m=1}^{\infty} \mathrm{Sym}^m\mathbb{P}^1 = \coprod_{m=1}^{\infty} \mathbb{P}^m$$

via the incidence subvariety $D_m \subset \mathbb{P}^1 \times \mathbb{P}^m$. D_m is defined by the bihomogeneous polynomial $\sum_{i=0}^m X_iT_0^{m-i}T_1^m$; evidently, D_m is smooth over k.

The structure map $D_m \to \mathrm{Spec}\, k$ determines the canonical map (of presheaves of sets) $D_{m+} \to S^0$. This in turn determines the canonical element $1_{D_m} \in S^0(D_m)$. Composing with the canonical map $\Sigma^\infty_s S^0 \to \mathbb{S}_0^{(0/1)}$ gives the section $1_{D_m} \in \mathbb{S}_0^{(0/1)}(D_m)$.

By taking bifibrant models, we may assume that $\mathbb{S}_0^{(0/1)}(D_m)$ and $(\mathbb{S}_1^{(1/2)})^{D_m}(\mathbb{P}^1 \times \mathbb{P}^m)$ are bifibrant spectra, so the Thom-space isomorphism

$$\mathbb{S}_0^{(0/1)}(D_m) \cong (\mathbb{S}_1^{(1/2)})^{D_m}(\mathbb{P}^1 \times \mathbb{P}^m)$$

in $\mathcal{SH}$ lifts to the weak equivalence

$$Th : \mathbb{S}_0^{(0/1)}(D_m) \xrightarrow{\sim} (\mathbb{S}_1^{(1/2)})^{D_m}(\mathbb{P}^1 \times \mathbb{P}^m).$$

Applying Th to $1_{D_m} \in \mathbb{S}_0^{(0/1)}(D_m)$ gives us the element

$$1^{D_m} \in (\mathbb{S}_1^{(1/2)})^{D_m}(\mathbb{P}^1 \times \mathbb{P}^m).$$

Similarly, we have the "base-point"

$$1^{\infty} \in (\mathbb{S}_1^{(1/2)})^{\infty}(\mathbb{P}^1).$$

Let $i_m : \mathrm{Sym}^m\mathbb{P}^1 \to \mathrm{Sym}^{m+1}\mathbb{P}^1$ be the map sending $\sum_{i=1}^m x_i$ to $\sum_{i=1}^m x_i + \infty$. We have

$$(\mathrm{id} \times i_m)^*(D_{m+1}) = D_m + \infty \times \mathbb{P}^m.$$

We have the isomorphism in $\mathcal{SH}$

$$(\mathbb{S}_1^{(1/2)})^{D_m \cup \infty \times \mathbb{P}^m}(\mathbb{P}^1 \times \mathbb{P}^m) \cong (\mathbb{S}_1^{(1/2)})^{D_m}(\mathbb{P}^1 \times \mathbb{P}^m) \oplus (\mathbb{S}_1^{(1/2)})^{\infty \times \mathbb{P}^m}(\mathbb{P}^1 \times \mathbb{P}^m),$$

and, via this isomorphism, the identity of maps in $\mathcal{SH}$

(8.1.1)
$$\Sigma_s^\infty S^0 \xrightarrow{(\mathrm{id} \times i_m)^*(1^{D_{m+1}})} (\mathbb{S}_1^{(1/2)})^{D_m \cup \infty \times \mathbb{P}^m}(\mathbb{P}^1 \times \mathbb{P}^m)$$
$$\Sigma_s^\infty S^0 \xrightarrow{1^{D_m} \oplus p_m^*(1^\infty)} (\mathbb{S}_1^{(1/2)})^{D_m \cup \infty \times \mathbb{P}^m}(\mathbb{P}^1 \times \mathbb{P}^m),$$

where $p : \mathbb{P}^1 \times \mathbb{P}^m \to \mathbb{P}^1$ is the projection.

Let $(\mathbb{S}_1^{(1/2)})_{\mathrm{fin}}(X, \mathbb{P}^1)$ be the limit

$$(\mathbb{S}_1^{(1/2)})_{\mathrm{fin}}(X, \mathbb{P}^1) := \underset{\overrightarrow{D}}{\mathrm{hocolim}} (\mathbb{S}_1^{(1/2)})^D(\mathbb{P}^1 \times X)$$

as D runs over all codimension one closed subsets of $\mathbb{P}^1 \times X$ which are finite over X and such that each irreducible component of D dominates a component of X.

As the pull-back of divisors gives an isomorphism

$$\coprod_{m=1}^{\infty} \mathrm{Hom}_{\mathbf{Sm}_k}(X, \mathbb{P}^m) \cong \mathrm{Cyc}^1(X),$$

we may send $\mathrm{Cyc}^1(X)$ to $(\mathbb{S}_1^{(1/2)})_{\mathrm{fin}}(X, \mathbb{P}^1)$ by sending $f : X \to \mathbb{P}^m$ to $f^*(1^{D_m})$. This gives us the natural transformation of functors $\mathbf{Sm}_k^{\mathrm{op}} \to$

Spt

$$\mathrm{cyc}^1 := \coprod_{n=1}^{\infty} \mathrm{cyc}^1_n :: \Sigma^\infty_s \mathrm{Cyc}^1 \to (\mathbb{S}_1^{(1/2)})_{\mathrm{fin}}(?, \mathbb{P}^1).$$

Since all the supports D defining $(\mathbb{S}_1^{(1/2)})_{\mathrm{fin}}(X, \mathbb{P}^1)$ are codimension one, we have the natural map

$$(\mathbb{S}_1^{(1/2)})_{\mathrm{fin}}(X, \mathbb{P}^1) \to (\mathbb{S}_1^{(1/2)})^{(1)}(X \times \mathbb{P}^1);$$

restricting to $X \times (\mathbb{P}^1 \setminus \{\infty\})$ and "forgetting supports" gives the map of functors $\mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{Spt}$

$$(\mathbb{S}_1^{(1/2)})_{\mathrm{fin}}(?, \mathbb{P}^1) \to (\mathbb{S}_1^{(1/2)})^{(1)}(? \times \mathbb{A}^1) \to \mathbb{S}_1^{(1/2)}(? \times \mathbb{A}^1),$$

which sends the portion of $(\mathbb{S}_1^{(1/2)})_{\mathrm{fin}}(X, \mathbb{P}^1)$ supported on $X \times \infty$ to the base-point. Composing this map with cyc^1_m gives the maps

$$\overline{\mathrm{cyc}}^1_m : \Sigma^\infty_s \Sigma^m \mathbb{P}^1 \to (\mathbb{S}_1^{(1/2)})^{\mathbb{A}^1}.$$

We have the map

$$\iota_m : \Sigma^\infty_s \mathrm{Sym}^m \mathbb{P}^1 \to \Sigma^\infty_s \mathrm{Sym}^{m+1} \mathbb{P}^1$$

induced by "adding ∞"; let $\Sigma^\infty_s \mathrm{Sym}^* \mathbb{P}^1$ denote the resulting homotopy colimit. As $(\mathbb{S}_1^{(1/2)})^{\mathbb{A}^1}$ is fibrant in $\mathbf{Spt}(\mathbf{Sm}_k)$, the identity of maps (8.1.1) implies that the maps $\overline{\mathrm{cyc}}^1_m$ descend to the map

$$\overline{cyc}^1 : \Sigma^\infty_s \mathrm{Sym}^* \mathbb{P}^1 \to (\mathbb{S}_1^{(1/2)})^{\mathbb{A}^1}.$$

As the evident map : $\Sigma^\infty_s \mathrm{Sym}^* \mathbb{P}^1 \to \Sigma^\infty_s \widetilde{\mathrm{Cyc}}^1$ is a weak equivalence of cofibrant objects in $\mathbf{Spt}(\mathbf{Sm}_k)$, $\overline{cyc}^1$ descends to

$$\widetilde{\mathrm{cyc}}^1 : \Sigma^\infty_s \widetilde{\mathrm{Cyc}}^1 \to (\mathbb{S}_1^{(1/2)})^{\mathbb{A}^1}.$$

Finally, since $\Sigma^\infty_s \widetilde{\mathrm{Cyc}}^1 \to \mathcal{H}\mathbb{Z}_1$ is a weak equivalence among cofibrant objects in $\mathbf{Spt}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathbf{Sm}_k)$, and $(\mathbb{S}_1^{(1/2)})^{\mathbb{A}^1}$ is fibrant in $\mathbf{Spt}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathbf{Sm}_k)$, the homotopy class of $\widetilde{\mathrm{cyc}}^1$ extends, uniquely up to homotopy, to the map in $\mathbf{Spt}(\mathbf{Sm}_k)$,

$$\mathrm{rev}^1 : \mathcal{H}\mathbb{Z}_1 \to (\mathbb{S}_1^{(1/2)})^{\mathbb{A}^1} = (\sigma_0 \mathbb{S})_1^{\mathbb{A}^1}.$$

For $d > 1$, let W be in $\mathrm{Cyc}^d(X)$. We first consider the case of semi-local X, with a finite set of chosen points $x_1, \ldots, x_s$, to explain the idea of the construction. Then $P := \cup_i W \cap (\mathbb{P}^1)^d \times x_i$ is a finite subset of $(\mathbb{P}^1)^d \times X$. Thus, there is a k-point $*$ in $\mathbb{P}^1(k)$ with $|W| \subset (\mathbb{P}^1 \setminus \{*\})^d \times X$, i.e., W is a finite cycle on $\mathbb{A}^d \times X$.

Choosing a general linear projection $\pi : \mathbb{A}^d \to \mathbb{A}^1$, we map W birationally to $\pi_*(W)$, and the set P isomorphically to $\pi(P)$. We have the push-forward weak-equivalence

$$\pi_* : \mathbb{S}_0^{(0/1)}(|W|) \to \mathbb{S}_0^{(0/1)}(|\pi_* W|),$$

defined as follows: Choosing suitable coordinates on $\mathbb{A}^d$ gives an isomorphism $\mathbb{A}^d \cong \mathbb{A}^1 \times \mathbb{A}^{d-1}$ for which π becomes identified with the projection on $\mathbb{A}^1$. We may therefore embed $\mathbb{A}^d$ as an open subset of $\mathbb{A}^1 \times (\mathbb{P}^1)^{d-1}$. Let $\bar{\pi} : \mathbb{A}^1 \times (\mathbb{P}^1)^{d-1} \to \mathbb{A}^1$ be the projection. We thus have the weak equivalences

$$\begin{aligned} \mathbb{S}_0^{(0/1)}(|W|) &:= (\mathbb{S}_0^{(d/d+1)})^{|W|}(X \times \mathbb{A}^d) \sim (\mathbb{S}_0^{(d/d+1)})^{|W|}(X \times \mathbb{A}^1 \times (\mathbb{P}^1)^{d-1}) \\ &\to (\mathbb{S}_0^{(d/d+1)})^{\bar{\pi}^{-1}(|\pi_* W|)}(X \times \mathbb{A}^1 \times (\mathbb{P}^1)^{d-1}) \sim (\mathbb{S}_0^{(1/2)})^{|\pi_* W|}(X \times \mathbb{A}^1) \\ &=: \mathbb{S}_0^{(0/1)}(|\pi_* W|), \end{aligned}$$

giving the definition of π_*.

Thus, the class $\mathrm{rev}^1(\pi_* W)$ gives the class $\mathrm{rev}^d(W) \in \mathbb{S}_0^{(0/1)}(|W|)$. This class is functorial with respect to restriction to the points $x_1, \ldots, x_s$.

To make this as canonical as possible, take $X \in \mathbf{Sm}_k$. Let $q : (\mathbb{P}^1)^d \times X \to X$ be the projection. Let $*$ be the generic point of $\mathbb{P}^1_k$, $K = k(\mathbb{P}^1)$, and let $(\mathbb{P}^1_K)^d_\infty = (\mathbb{P}^1_K)^d \setminus (\mathbb{P}^1_K - \{*\})^d$.

Let W be a finite cycle, let $U = X_K \setminus q(|W|_K \cap (\mathbb{P}^1_K)^d_\infty)$ and let $W^0_K = (\mathbb{P}^1_K)^d_\infty \times U \cap W_K$. Then W^0_K is a finite cycle over U.

Let $L \supset K$ be the field of coefficients of the generic affine-linear map $\pi : (\mathbb{P}^1 \setminus \{*\})^d \to \mathbb{A}^1$, together with choice of affine-linear isomorphism

$$(\mathbb{P}^1 \setminus \{*\})^d \cong \mathbb{A}^1 \times \mathbb{A}^{d-1} \tag{8.1.2}$$

over $\mathbb{A}^1$. Then for each point $x \in X$, there is a unique point $x_K \in U$ lying over x, and π gives an isomorphism from $W_L \cap (\mathbb{P}^1_L)^d \times x_L$ to its image $\pi(W_L \cap (\mathbb{P}^1_L - \{*\})^d \times x_L)$ in $\mathbb{A}^1 \times x_L$.

We now apply the results of Corollary 3.7.5 and Lemma 3.8.1, giving us the sequence of isomorphisms in $\mathcal{SH}$

$$\begin{aligned} (\mathbb{S}_d^{(d/d+1)})^{|W|}((\mathbb{P}^1)^d \times X) &\cong (\mathbb{S}_d^{(d/d+1)})^{|W_L|}((\mathbb{P}^1)^d \times X_L) \\ &\cong (\mathbb{S}_d^{(d/d+1)})^{|W^0_L|}((\mathbb{P}^1)^d \times U_L) \\ &\xrightarrow{\pi_*} (\mathbb{S}_1^{(1/2)})^{\pi(|W^0_L|)}(U_L) \\ &\cong (\mathbb{S}_1^{(1/2)})^{\overline{\pi(|W^0_L|)}}(X_L). \end{aligned} \tag{8.1.3}$$

Define $(\mathbb{S}_d^{(d/d+1)})_{\rm fin}(X,(\mathbb{P}^1)^d)$ to be the limit

$$(\mathbb{S}_d^{(d/d+1)})_{\rm fin}(X,(\mathbb{P}^1)^d)=\operatorname*{hocolim}_W(\mathbb{S}_d^{(d/d+1)})^W(((\mathbb{P}^1)^d\times X)$$

as W runs over codimension d closed subsets of $((\mathbb{P}^1)^d\times X$, finite over X, such that each irreducible component of W dominates some component of X. Sending X to $(\mathbb{S}_d^{(d/d+1)})_{\rm fin}(X,(\mathbb{P}^1)^d)$ defines the functor

$$(\mathbb{S}_d^{(d/d+1)})_{\rm fin}(?,(\mathbb{P}^1)^d):\mathbf{Sm}_k^{\rm op}\to\mathbf{Spt}.$$

Let $(\mathbb{S}_1^{(1/2)})_{\rm fin}(X,\mathbb{P}^1)_{L,d}$ be the limit

$$(\mathbb{S}_1^{(1/2)})_{\rm fin}(X,\mathbb{P}^1)_{L,d}:=\operatorname*{hocolim}_{\overrightarrow{D}}(\mathbb{S}_1^{(1/2)})^D(\mathbb{P}^1\times X_L),$$

where now D runs over the closed subsets of the form $\overline{\pi(|W_L^0|)}$, where W runs over all codimension d closed subsets of $(\mathbb{P}^1)^d\times X$, with the finiteness and dominance conditions as above.

Sending X to $(\mathbb{S}_1^{(1/2)})_{\rm fin}(X,\mathbb{P}^1)_{L,d}$ defines the functor

$$(\mathbb{S}_1^{(1/2)})_{\rm fin}(?,\mathbb{P}^1)_{L,d}:\mathbf{Sm}_k^{\rm op}\to\mathbf{Spt}.$$

and the sequence of maps (8.1.3) gives us the weak equivalence

$$\hat\pi_*^{(d)}:(\mathbb{S}_d^{(d/d+1)})_{\rm fin}(?,(\mathbb{P}^1)^d)\to(\mathbb{S}_1^{(1/2)})_{\rm fin}(?,\mathbb{P}^1)_{L,d}$$

in $\mathcal{SH}_s^{\mathbb{A}^1}(k)$.

Replacing $(\mathbb{S}_1^{(1/2)})_{\rm fin}(?,\mathbb{P}^1)_{L,d}$ with the functorial bifibrant model in $\mathbf{Spt}^{\mathbb{A}^1}_{\rm Nis}(\mathbf{Sm}_k)$, and changing notation, we may assume that the functor $(\mathbb{S}_1^{(1/2)})_{\rm fin}(?,\mathbb{P}^1)_{L,d}$ is bifibrant. Similarly, we may assume that $(\mathbb{S}_d^{(d/d+1)})_{\rm fin}(?,(\mathbb{P}^1)^d)$ is bifibrant. Thus, $(\hat\pi_*^{(d)})^{-1}$ lifts to the weak equivalence in $\mathbf{Spt}(\mathbf{Sm}_k)$

$$\pi_*^{-1}:(\mathbb{S}_1^{(1/2)})_{\rm fin}(?,\mathbb{P}^1)_{L,d}\to(\mathbb{S}_d^{(d/d+1)})_{\rm fin}(?,(\mathbb{P}^1)^d).$$

We perform the same construction to Cyc^d. Let $\mathrm{Cyc}^1_{L.d}(X)$ be the set of finite cycles on $\mathbb{P}^1\times X_L$ of the form $\overline{\pi_*(W_L^0)}$ for some $W\in\mathrm{Cyc}^d(X)$. Thus, we have the functor $\mathrm{Cyc}^1_{L,d}:\mathbf{Sm}_k^{\rm op}\to\mathbf{Sets}$, the natural transformation

$$\mathrm{cyc}^1_L:\mathrm{Cyc}^1_{L,d}\to(\mathbb{S}_1^{(1/2)})_{\rm fin}(?,\mathbb{P}^1)_{L,d}$$

and the natural isomorphism

$$\pi_*:\mathrm{Cyc}^d\to\mathrm{Cyc}^1_{L,d}.$$

We thus have the natural transformation

$$\mathrm{cyc}^d:\Sigma_s^\infty\mathrm{Cyc}^d\to(\mathbb{S}_d^{(d/d+1)})_{\rm fin}(?,(\mathbb{P}^1)^d)$$

defined by

$$\mathrm{cyc}^d := \pi_*^{-1} \circ \mathrm{cyc}_L^1 \circ \pi_*.$$

Following cyc^d with the restriction to $X \times (\mathbb{P}^1 \setminus \infty)^d$ and then the map "forget supports" gives the map

$$\widetilde{\mathrm{cyc}}^d : \Sigma_s^\infty \widetilde{\mathrm{Cyc}}^d \to (\mathbb{S}_d^{(d/d+1)})^{\mathbb{A}^d},$$

which by universality extends to

$$\mathrm{rev}^d : \mathcal{H}\mathbb{Z}_d \to (\sigma_0 \mathbb{S})_d^{\mathbb{A}^d}.$$

8.2. **The extension to a map of $\mathbb{P}^1$-spectra.**

Proposition 8.2.1. *The maps* $\mathrm{rev}^d(X)$ *give rise to a natural map of* $\mathbb{P}^1$*-spectra* $\mathrm{rev} : \mathcal{H}\mathbb{Z} \to (\sigma_0\mathbb{S})^{\mathbb{A}^*}$*.*

Proof. The connecting maps for $(\sigma_0\mathbb{S})^{\mathbb{A}^*}$

$$(\sigma_0\mathbb{S})_d^{\mathbb{A}^d} \to \Omega_{\mathbb{P}^1}(\sigma_0\mathbb{S})_{d+1}^{\mathbb{A}^{d+1}}$$

are adjoint to maps

$$\mathbb{P}^1 \wedge (\sigma_0\mathbb{S})_d^{\mathbb{A}^d} \to (\sigma_0\mathbb{S})_{d+1}^{\mathbb{A}^{d+1}}$$

which in turn are induced by the natural maps

$$\psi_d : \mathbb{P}^1(X) \times (\sigma_0\mathbb{S})_d(X \times \mathbb{A}^d) \to (\sigma_0\mathbb{S})_{d+1}(X \times \mathbb{A}^{d+1})$$

defined by the following: Let $f : X \to \mathbb{P}^1$ be a morphism. The graph of f gives the inclusions $X \times \mathbb{A}^d \to X \times \mathbb{P}^1 \times \mathbb{A}^d$, which then gives the map

$$f_* : (\mathbb{S}_d)^{(d/d+1)}(X \times \mathbb{A}^d) \to (\mathbb{S}_{d+1})^{(d+1/d+2)}(X \times \mathbb{P}^1 \times \mathbb{A}^d).$$

Composing with the restriction

$$(\mathbb{S}_{d+1})^{(d+1/d+2)}(X \times \mathbb{P}^1 \times \mathbb{A}^d) \to (\mathbb{S}_{d+1})^{(d+1/d+2)}(X \times \mathbb{A}^1 \times \mathbb{A}^d)$$

and using the canonical weak equivalences

$$(\mathbb{S}_{d+1})^{(d+1/d+2)}(X \times \mathbb{P}^1 \times \mathbb{A}^d) \sim (\sigma_0\mathbb{S})_d(X \times \mathbb{A}^d)$$
$$(\mathbb{S}_{d+1})^{(d+1/d+2)}(X \times \mathbb{A}^1 \times \mathbb{A}^d) \sim (\sigma_0\mathbb{S})_{d+1}(X \times \mathbb{A}^{d+1})$$

completes the definition of ψ_d.

The connecting maps for $\mathcal{H}\mathbb{Z}$ are induced by maps

$$\mathbb{P}^1(X) \times \mathrm{Cyc}^d(X) \to \mathrm{Cyc}^{d+1}(X)$$

which are defined similarly, by taking the fiber product of a cycle $W \in \mathrm{Cyc}^d(X)$ with the graph of a map $f : X \to \mathbb{P}^1$ to define the resulting cycle $\Gamma_f \times_X W \in \mathrm{Cyc}^{d+1}(X)$.

We need to see that the maps rev^d are compatible with these connecting maps, up to a compatible family of homotopies.

For this, note that the projection p_d on the last d factors gives an isomorphism of $|\Gamma_f \times_X W|$ with $|W|$. Thus, the only difference between $\psi_d(f \times \mathrm{rev}^d(W))$ and $\mathrm{rev}^{d+1}(\Gamma_f \times_X W)$ arises in the use of two different projections: $\pi_d \circ p_d$ and π_{d+1}, where π_d and π_{d+1} are the generic projections used in the definition of rev^d and rev^{d+1}.

Let L_{d+1} be the field extension used to define π_{d+1}. Clearly, we can define a family of linear projections

$$\pi_{d,1} : (\mathbb{P}^1 - \{*\})^d \times \Delta^1_{0,L_{d+1}} \to \mathbb{A}^1 \times \Delta^1_{0,L_{d+1}}$$

which agrees with $\pi_d \circ p_d$ at $(0,1)$ and π_{d+1} at $(1,0)$, by making a linear interpolation. Similarly, we may interpolate between the different isomorphisms

$$(\mathbb{P}^1 - \{*\})^{d+1} \cong \mathbb{A}^1 \times \mathbb{A}^d$$

used to construct the pushforward π_*.

Using this family, via the rational invariance of $\mathbb{S}_0^{(0/1)}$, gives a homotopy between $\psi_d(f \times \mathrm{rev}^d(W))$ and $\mathrm{rev}^{d+1}(\Gamma_f \times_X W)$.

Similarly, for each n, we construct by linear interpolation a family of linear projections

$$\pi_{d,n} : (\mathbb{P}^1 - \{*\})^{d+n} \times \Delta^n_{0,L_{d+n}} \to \mathbb{A}^1 \times \Delta^n_{0,L_{d+n}}$$

and isomorphisms

$$(\mathbb{P}^1 - \{*\})^{d+n} \cong \mathbb{A}^1 \times \mathbb{A}^{d+n-1}$$

such that

$$\pi_{d,n}(v_i) = \pi_{d+i} \circ p_{d+i} \circ p_{d+i+1} \circ \ldots \circ p_{d+n-1},$$

where v_i is the vertex $t_i = 1$, $t_j = 0$, $j \neq i$. These give the higher homotopies required to define the desired map of $\mathbb{P}^1$-spectra. □

8.3. **The cycle class map.** We denote the Bloch motivic cohomology spectrum $(\mathcal{Z}^0, \Sigma^2\mathcal{Z}^1, \ldots)$ by $\Sigma\mathcal{Z}$.

Lemma 8.3.1.

$$\sigma_q \Sigma^p_{\mathbb{P}^1} \Sigma\mathcal{Z} \sim \begin{cases} 0 & \text{for } q \neq p \\ \Sigma^p_{\mathbb{P}^1}\Sigma\mathcal{Z} & \text{for } q = p \end{cases}$$

Proof. By Remark 6.4.1, it suffices to prove the case $p = 0$. This follows directly from the identification

$$\mathcal{H}\mathbb{Z} \sim (\mathcal{Z}^0, \Sigma^2\mathcal{Z}^1, \ldots)$$

and Theorem 5.5.1. □

The canonical map $\mathbb{S} \to \Sigma\mathcal{Z}$ thus induces the map

$$\mathrm{cl} : \sigma_0\mathbb{S} \to \sigma_0\Sigma\mathcal{Z} \sim \Sigma\mathcal{Z}.$$

On the zero-spaces, this is a natural transformation

$$\mathrm{cl}^0 : \mathbb{S}_0^{(0/1)} \to \mathcal{Z}^0.$$

Note that $\mathcal{Z}^0$ is the constant sheaf $\mathbb{Z}$ (for the Zariski topology) on $\mathbf{Sm}_k$.

By the naturality of cl, and the explicit description of the dth space $\mathbb{S}_d^{(d/d+1)}$ of $\sigma_0\mathbb{S}$ given by Corollary 4.3.2, we find that, for $X \in \mathbf{Sm}_k$,

$$\mathrm{cl}^d : \mathbb{S}_d^{(d/d+1)}(X, -) \to z^d(X, -)$$

is induced by the map on n-simplices

$$\coprod_{x \in X^{(d)}(n)} \mathbb{S}_0^{(0/1)}(k(x)) \xrightarrow{\amalg_x \mathrm{cl}^0} \coprod_{x \in X^{(d)}(n)} \mathbb{Z}.$$

Replacing $\mathcal{Z}^d(X)$ with $\mathcal{Z}^d(X \times \mathbb{A}^d)$, we have the modified spectrum $\Sigma\mathcal{Z}^{\mathbb{A}^*}$, and the map $\mathrm{cl} : \sigma_0^{\mathbb{A}^*}\mathbb{S} \to \Sigma\mathcal{Z}^{\mathbb{A}^*}$.

8.4. $\sigma_0\mathbb{S}$ and $\mathcal{H}\mathbb{Z}$.

Theorem 8.4.1. *The maps* cl *and* rev *are weak equivalences.*

Proof. We first consider the composition ψ:

$$\Sigma^\infty_{\mathbb{P}^1} S^0_k \to \mathcal{H}\mathbb{Z} \xrightarrow{\mathrm{rev}} \sigma_0^{\mathbb{A}^*}\mathbb{S}.$$

Looking at the dth spaces gives the map

$$\Sigma^d_{\mathbb{P}^1} S^0_k \to (\sigma_0^{\mathbb{A}^*}\mathbb{S})_d,$$

i.e., a section

$$p_d \in \mathbb{S}_d^{(d/d+1)}(\Sigma^d_{\mathbb{P}^1}\mathbb{A}^d_+).$$

It follows directly from the definition of rev^d that p_d is the same, after composing with canonical weak equivalences, as the map $\iota_d(\mathrm{Spec}\, k)$. Thus, by Lemma 7.2.1 and Theorem 7.3.1, $\psi(X)$ is homotopic to the composition

$$\Sigma^\infty_{\mathbb{P}^1} X_+ \xrightarrow{\iota_{\mathrm{Spec}\, k}} \phi_0(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1} X_+) \xrightarrow{p} \sigma_0(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1} X_+)$$

where p is the canonical map. Passing to $\mathbb{P}^1$-Ω-spectra and applying σ_0 thus gives the weak equivalence (Corollary 7.4.6)

$$\sigma_0(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1} X_+) \xrightarrow{\sigma_0\psi(X)} \sigma_0(\sigma_0(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1} X_+)).$$

We now consider the composition $\phi(X)$:

$$\mathcal{H}\mathbb{Z}(X) \xrightarrow{\mathrm{rev}} \sigma_0^{\mathbb{A}^*}\mathbb{S}(X) \xrightarrow{\mathrm{cl}} \Sigma\mathcal{Z}^{\mathbb{A}^*}(X).$$

From the explicit description of cl given in §8.3, we see that $\phi(X)$ is given by the map (on the dth space) which associates to a cycle W on $X \times \Delta^n \times (\mathbb{P}^1)^d$ the restriction to $X \times \Delta^n \times \mathbb{A}^d$. This map is the weak equivalence described above in Example 6.2.2(2). □

Corollary 8.4.2. *Let $\mathcal{E}$ be a $\mathbb{P}^1$-Ω-spectrum. Then there is a natural $\mathcal{H}\mathbb{Z}$-module structure on $\sigma_n\mathcal{E}$.*

Proof. This follows from Theorem 8.4.1 and Proposition 7.5.1. □

9. The motivic Atiyah-Hirzebruch spectral sequence

We collect our results on the homotopy coniveau spectral sequence. For the results on $\mathcal{DM}(k)$ and $\mathcal{SH}(k)$ we use in this section, we refer the reader to the lectures of Morel [14, 15] and Voevodsky [21].

9.1. **$\mathcal{H}\mathbb{Z}$-modules and $\mathcal{DM}$.** We have the "big" triangulated tensor category of motives $\mathcal{DM}(k)$, containing the triangulated category of effective motives $\mathcal{DM}_-^{\mathrm{eff}}(k)$. There is an "Eilenberg-Maclane" functor

$$\mathcal{H} : \mathcal{DM}(k) \to \mathcal{SH}(k)$$

sending $\mathcal{DM}_-^{\mathrm{eff}}(k)$ to $\mathcal{SH}^{\mathrm{eff}}(k)$, and a "Suslin homology" functor

$$h_S : \mathcal{SH}(k) \to \mathcal{DM}(k),$$

which is left adjoint to $\mathcal{H}$ and sends $\mathcal{SH}^{\mathrm{eff}}(k)$ to $\mathcal{DM}_-^{\mathrm{eff}}(k)$. We denote the unit object of $\mathcal{DM}(k)$ by $\mathbb{Z}$. There are canonical isomorphisms $\mathcal{H}(\mathbb{Z}) \cong \mathcal{H}\mathbb{Z}$, $h_S(\mathbb{S}) \cong \mathbb{Z}$. Thus, for each $M \in \mathcal{DM}(k)$, $\mathcal{H}(M)$ comes with an $\mathcal{H}\mathbb{Z}$-module structure, and we have a natural isomorphism

$$\mathrm{Hom}_{\mathcal{SH}(k)}(\mathbb{S}, \mathcal{H}(M)) \cong \mathrm{Hom}_{\mathcal{DM}(k)}(\mathbb{Z}, M).$$

Consider the following

Conjecture 9.1.1. *Let $\mathcal{E}$ be a $\mathbb{P}^1$-Ω-spectrum with an $\mathcal{H}\mathbb{Z}$-module structure. Then there is an object $\mathcal{ME}$ of $\mathcal{DM}(k)$ and an isomorphism of $\mathcal{H}\mathbb{Z}$-modules $\phi : \mathcal{E} \to \mathcal{H}(\mathcal{ME})$. The isomorphism class of $\mathcal{ME}$ is unique.*

An approach to a proof of this conjecture, for k a field of charactistic zero is sketched in [21].

9.2. **The spectral sequence.** In this section, we assume that Conjecture 9.1.1 holds.

Definition 9.2.1. Let $\mathcal{E} = (E_0, E_1, \ldots)$ be a $\mathbb{P}^1$-Ω-spectrum. Define the object $\pi_p^\mu\mathcal{E}$ of $\mathcal{DM}(k)$ by

$$\pi_p^\mu\mathcal{E} := \mathcal{M}(\sigma_p\mathcal{E})[-p].$$

We have the functor

$$m : \mathbf{Sm}_k \to \mathcal{DM}(k),$$
$$m(X) = h_S(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+).$$

For an object M of $\mathcal{DM}(k)$ and $X \in \mathbf{Sm}_k$, we have the *motivic cohomology*

$$\mathbb{H}^n(X, M) := \mathrm{Hom}_{\mathcal{DM}}(m(X), M[n]).$$

We have the natural isomorphism

$$\mathbb{H}^n(X, M) \cong \mathrm{Hom}_{\mathcal{SH}(k)}(\Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1}X_+, \Sigma_s^{-n}\mathcal{H}(M)).$$

Theorem 9.2.2. *Let $\mathcal{E}$ be a $\mathbb{P}^1$-Ω-spectrum, let $E : \mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{Spt}$ be the 0th spectrum of $\mathcal{E}$ and let X be in $\mathbf{Sm}_k$. Then the homotopy coniveau spectral sequence for $E(X)$ is*

$$E_2^{p,q} = \mathbb{H}^p(X, \pi^\mu_{-q}\mathcal{E}) \Longrightarrow \pi_{-p-q}\hat{E}(X).$$

Proof. The E_1-term is given by

$$E_1^{p,q} = \pi_{-p-q}(E^{(p/p+1)}(X)).$$

Also $E^{(p/p+1)}$ is the 0th spectrum in $\sigma_p(\mathcal{E})$, so we have

$$\begin{aligned}\pi_{-p-q}(E^{(p/p+1)}(X)) &= \mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma_s^{-p-q}\Sigma^\infty_{\mathbb{P}^1}X_+, \sigma_p(\mathcal{E}))\\ &= \mathrm{Hom}_{\mathcal{DM}(k)}(m(X)[-p-q], \pi^\mu_p(\mathcal{E})[p])\\ &= \mathbb{H}^{2p+q}(X, \pi^\mu_p(\mathcal{E})).\end{aligned}$$

As the transformation $(p, q) \mapsto (p + r, q - r + 1)$ sends $(2p + q, p)$ to $(2p + q + r + 1, p + r)$, we can reindex to form an E_2-spectral sequence by replacing $\mathbb{H}^{2p+q}$ with $\mathbb{H}^p$ and π^μ_p with π^μ_{-q}:

$$E_2^{p,q} := \mathbb{H}^p(X, \pi^\mu_{-q}(\mathcal{E})) \Longrightarrow \pi_{-p-q}\hat{E}(X).$$

□

To aid in concrete computations, we use:

Lemma 9.2.3. *We have an isomorphism in $\mathcal{DM}(k)$:*

$$\pi^\mu_{p+q}\mathcal{E} \cong \pi^\mu_q(\Sigma^{-p}_{\mathbb{P}^1}\mathcal{E}) \otimes \mathbb{Z}(p)[p].$$

Proof. The functor $\mathcal{H}$ is a tensor functor and there is a canonical weak equivalence

$$\mathcal{H}(\mathbb{Z}(q)[2q] \otimes M) \sim \Sigma^q_{\mathbb{P}^1}\mathcal{H}(M).$$

where $\mathbb{Z}(q)$ is the Tate object in $\mathcal{DM}(k)$. Thus, if $\mathcal{F}$ is an $\mathcal{H}\mathbb{Z}$-module, we have the canonical isomorphism

$$\mathcal{M}(\Sigma^q_{\mathbb{P}^1}\mathcal{F}) \cong \mathbb{Z}(q)[2q] \otimes \mathcal{M}(\mathcal{F}).$$

By Remark 6.4.1, we have the canonical isomorphisms in $\mathcal{DM}(k)$

$$\begin{aligned}\pi^\mu_{p+q}\mathcal{E} = \mathcal{M}(\sigma_{p+q}\mathcal{E})[-p-q] &\cong \mathcal{M}(\sigma_{p+q}(\Sigma^p_{\mathbb{P}^1}\Sigma^{-p}_{\mathbb{P}^1}\mathcal{E}))[-p-q]\\ &\cong \mathcal{M}(\Sigma^p_{\mathbb{P}^1}\sigma_q(\Sigma^{-p}_{\mathbb{P}^1}\mathcal{E}))[-p-q]\\ &\cong \mathcal{M}(\sigma_q(\Sigma^{-p}_{\mathbb{P}^1}\mathcal{E}))\otimes\mathbb{Z}(p)[p-q]\\ &\cong \pi^\mu_q(\Sigma^{-p}_{\mathbb{P}^1}\mathcal{E})\otimes\mathbb{Z}(p)[p].\end{aligned}$$

□

So, for example, if $\mathcal{E}$ is the K-theory $\mathbb{P}^1$-Ω-spectrum $\mathcal{K} := (K, K, \ldots)$, then $\Sigma^q_{\mathbb{P}^1}\mathcal{K} = \mathcal{K}$, and $\sigma_0\mathcal{K} = \mathcal{H}\mathbb{Z}$. Thus $\pi^\mu_0(\mathcal{K}) = \mathbb{Z}$ and

$$\pi^\mu_p(\mathcal{K}) = \pi^\mu_0(\mathcal{K})\otimes\mathbb{Z}(p)[p] = \mathbb{Z}(p)[p].$$

Remark 9.2.4. Note that this identity does not rely on Conjecture 9.1.1, rather, we have the isomorphism

$$\mathcal{H}(\pi^\mu_p(\mathcal{K})[p]) \cong \sigma_p\mathcal{K}$$

by direct computation.

Thus, our E_2-spectral sequence is the Bloch-Lichtenbaum, Friedlander-Suslin spectral sequence [3, 6]

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}(-q)) \Longrightarrow K_{-p-q}(X).$$

Remark 9.2.5. The $\mathbb{P}^1$-Ω-spectrum $\phi_0\mathcal{K}$:

$$\phi_0\mathcal{K} = (K, K^{(1)}, K^{(2)}, \ldots)$$

gives an explicit model for $\mathbb{P}^1$-connected algebraic K-theory.

As a second example, we recall that, for $\mathcal{E} \in \mathcal{SH}(k)$, $X \in \mathbf{Sm}_k$, we have the bi-graded homotopy groups

$$\mathcal{E}_{a,b}(X) := \mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma^{a-2b}\Sigma^b_{\mathbb{P}^1}\Sigma^\infty X_+, \mathcal{E}).$$

Letting $E(b)$ be the 0th spectrum of $\Sigma^b_{\mathbb{P}^1}\mathcal{E}$, we thus have the identity

$$\mathcal{E}_{a,-b}(X) = \pi_{a+2b}(E(b)(X)).$$

Thus we have the spectral sequence

$$E_2^{p,q} = \mathbb{H}^p(X, \pi^\mu_{-q}(\Sigma^b_{\mathbb{P}^1}\mathcal{E})) \Longrightarrow \pi_{-p-q}(\hat{E}(b)(X)) = \hat{\mathcal{E}}_{-p-q-2b,-b}(X).$$

Via Lemma 9.2.3, we have

$$\begin{aligned}\mathbb{H}^p(X, \pi^\mu_{-q}(\Sigma^b_{\mathbb{P}^1}\mathcal{E})) &= \mathbb{H}^p(X, \pi^\mu_{-b-q}(\mathcal{E})\otimes\mathbb{Z}(b)[b])\\ &= \mathbb{H}^{p+b}(X, \pi^\mu_{-b-q}(\mathcal{E})\otimes\mathbb{Z}(b)).\end{aligned}$$

Thus, making the translation $(p, q) \mapsto (p-b, q-b)$, we have the spectral sequence

$$E_2^{p,q} = \mathbb{H}^p(X, \pi^\mu_{-q}(\mathcal{E})\otimes\mathbb{Z}(b)) \Longrightarrow \hat{\mathcal{E}}^{p+q,b}(X).$$

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