# HOMOLOGY OF ALGEBRAIC VARIETIES: AN INTRODUCTION TO RECENT RESULTS OF SUSLIN AND VOEVODSKY

MARC LEVINE

## 1. Introduction

The recent series of papers by Suslin, Voevodsky, Suslin-Voevodsky and Friedlander-Voevodsky: [49], [50], [51], [52], [53] and [20], has developed a remarkable new viewpoint in the study of algebraic cycles. A new "topology" defined by Voevodsky, the qfh-topology, relates algebraic cycles to certain representing sheaves for this topology, and thus allows a systematic application of the powerful methods of sheaf-theory in areas which heretofore have been approached by essentially algebro-geometric or homotopy theoretic means. We hope to give here, not a full overview, but rather a sample of these new techniques and the results they have made possible. We will concentrate on the applications to the mod-$n$ theory, for which the most striking applications have as yet appeared.

## 2. Historical background

Let $X$ be a CW-complex. The topological $K_0$ (see [2], [3]) of $X$, $K_0^{\mathrm{top}}(X)$, is defined as an abelian group via generators and relations: the generators are the isomorphism classes $[E]$ of complex vector bundles $E \to X$, with relations given by *stable equivalence*:

$$[E] \equiv [E'] \iff E \oplus e^n \cong E' \oplus e^n \text{ for some } n,$$

where $e^n$ is the trivial rank $n$ bundle on $X$. For $X$ a finite CW complex, this group is also given as the homotopy classes of maps of $X$ to the classifying space $B\mathrm{GL}_{\mathbb{C}}$ of the topological group

$$\mathrm{GL}_{\mathbb{C}} := \varinjlim_N \mathrm{GL}_{N,\mathbb{C}},$$

where $\mathrm{GL}_{N,\mathbb{C}}$ is the topological group of invertible $n$ by $n$ matrices over $\mathbb{C}$. The higher topological $K$-theory of a finite CW-complex $X$ can then be defined as the homotopy groups of a function space:

$$K_n^{\mathrm{top}}(X) := \pi_n(\mathrm{Hom}(X, B\mathrm{GL}_{\mathbb{C}})).$$

The filtration of $X$ via its $k$-skeleta, together with Bott periodicity:

$$K_n^{\mathrm{top}}(X) \cong K_{n+2}^{\mathrm{top}}(X),$$

*Key words and phrases.* motives, cycles.
research supported by the NSF

gives rise to the *Atiyah-Hirzebruch spectral sequence* relating singular cohomology and topological $K$-theory; this spectral sequence degenerates rationally, giving the isomorphisms

$$K^{\mathrm{top}}_{\mathrm{even}}(X) \otimes \mathbb{Q} \cong \oplus_n H^{2n}(X, \mathbb{Q})$$
$$K^{\mathrm{top}}_{\mathrm{odd}}(X) \otimes \mathbb{Q} \cong \oplus_n H^{2n+1}(X, \mathbb{Q}).$$

The algebraic $K_0$ of an algebraic variety $X$ is defined as the abelian group with generators the isomorphism classes $[E]$ of algebraic vector bundles $E$ on $X$, with relations given by setting

$$[E] = [E'] + [E'']$$

if there exists an exact sequence

$$0 \to E' \to E \to E'' \to 0.$$

If $X$ is affine, all such exact sequences split, and one gets the same group by imposing the relation of stable equivalence as in the topological case, but in general, stable equivalence is a weaker relation (they agree, however, in the topological setting). In fact, the algebraic $K_0$ was defined (by Grothendieck) *before* the topological case, but the higher $K$-theory in the topological setting was defined before the algebraic case.

Pursuing the analogy with the topological situation, Karoubi and Villamayor [31] gave a definition of higher algebraic $K$-theory of a ring $R$ by means of the *discrete* group $\mathrm{GL}(R)$, where the topology of $\mathrm{GL}_{\mathbb{C}}$ is replaced by a certain simplicial structure. As this idea is central to our whole discussion, we give a description in a somewhat more general setting.

The algebraic version of homotopy is gotten by replacing the unit interval with the affine line $\mathbb{A}^1$. Following this further, one considers the cosimplicial variety $\Delta^*$, with $n$-cosimplices $\Delta^n$ given as the hyperplane in $n+1$-space defined by the linear equation

$$\sum_{i=0}^{n} t_i = 1.$$

If one takes the $t_i$ to be real numbers, with $0 \le t_i \le 1$, this is the usual $n$-simplex; as the usual expressions for the co-face and co-degeneracy maps between the $n$-simplex and the $n-1$-simplex are linear functions of the $t_i$, one obtains the structure of a cosimplicial variety on $\Delta^*$ by using the same formulas as in the real case.

Starting with the map

$$\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$$
$$(s,t) \mapsto st$$

one can construct a map

$$H\colon \mathbb{A}^1 \times \Delta^* \to (\mathbb{A}^1 \times \Delta^*)^{[0,1]}, \tag{2.1}$$

having the formal properties of a "homotopy" between the identity map on $\mathbb{A}^1 \times \Delta^*$ and the map

$$(x,t) \mapsto (0,t).$$

Now, suppose we have a functor (here **Sch** is the category of schemes)

$$F\colon \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Sets}$$

We may form the new functor

$$F_h\colon \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Simplicial\ Sets} \tag{2.2}$$

by

$$F_h(X) = F(X \times \Delta^*).$$

The simplicial set $F_h(X)$ then satisfies the *homotopy property*: by applying $F$ to (2.1), one shows that the natural map

$$F_h(X) \to F_h(X \times \mathbb{A}^1)$$

is a homotopy equivalence on the geometric realization. Similarly, if we have a functor

$$F\colon \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Ab}$$

we may form the new functor with values in chain complexes

$$F_{h\mathbb{Z}}\colon \mathbf{Sch}^{\mathrm{op}} \to \mathbf{C}(\mathbf{Ab}) \tag{2.3}$$

by taking the chain complex associated to the simplicial abelian group $F_h$. The functor $F_{h\mathbb{Z}}$ is also homotopy invariant: the natural map

$$F_{h\mathbb{Z}}(X) \to F_{h\mathbb{Z}}(X \times \mathbb{A}^1)$$

is a quasi-isomorphism.

Now back to Karoubi-Villamayor $K$-thery. Suppose $X$ is an affine variety with ring of functions $R$. The group $\mathrm{GL}_N(R)$ is the group of algebraic maps

$$X \to \mathrm{GL}_N,$$

where $\mathrm{GL}_N$ is the open subscheme of affine $N^2$ space defined as the locus where the determinant function is non-zero. Replacing $X$ with $X \times \Delta^*$ gives the simplicial ring $\Delta_*(R)$ with

$$\Delta_n(R) = R[t_0, \dots, t_n]/(\sum_i t_i - 1)$$
$$\mathrm{Spec}\ \Delta_*(R) = X \times \Delta^*;$$

and the simplicial group $\mathrm{GL}_N(\Delta_*(R))$:

$$\mathrm{GL}_N(\Delta_n(R)) = \mathrm{Hom}_{alg}(X \times \Delta^n, \mathrm{GL}_N).$$

One then takes the geometric realization $|\mathrm{GL}_N(\Delta_*(R))|$ of this simplicial set, passes to the limit over $N$, $|\mathrm{GL}(\Delta_*(R))|$, and defines the *Karoubi-Villamayor $K$-theory of $R$* by

$$KV_n(R) = \pi_{n-1}(|\mathrm{GL}(\Delta_*(R))|).$$

One can extend this definition to an arbitrary scheme by a sheafification process.

Quillen gave another definition of higher algebraic $K$-theory, first for rings by using his plus-construction [38], then for arbitrary exact categories via his categorical $Q$-construction [39]; this definition gained wide acceptance as the "correct" one in the general setting. It turns out that, for a regular scheme $X$ (the algebro-geometric version of a manifold), the Karoubi-Villamayor $K$-theory agrees with the Quillen $K$-theory, although for singular schemes, the two definitions are not in general the same.

In contrast with the topological case, the algebraic version of singular cohomology, the so-called "motivic cohomology", was not completely defined until some fifteen years after the definition of algebraic $K$-theory. The first step towards the full definition of motivic cohomology was, however, taken quite a bit earlier, arising in the 50's with the construction of the Chow ring of algebraic cycles modulo rational equivalence (see e.g. [56], [45], [8], [9], [7], [12] and [25]). What emerged from these constructions was an algebraic homology-like theory built out of the free abelian group on the algebraic subvarieties of $X$, the *algebraic cycles on* $X$. Replacing the unit interval with the affine line as above leads to the relation of *rational equivalence*, an algebraic version of homology. More precisely, a cycle is *rationally equivalent to zero* if it is of the form

$$\mathrm{pr}_X(W \cdot (X \times 0) - W \cdot (X \times 1))$$

for $W$ a cycle on $X \times \mathbb{A}^1$, where $\mathrm{pr}_X$ is the projection on $X$, and $\cdot$ is the intersection product (one requires that $W$ contains no component of the form $W_0 \times t$, so that the intersection product is defined). One then defines the *Chow group* of $X$ as the group of algebraic cycles modulo rational equivalence.

For smooth quasi-projective varieties, this homology-like theory has a cohomological flavor as well: the intersection of subvarieties extends to the *intersection product* on the group of cycles mod rational equivalence, defining the *Chow ring*

$$\mathrm{CH}^*(X) := \oplus_q \mathrm{CH}^q(X),$$

with the grading by codimension. In addition, the Chow ring admits functorial pull-back maps

$$f^*\colon \mathrm{CH}^*(X) \to \mathrm{CH}^*(Y)$$

for arbitrary maps $f\colon Y \to X$ between smooth, quasi-projective varieties. Grothendieck's theory of Chern classes with values in the Chow ring [25], together with the Grothendieck-Riemann-Roch theorem [7], gives the isomorphism

$$\sum_q c_q\colon K_0(X) \otimes \mathbb{Q} \cong \oplus_q \mathrm{CH}^q(X) \otimes \mathbb{Q}.$$

The major inadequacy of both the algebraic $K_0$ and the Chow ring is the lack of good *localization sequence*, relating the theory for $X$, an open subscheme $U$, and the closed complement $Z := X \setminus U$. Both theories have the beginning (or end, depending on your point of view) of such a sequence; Quillen's uniform definition of the higher $K$-groups, together with his localization theorem (see [39]), filled this gap on the $K$-theory side, giving the long exact sequence

$$\ldots \to K_p(Z) \to K_p(X) \to K_p(U) \to K_{p-1}(Z) \to \ldots \to K_0(X) \to K_0(U) \to 0$$

(this is for $X$ and $Z$ smooth; there is a similar sequence in general).

The first successful definition of motivic cohomology was supplied by Bloch in 1985, with his construction of the *higher* Chow groups [4]. The idea is to make sense of the algebraic cycles on the co-simplicial variety $X \times \Delta^*$. The technical problem here is that an arbitrary codimension $q$ subvariety of $X \times \Delta^p$ may not intersect a face $X \times \Delta^{p'}$ in codimension $q$, and thus the intersection product with this face is not defined; Bloch solves this by considering the subgroup generated by subvarieties which *do* intersect all faces in the correct codimension. This gives the homological complex $\mathcal{Z}^q(X,*)$, with $\mathcal{Z}^q(X,p)$ being the subgroup of the cycles on $X \times \Delta^p$ just described, and differential given by the pull-back via the coboundary maps in $X \times \Delta^*$ (i.e., intersection with the codimension one faces $X \times \Delta^{p-1}$ of $X \times \Delta^p$). The higher Chow groups are then defined as the homology

$$\mathrm{CH}^q(X,p) := H_p(\mathcal{Z}^q(X,*))$$

This extends the definition of the Chow ring $\mathrm{CH}^q(X)$, via the identity

$$\mathrm{CH}^q(X) = \mathrm{CH}^q(X,0).$$

It wasn't until around 1992 that the localization property for $\mathrm{CH}^q(X,*)$ was proved (see [5]), and this only for varieties over a field of characteristic 0. We do know, however (see e.g. [32]), that there is a close relation between the higher algebraic $K$-theory and the higher Chow groups of a variety $X$ which is smooth over a field, so the higher Chow groups do seem to be the "correct" groups for motivic cohomology.

## 3. The Quillen-Lichtenbaum conjecture.

Let $X$ be a variety defined over $\mathbb{C}$, $X(\mathbb{C})$ the topological space of solutions to the equations defining $X$. Passing from algebraic maps to continous maps defines the comparison map from algebraic to topological $K$-theory

$$K_p^{\mathrm{alg}}(X) \to K_p^{\mathrm{top}}(X(\mathbb{C})).$$

In general, this map is far from being an isomorphism. However, the situation seems to get quite a bit simpler if one takes $K$-theory with mod $n$ coefficients (defined via mod $n$ homotopy groups of the appropriate spaces):

$$K_p^{\mathrm{alg}}(X;\mathbb{Z}/n) \to K_p^{\mathrm{top}}(X(\mathbb{C});\mathbb{Z}/n).$$

The so-called *Quillen-Lichtenbaum conjecture* asserts (in one of its forms) that this map is an isomorphism for $p \geq 2\dim_{\mathbb{C}}(X)$.

One can extend this to varieties over more general fields as follows. Grothendieck *et al.* [26] have defined the *étale cohomology*, $H^p_{\text{ét}}(X,\mathbb{Z}/n)$, of a scheme $X$, which (for $n$ invertible on $X$) has many of the formal properties of the mod $n$ singular cohomology of a space (see §5 for more information). In fact, the *comparison theorem* of Artin gives a natural isomorphism between the étale cohomology $H^p_{\text{ét}}(X,\mathbb{Z}/n)$ of a scheme $X$ defined over $\mathbb{C}$, and the singular cohomology $H^p(X(\mathbb{C}),\mathbb{Z}/n)$ of the analytic space of solutions $X(\mathbb{C})$. Building on the methods of Grothendieck, Friedlander [14], [15], and Dwyer-Friedlander [13] constructed an algebro-geometric version of mod $n$ topological $K$-theory, the *étale $K$-theory* $K_*^{\text{ét}}(X;\mathbb{Z}/n)$. There is an Atiyah-Hirzebruch spectral sequence in the étale case as well, relating $H^*_{\text{ét}}(X,\mathbb{Z}/n)$ and $K_*^{\text{ét}}(X;\mathbb{Z}/n)$.

The Quillen-Lichtenbaum conjecture then asserts that the natural map of mod-$n$ algebraic $K$-theory to mod-$n$ étale $K$-theory of a variety $X$ over a field $k$:

$$K_p^{\mathrm{alg}}(X;\mathbb{Z}/n) \to K_p^{\text{ét}}(X;\mathbb{Z}/n)$$

is an isomorphism for $p \geq 2\dim_k(X) + \text{c.d.}(k)$, where c.d.$(k)$ the mod $n$ cohomological dimension of the Galois group of $\bar{k}/k$. Via the theory of Chern classes to étale cohomology, this can be interpreted as stating that the Chern class maps induce isomorphisms (for $F$ a field)

$$\sum_{2q-p=m} c_{q,p}\colon K_m(F;\mathbb{Z}/n) \to \oplus_{2q-p=m} H^p_{\text{ét}}(F,\mathbb{Z}/n(q)); \quad 2q \geq p,$$

for $n$ prime to "small primes" (depending on $m$) and prime to the characteristic of $F$.

We should mention here that the original conjecture of Quillen and Lichtenbaum (see e.g. [34]) concerned itself only with the $K$-groups and étale cohomology of *number rings*, and that the conjecture stated above is a slight modification of Conjecture 3.9 in [15].

Thus, one hopes for natural isomorphisms,

$$H^p_\mu(X,\mathbb{Z}/n(q)) \to H^p_{\text{ét}}(X,\mathbb{Z}/n(q)); \quad 2q \geq p,\ q \geq \dim_k(X), \tag{3.1}$$

where $H^p_\mu(X,\mathbb{Z}/n(q))$ is the mod-$n$ motivic cohomology of a smooth $k$-scheme $X$; at least provisionally, this can be defined via Bloch's cycle complex as

$$H^p_\mu(X,\mathbb{Z}/n(q)) = H_{2q-p}(\mathcal{Z}^q(X,*)\otimes\mathbb{Z}/n).$$

One main result of Suslin and Voevodsky ([49] and [48]) is that, for varieties over an algebraically closed field, such isomorphisms exist.

In fact, the major player in the proof is not Bloch's cycle complex, but a homological version introduced by Suslin (in a 1988 lecture at Luminy). The *Suslin homology* of a scheme $X$ is defined in terms of families of zero cycles on $X$, parametrized by the co-simplicial scheme $\Delta^*$. There was no serious progress in the study or application of Suslin homology until Voevodsky introduced his qfh-topology and h-topology in 1992; this provided the needed breakthrough by allowing an interpretation of the mod $n$ Suslin cohomology (the dual of Suslin homology) as a cohomology theory arising from sheaves on a Grothendieck site.

## 4. Relative 0-cycles and Suslin homology of schemes

A pointed topological space $(X,*)$ freely generates a commutative monoid with $*$ acting as identity, the *pointed infinite symmetric product* $Sp^\infty X$, whose points are the finite formal sums $\sum_i x_i$ with $x_i \in X$, modulo the relation

$$n\cdot * + \sum_i x_i \sim \sum_i x_i.$$

If $(X,*)$ is a connected CW complex, the theorem of Dold and Thom [11] shows that $Sp^\infty X$ represents the homology of $X$, via a natural isomorphism

$$\pi_n(Sp^\infty X) \cong H_n(X;\mathbb{Z}).$$

One can mimick this in the algebraic setting. Rather than attempting to deal with the infinite dimensional scheme $Sp^\infty X$, Suslin considers a functor which is in principle the one represented by the group completion of $Sp^\infty X$. As in §2, one replaces the topology on $Sp^\infty X$ induced by that of $X$ with the simplicial structure induced by $\Delta^*$.

Definition 4.1. Let $X$ and $S$ be $k$-schemes, with $S$ smooth and irreducible. Define the group $C_0(S;X)$ to be the free abelian group on the subvarieties $W$ of $X \times S$ such that the projection $p_2\colon W \to S$ is finite and surjective (recall that a map $f\colon Y \to Z$ is finite if $f$ is proper and each fiber of $f$ is a finite set). Set

$$C_n(S;X) = C_0(S \times \Delta^n; X).$$

The group $C_0(S;X)$ is covariantly functorial in $X$ and contravariant in $S$; thus, we may form the complex (see (2.2))

$$C_*(S;X) := C_0(S \times \Delta^*; X) = C_0(S;X)_{h\mathbb{Z}};$$

and the homology

$$H_p^{\text{sing}}(S;X) := H_p(C_*(S;X)).$$

The groups $H_p^{\text{sing}}(S;X)$ are thus covariant in $X$ and contravariant in $S$.

Example 4.2. Take $S = \operatorname{Spec} k$. Then $H_p^{\text{sing}}(k;X)$ is the *Suslin homology of* $X$. Sending $X$ to $H_p^{\text{sing}}(X;\mathbb{Z}) := H_p^{\text{sing}}(k;X)$ defines the functor

$$H_p^{\text{sing}}(-;\mathbb{Z})\colon \mathbf{Sch}/k \to \mathbf{Ab}$$

Let $Y$ be a smooth irreducible $k$-scheme, and define $C_0(X)(Y)$ by

$$C_0(X)(Y) := C_0(Y;X). \tag{4.1}$$

This defines the (contravariant) functor $C_0(X)$ on the category $\mathbf{Sm}/k$ of smooth, finite-type $k$-schemes.

## 5. Grothendieck topologies

Grothendieck topologies play a central role in the arguments of Suslin-Voevodsky; we give here a brief overview. For more details see, e.g., [1], [35], [10], and [26].

Grothendieck introduced the notion of a Grothendieck topology, and the associated category of sheaves for the topology, in the construction of étale cohomology, the algebro-geometric replacement of singular cohomology. To form a topology on a set $X$, one selects a collection of subsets of $X$, the *open* subsets for the topology, subject to certain axioms. Grothendieck extended this notion by considering the inclusion of an open subset as a special case of a morphism in a category, and considering, for a category $\mathcal{C}$, families of morphisms in $\mathcal{C}$

$$f_\alpha\colon U_\alpha \to X,$$

satisfying axioms which generalize the notion of an open cover.

More specifically, for each object $U$ of $\mathcal{C}$, one must define when a family of morphisms

$$\{U_\alpha \to U \mid \alpha \in A\}$$

in $\mathcal{C}$ is a *cover* of $U$; one requires

(1) *stability under base-change*: if

$$\{U_\alpha \to U \mid \alpha \in A\}$$

is a cover of $U$, and $V \to U$ is a morphism in $\mathcal{C}$, then the fiber product $U_\alpha \times_U V$ exists for each $\alpha$, and the famiy

$$\{p_2 \colon U_\alpha \times_U V \to V \mid \alpha \in A\}$$

is a cover of $V$,

(2) *local determination*: if

$$\{U_\alpha \to U \mid \alpha \in A\}$$

is a cover of $U$, and if

$$\mathcal{V} := \{V_\beta \to U \mid \beta \in B\}$$

is a collection of morphisms in $\mathcal{C}$, such that

$$\{U_\alpha \times_U V_\beta \to U_\alpha \mid \beta \in B\}$$

is a cover of $U_\alpha$ for each $\alpha$, then $\mathcal{V}$ is a cover of $U$.

(3) *stability under composition*: if

$$\{U_\alpha \xrightarrow{f_\alpha} U \mid \alpha \in A\}$$

is a cover of $U$, and if

$$\{V_\beta \xrightarrow{g_{\alpha\beta}} U_\alpha \mid \beta \in B_\alpha\}$$

is a cover of $U_\alpha$ for each $\alpha$, then

$$\{V_\beta \xrightarrow{f_\alpha \circ g_{\alpha\beta}} U \mid \beta \in B_\alpha, \alpha \in A\}$$

is a cover of $U$.

(4) each isomorphism

$$U' \to U$$

forms a cover of $U$.

As category together with a Grothendieck topology is called a *Grothendieck site*.

We have actually defined above the notion of a Grothendieck *pre-topology* on $\mathcal{C}$; this suffices to define the primary objects of interest: presheaves and sheaves. A Grothendieck pre-topology generates a Grothendieck topology, just as a basis for a topology generates a topology, so we will freely confuse the two notions in what follows.

One can view a topology in the usual sense on a set $X$ as a Grothendieck topology, by defining $\mathcal{C}$ to be the category formed by the inclusion maps $V \to U$ among the open subsets of $X$, and defining the covers of $U \subset X$ to be collections

$$\{U_\alpha \to U\},$$

with

$$\cup_\alpha U_\alpha = U,$$

i.e., open covers of $U$.

Conversely, if one has a Grothendieck topology on a category $\mathcal{C}$, one can consider the morphisms

$$f\colon V \to U$$

which occur in some cover $\{f_\alpha\colon U_\alpha \to U\}$ as an "open subset" of $U$. If the category $\mathcal{C}$ has a final object $X$, and if each morphism $V \to U$ occurs in some cover of $U$, then one may consider the Grothendieck topology on $\mathcal{C}$ as a "Grothendieck topology on $X$", and the objects of $\mathcal{C}$ as the opens for the Grothendieck topology.

In the general case, a Grothendieck topology $\mathfrak{T}$ on a category $\mathcal{C}$ induces the Grothendieck topology $\mathfrak{T}_X$ on each object $X$ of $\mathcal{C}$ by forming the category $\mathcal{C}_X$ of maps $U \to X$ which occur in a cover of $X$, where a map $(U \to X) \to (V \to X)$ is a commutative triangle

$$\begin{array}{ccc} U & \to & V \\ & \searrow \ \swarrow & \\ & X, & \end{array}$$

and using the covers in $\mathfrak{T}$ to define the covers in $\mathfrak{T}_X$. The operation of restricting to an object of $\mathcal{C}$ is like taking the induced topology on a subset of a topological space. One can consider a Grothendieck topology on a category $\mathcal{C}$ as being defined by giving a Grothendieck topology on $X$ for each object $X$ of $\mathcal{C}$, with this assignment being natural in $X$.

Once one has a Grothendieck topology $\mathfrak{T}$ on a category $\mathcal{C}$, one defines a presheaf $F$ (say of abelian groups) for the topology as a contravariant functor from $\mathcal{C}$ to **Ab**. A sheaf is a presheaf $F$ that satisfies the sheaf axiom: if

$$\{i_\alpha\colon U_\alpha \to U\}$$

is a cover of $U$, then the sequence

$$0 \to F(V) \xrightarrow{\prod_\alpha F(i_\alpha)} \prod_\alpha F(U_\alpha) \overset{F(p_1)}{\underset{F(p_2)}{\rightrightarrows}} \prod_{\alpha,\beta} F(U_\alpha \times_V U_\beta)$$

is exact. The category $\mathrm{Sh}_{\mathfrak{T}}$ of sheaves of abelian groups for the topology $\mathfrak{T}$ is then an abelian category, so one can perform the usual operations of homological algebra in this setting. If $X$ is in $\mathcal{C}$, using the topology $\mathfrak{T}_X$ on $\mathcal{C}_X$, one has the notions of a pre-sheaf or a sheaf on $X$, giving the category of sheaves on $X$ for the topology $\mathfrak{T}$, $\mathrm{Sh}_{\mathfrak{T}}(X)$.

There is also the operation of sheafifying a presheaf, so one has in particular the "constant" sheaf $A_X$ on $X$ for each abelian group $A$. As one has the canonical isomorphism

$$\mathrm{Hom}_{\mathrm{Sh}_{\mathfrak{T}}(X)}(\mathbb{Z}_X, \mathcal{F}) \cong \mathcal{F}(X),$$

the groups $\mathrm{Ext}^p(\mathbb{Z}_X, \mathcal{F})$ are the higher derived functors of the global sections functor; one can therefore define the cohomology of a sheaf $\mathcal{F}$ on $X$ for the topology $\mathfrak{T}$ by

$$H^p_{\mathfrak{T}}(X; \mathcal{F}) = \mathrm{Ext}^p_{\mathrm{Sh}_{\mathfrak{T}}(X)}(\mathbb{Z}_X, \mathcal{F}). \tag{5.1}$$

EXAMPLE 5.1.

(1) The *Zariski* topology on a scheme $X$ is a topology in the classical sense, with opens being complements of algebraic subsets $F$ of $X$. One makes the Zariski topology into a Grothendieck topology as described above.

(2) For $X$ a smooth scheme of finite type over an algebraically closed field $k$, an étale map is a morphism $i\colon U \to X$ whose differential $di_u$ is an isomorphism for all $k$ points $u \in U$. For example, if $X$ is a smooth variety over $\mathbb{C}$, and $f\colon Y \to X$ is *proper*, then $f$ is étale if and only if the map of complex manifolds $f(\mathbb{C})\colon Y(\mathbb{C}) \to X(\mathbb{C})$ is a covering space (necessarily finite). More generally, if $X$ is still over $\mathbb{C}$, but not necessarily smooth, then a map $f\colon Y \to X$ is étale if and only if, for each point $y$ of $Y$, there are neighborhoods $U$ of $y \in Y(\mathbb{C})$ and $V$ of $f(y) \in X(\mathbb{C})$ (in the $\mathbb{C}$-topology) such that $f$ gives an isomorphism

$$f(\mathbb{C})\colon U \to V.$$

We omit the general definition of an étale morphism.
The étale topology on a scheme $X$ is given by taking the category of "opens of $X$" to be the étale maps $U \to X$ (of finite type), and saying that a cover of $U \to X$ is a collection of étale maps

$$\{f_\alpha\colon U_\alpha \to U\}$$

such that

$$U = \cup_\alpha f_\alpha(U_\alpha).$$

Using the methods discussed above, one has the étale cohomology of a scheme $X$ with coefficients in an étale sheaf $\mathcal{F}$, $H^*_{\text{ét}}(X, \mathcal{F})$.

Remark 5.2. Let $x$ be a point of a variety $X$. The *henselization* $X^h_x$ of $X$ at $x$ plays the role, for the étale topology, of a small neighborhood of $x$ in $X$ in the classical topology. The henselization is gotten by taking the inverse limit over the the collection of pointed étale map

$$(Y, y) \to (X, x).$$

For instance, if the base field $k$ is algebraically closed, and $x$ is a smooth $k$-point of a smooth $k$-variety $X$ of dimension $d$, then $X^h_x$ is isomorphic to the henselization $(\mathbb{A}^d)^h_0$ of affine $d$-space at the origin.

The examples in 5.1 describe the so-called *small* site: giving a Grothendieck topology for a single scheme $X$; giving a Grothendieck topology on the category $\mathbf{Sch}/k$ of schemes over over a fixed base field $k$ defines what is known as a *big* site.

For example, the étale topology on a scheme $X$ is natural in $X$: if $U \to X$ is étale, and $f\colon Y \to X$ is a morphism of schemes, then $U \times_X Y \to Y$ is étale. This gives us the big étale site on $\mathbf{Sch}/k$. The big Zariski site is defined similarly.

Now, suppose we have a sheaf $\mathcal{F}$ on $\mathbf{Sch}/k$ for some topology $\mathfrak{T}$. For a $k$-scheme $X$, one can restrict $\mathcal{F}$ to $X$ to give the sheaf $\mathcal{F}_X$ for the topology $\mathfrak{T}_X$, and the cohomology $H^*(X, \mathcal{F}_X)$, defined as in (5.1) as the Ext groups

$$H^*(X, \mathcal{F}_X) = \mathrm{Ext}^*_{\mathrm{Sh}_{\mathfrak{T}_X}(X)}(\mathbb{Z}_X, \mathcal{F}_X).$$

One can also define the cohomology on $X$ entirely in the category $\mathrm{Sh}_{\mathfrak{T}}/k$ of sheaves on $\mathbf{Sch}/k$ in the following way: Let

$$\mathrm{Hom}_k(-, X)\colon \mathbf{Sch}/k^{\mathrm{op}} \to \mathbf{Sets}$$

be the functor represented by $X$

$$Y \mapsto \mathrm{Hom}_k(Y, X).$$

Form the free abelian group on $\mathrm{Hom}_k(-, X)$, giving the presheaf

$$\mathbb{Z}[\mathrm{Hom}_k(-, X)] \colon \mathbf{Sch}/k^{\mathrm{op}} \to \mathbf{Ab}.$$

Let $\mathbb{Z}_{\mathfrak{T}}(X)$ denote the sheafification of $\mathbb{Z}[\mathrm{Hom}_k(-, X)]$ for the topology $\mathfrak{T}$. It follows from the Yoneda lemma that there is an natural isomophism

$$H^*_{\mathfrak{T}_X}(X, \mathcal{F}_X) = \mathrm{Ext}^*_{\mathrm{Sh}_{\mathfrak{T}_X}(X)}(\mathbb{Z}_X, \mathcal{F}_X) \cong \mathrm{Ext}^*_{\mathrm{Sh}_{\mathfrak{T}}/k}(\mathbb{Z}_{\mathfrak{T}}(X), \mathcal{F}), \tag{5.2}$$

which is the interpretation of sheaf cohomology on $X$ we wanted.

We shall see in the next section that two new topologies, both finer than the étale topology, provide via (5.2) the link between mod $n$ étale cohomology and Suslin homology.

## 6. The h-topology and the qfh-topology

Voevodsky's qfh-topology allows one to use the machinery of sheaf cohomology in the study of algebraic cycles; the h-topology is perhaps a more natural construction, but the resulting sheaf cohomology is not so obviously related to cycles. As we shall later see, the mod-$n$ cohomologies in the h- and qfh-topologies agree, so the two points of view complement each other.

We fix a base field $k$, and write $\mathbf{Sch}/k$ for the category of schemes of finite type over $k$.

Definition 6.1.

(1) Let $f \colon X \to Y$ be a map of $k$-schemes. $f$ is a *topological epimorphism* if $f$ is surjective on points and if $Y$ has the quotient topology, i.e., a subset $U$ of $Y$ is open if and only if $f^{-1}(U)$ is open in $X$. $f$ is a *universal topological epimorphism* if, for each map of schemes $Z \to Y$, the projection

$$Z \times_Y X \to Z$$

is a topological epimorphism.

(2) The *h-topology* on $\mathbf{Sch}/k$ is the Grothendieck topology for which an h-cover of $Y$ is a universal topological epimorphism $X \to Y$.

(3) the *qfh-topology* on $\mathbf{Sch}/k$ is the Grothendieck topology for which a qfh-cover of $Y$ is a universal topological epimorphism $X \to Y$ which is quasi-finite over $Y$ (the inverse image of each point of $Y$ is a finite set).

The following structure theorem gives a more concrete idea of the h-topology and qfh-topology.

**Theorem 6.2.** *i) Let $V \to Y$ be an h-cover. There is a refinement of $V$,*

$$\begin{array}{ccc} U & \to & V \\ & \searrow \; \swarrow & \\ & Y & \end{array}$$

*and a factorization of* $U \to Y$ *as*

$$U = \coprod_i U_i \xrightarrow{j} X \xrightarrow{p} Z \xrightarrow{q} Y \tag{6.1}$$

*where* $p$ *is a finite morphism,* $q$ *is the blow-up of a closed subscheme of* $Y$*, and* $j$ *is a Zariski open cover. Conversely, each morphism* $U \to Y$ *which factors as in* (6.1) *is an h-cover.*

*ii) Let* $V \to Y$ *be a qfh-cover. There is a refinement of* $V$*,*

$$\begin{array}{ccc} U & \to & V \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

*and a factorization of* $U \to Y$ *as*

$$U = \coprod_i U_i \xrightarrow{j} X \xrightarrow{p} Y \tag{6.2}$$

*where* $p$ *is a finite morphism and* $j$ *is a Zariski open cover. Conversely, each morphism* $U \to Y$ *which factors as in* (6.2) *is a qfh-cover.*

Remark 6.3. i) Clearly the h-topology is finer than the qfh-topology. The qfh-topology is finer than the étale topology.

ii) If the characteristic of $k$ is zero, one may use Hironaka's resolution of singularities to show that each h cover of a $k$-scheme $Y$ has a refinement $U \to Y$ with $U$ smooth over $k$. In characteristic $p > 0$, the recent work of de Jong [29] gives the same result. This is certainly *not* the case for qfh-covers, even of a smooth $k$-scheme. This points out an important technical advantage of the h-topology over the qfh-topology: *every* $k$-scheme is locally smooth in the h-topology, while only 0 and 1-dimensional $k$-schemes are locally smooth in the qfh-topology.

## 7. families of 0-cycles and the qfh-topology

We have seen in (5.2) that the sheafification $\mathbb{Z}_{\mathfrak{T}}(X)$ of the abelian group generated by the representing presheaf $\mathrm{Hom}(-, X)$ plays a central role in understanding sheaf cohomology. We have the "family of 0-cycles on X" functor of example 4.1:

$$C_0(X)\colon \mathbf{Sm}/k^{\mathrm{op}} \to \mathbf{Ab}$$

The qfh-topology links sheaf cohomology with algebraic cycles via

**Theorem 7.1.** *Let* $p$ *be the exponential characteristic of the base field* $k$ *(*$p = \mathrm{char}(k)$ *if* $\mathrm{char}(k) > 0$*,* $p = 1$ *if* $\mathrm{char}(k) = 0$*). The functor* $C_0(X)[1/p]$ *on* $\mathbf{Sm}/k$ *extends uniquely to a qfh-sheaf (denoted* $c_0(X)$*) on* $\mathbf{Sch}/k$*. Moreover,* $c_0(X)$ *is naturally isomorphic the "representing" qfh-sheaf* $\mathbb{Z}_{qfh}(X)[1/p]$*.*

The map relating $\mathbb{Z}_{\mathrm{qfh}}(X)[1/p]$ and $C_0(X)[1/p]$ is gotten by sending a morphism

$$f\colon Y \to X$$

to the transpose of the graph in $X \times Y$. The proof of first part of theorem 7.1 can be divided into two steps: the first step consists in extending $C_0(X)[1/p]$ to normal schemes. This is accomplished in [48] by Galois-theoretic methods, and in [50] by a type of limit process, reminiscent of the method of Weil [56] for defining intersection multiplicities. The extension from normal schemes to arbitrary schemes of finite type is then a fairly formal process, using the fact that a reduced scheme of finite type is a direct limit of normal schemes.

Another crucial property of qfh-sheaves is that they admit *transfers*. Recall that a map of schemes $f\colon X \to Y$ is *finite* if $f$ is proper and quasi-finite ($f^{-1}(y)$ is a finite set for each $y \in Y$). For instance, a finite extension of field $K \to L$ defines a finite morphism Spec $L \to$ Spec $K$. If $P$ is a presheaf on $\mathbf{Sch}/k$, a transfer on $P$ is gotten by giving a map

$$\mathrm{Tr}_{X/S} = \mathrm{Tr}_p\colon P(X) \to P(S),$$

for each finite morphism $p\colon X \to S$ with $X$ and $S$ reduced and irreducible, and $S$ smooth. These maps should be compatible with pull-back in cartesian squares in a sense which we will leave vague. Now, in the qfh-topology, each finite map $X \to S$ is an open cover, and the sheaf axiom shows that, at least if $X \to S$ is Galois with group of automorphisms $G$, taking the trace

$$\begin{aligned} \mathrm{Tr}\colon P(X) &\to P(X) \\ x &\mapsto \sum_{g \in G} x^g \end{aligned}$$

defines a map $P(X) \to P(S)$. The general case follows with a bit of additional work.

## 8. The h-topology and the rigidity theorem

As the h-topology is finer that the qfh-topology, an h-sheaf also has transfers. The crucial property of sheaves for the h-topology is expressed by the *rigidity theorem*.

The original precursor of the rigidity theorem may be found on Roitman's work on zero-cycles. The work [42] considers the torsion subgroup of the group of zero-cycles modulo rational equivalence on a smooth projective variety $X$ over an algebraically closed field $k$, and the behavior of this group under the *Albanese mapping*:

$$\alpha_X\colon \mathrm{CH}_0(X) \to \mathrm{Alb}(X).$$

Here $\mathrm{Alb}(X)$ is the Albanese variety of $X$, a finite dimensional projective algebraic group. Roitman [41], extending the work of Mumford [37] for surfaces, had shown that the map $\alpha_X$ has a "huge" kernel, assuming that $X$ admits a global algebraic $p$-form with $p$ at least 2 (and that $k = \mathbb{C}$). In contrast with this, Roitman shows in [42] that $\alpha_X$ induces an isomorphism on the torsion subgroups (prime to the characteristic of $k$).

The main point in his argument is the rigidity result: Suppose we have a family of zero-cycles on $X$, parametized by an algebraic curve $C$, which are $n$-torsion mod rational equivalence, with $n$ prime to the characteristic of $k$. Then the family is *constant* mod rational equivalence. The argument is quite simple: Let $C(k)$ be the set of $k$-points of $C$ and let $\mathrm{CH}_0(X)_n$ be the $n$-torsion subgroup of $\mathrm{CH}_0(X)$. Sending $x \in C(k)$ to the corresponding 0-cycle gives the map

$$\rho\colon C(k) \to \mathrm{CH}_0(X)_n;$$

one easily shows that $\rho$ extends to a group homomorphism from the $k$-points of the Jacobian variety

$$J(\rho)\colon J(C)(k) \to \mathrm{CH}_0(X)_n.$$

Since $k$ is algebraically closed, the group $J(C)(k)$ is $n$-divisible, hence $J(\rho)$ is the zero map. Since $J(C)(k)$ is generated by the differences $x - y$, where $x$ and $y$ are $k$-points of $C$, this shows that $\rho(x) = \rho(y)$ for all $x, y \in C(k)$. The fundamental rigidity result, theorem 8.2 below, is essentially a formal version of this result; various other extensions, with application to $K$-theory, had been given by Suslin [46], [47], Gillet-Thomason [27] and Gabber [22].

Definition 8.1. A presheaf $\mathcal{F}$ on $\mathbf{Sch}/k$ is called *homotopy invariant* if the map

$$p_1^*\colon \mathcal{F}(X) \to \mathcal{F}(X \times \mathbb{A}^1)$$

is an isomorphism for all $X$.

**Theorem 8.2.** *Let $\mathcal{F}$ be presheaf on $\mathbf{Sch}/k$ which*

(1) *is homotopy invariant*
(2) *has transfers*
(3) *is $n$-torsion: $n\mathcal{F}(T) = 0$ for all $k$-schemes $T$, where $n$ is prime to the exponential characteristic of $k$.*

*Let $x$ be a smooth point on a $k$-variety $X$, $X^h_x$ the henselization of $X$ at $x$ (see remark 5.2) and let*

$$i_x\colon \mathit{Spec}\ k \to X^h_x$$

*be the inclusion. Then the map*

$$i_0^*\colon \mathcal{F}(X^h_x) \to \mathcal{F}(\mathit{Spec}\ k).$$

*is an isomorphism.*

It follows from theorem 8.2 and remark 6.3 that the h-sheaf associated to a homotopy invariant presheaf with transfers $\mathcal{F}$ and the the constant h-sheaf with value $\mathcal{F}(\mathrm{Spec}\ k)$ have the same $n$-torsion and $n$-cotorsion, assuming that $k$ is algebraically closed. This implies the fundamental cohomological rigidity theorem

**Theorem 8.3.** *Let $k$ be an algebraically closed field, and let $\mathcal{F}$ be a homotopy invariant presheaf on $\mathbf{Sch}/k$ which admits transfers. Denote by $\tilde{\mathcal{F}}_h$ the sheafification of $\mathcal{F}$ for the h-topology. Then for all $n$ prime to char($k$), there is a canonical isomorphism*

$$\mathit{Ext}^*_h(\tilde{\mathcal{F}}_h, \mathbb{Z}/n) \to \mathit{Ext}^*_{Ab}(\mathcal{F}(\mathit{Spec}\ k), \mathbb{Z}/n).$$

Remark 8.4. The proof in [49] assumes characteristic zero to allow the use of resolution of singularities, which enables one to show that each $k$-scheme of finite type admits an h-cover which is smooth over $k$; the result of de Jong noted in remark 6.3 allows the proof to go through in arbitrary characteristic.

## 9. Change of topology

Suppose we have two Grothendieck topologies on $\mathbf{Sch}/k$, $\mathfrak{T}$ and $\mathfrak{T}'$, such that $\mathfrak{T}$ is finer than $\mathfrak{T}'$, i.e., each cover in $\mathfrak{T}'$ is a cover in $\mathfrak{T}$.

We may take a sheaf for the topology $\mathfrak{T}'$ and sheafify it for the topology $\mathfrak{T}$, defining the functor

$$i^*\colon \mathrm{Sh}_{\mathfrak{T}'} \to \mathrm{Sh}_{\mathfrak{T}}$$

For example, we may compare the h-, qfh- and étale topologies,

$$\mathbf{Sch}/k_{\mathrm{h}} \xrightarrow{\alpha} \mathbf{Sch}/k_{\mathrm{qfh}} \xrightarrow{\beta} \mathbf{Sch}/k_{\text{ét}}.$$

The main comparison result is

**Theorem 9.1.** *Let $\mathcal{F}$ be an étale sheaf and let $\mathcal{G}$ be a qfh-sheaf on $\mathbf{Sch}/k$. Then*

$$Ext^*_{\acute{e}t}(\mathcal{F}, \mathbb{Z}/n) = Ext^*_{qfh}(\beta^*\mathcal{F}, \mathbb{Z}/n)$$
$$Ext^*_{qfh}(\mathcal{G}, \mathbb{Z}/n) = Ext^*_{h}(\alpha^*\mathcal{G}, \mathbb{Z}/n)$$

The proof is fairly straightforward: the comparison of the étale and qfh cohomology relies on the structure of qfh-covers (theorem 6.2(ii)) and some elementary facts on finite covers of strictly hensel schemes. One then compares the h-cohomology with étale cohomology, using the structure of h-covers (theorem 6.2(i)) and the Künneth formula for étale cohomology.

## 10. Suslin cohomology of qfh-sheaves

We now have all the main ingredients needed to relate Suslin homology and étale cohomology: the representability theorem 7.1 and the rigidity theorem 8.3. Helped along by the comparison theorem 9.1, a straightforward spectral sequence argument completes the proof. It is technically useful to work in a somewhat more general setting, working with an arbitrary presheaf rather than the constant presheaf $\mathbb{Z}$.

Let $\mathcal{F}$ be a presheaf on $\mathbf{Sch}/k$, and let $X$ be in $\mathbf{Sch}/k$. Applying $\mathcal{F}$ to the cosimplicial scheme $X \times \Delta^*$ (as in §2) gives the simplicial abelian group $\mathcal{F}(X \times \Delta^*)$, and the associated (homological) complex $\mathcal{F}_*(X)$:

$$\mathcal{F}_n(X) = \mathcal{F}(X \times \Delta^n).$$

This forms the presheaves $\mathcal{F}_n$ on $\mathbf{Sch}/k$, and the complex of presheaves $\mathcal{F}_*$. For an abelian group $A$, set

$$C_*(\mathcal{F}) := \mathcal{F}_*(\mathrm{Spec}\ k) = \mathcal{F}(\Delta^*)$$
$$H_*^{\mathrm{sing}}(\mathcal{F}, A) := H_*(C_*(\mathcal{F}) \otimes^L A)$$
$$H^*_{\mathrm{sing}}(\mathcal{F}, A) := H^*(\mathrm{RHom}(C_*(\mathcal{F}), A)).$$

Let $\tilde{\mathcal{F}}_{n\ \mathrm{qfh}}$ denote the qfh-sheaf associated to $\mathcal{F}_n$ and $\tilde{\mathcal{F}}_{\mathrm{qfh}}$ the qfh-sheaf associated to F. We also write $C_*(\mathcal{F})$ for the complex of constant sheaves.

**Theorem 10.1.** *Let $k$ be an algebraically closed field, and let $n > 0$ be prime to $char(k)$. Let $\mathcal{F}$ be a presheaf on $\mathbf{Sch}/k$ which admits transfers. Then there is a canonical isomorphism*

$$H^*_{sing}(\mathcal{F}, \mathbb{Z}/n) \cong Ext^*_{qfh}(\tilde{\mathcal{F}}_{qfh}, \mathbb{Z}/n).$$

Using theorem 9.1, and the fact that qfh-sheaves admit transfers (§7), theorem 10.1 implies

**Corollary 10.2.** *Let $\mathcal{F}$ be a qfh-sheaf on* $\mathbf{Sch}/k$*, with $k$ algebraically closed. Then there is a canonical isomorphism*

$$H^*_{sing}(\mathcal{F},\mathbb{Z}/n) \cong Ext^*_{\acute{e}t}(\mathcal{F},\mathbb{Z}/n)$$

*for all $n$ prime to $char(k)$.*

Taking $\mathcal{F} = \mathbb{Z}_{\rm qfh}(X)$ and using theorem 7.1 and formula (5.2) gives the main result:

**Corollary 10.3.** *Let $X$ be a scheme of finite type over an algebraically closed field $k$, and let $n$ be prime to $char(k)$. Then there is a natural isomorphism*

$$H^*_{sing}(X,\mathbb{Z}/n) \cong H^*_{\acute{e}t}(X,\mathbb{Z}/n).$$

The proof of theorem 10.1 starts by considering the two spectral sequences

$$I_1^{p,q} = \mathrm{Ext}^p_{\rm qfh}((\tilde{\mathcal{F}}_q)_{\rm qfh},\mathbb{Z}/n) \Longrightarrow \mathrm{Ext}^{p-q}((\tilde{\mathcal{F}}_*)_{\rm qfh},\mathbb{Z}/n),$$

$$II_2^{p,q} = \mathrm{Ext}^p_{\rm qfh}(H_q((\tilde{\mathcal{F}}_*)_{\rm qfh}),\mathbb{Z}/n) \Longrightarrow \mathrm{Ext}^{p-q}((\tilde{\mathcal{F}}_*)_{\rm qfh},\mathbb{Z}/n).$$

The comparison of qfh-cohomology and étale cohomology (theorem 9.1), and the homotopy invariance of étale cohomology

$$H^*_{\rm \acute{e}t}(X,\mathbb{Z}/n) \cong H^*_{\rm \acute{e}t}(X\times\mathbb{A}^1,\mathbb{Z}/n)$$

lead to a proof that the projections $X\times\Delta^p \to X$ induces isomorphisms

$$\mathrm{Ext}^p_{\rm qfh}(\tilde{\mathcal{F}}_{\rm qfh},\mathbb{Z}/n) \cong \mathrm{Ext}^p_{\rm qfh}((\tilde{\mathcal{F}}_q)_{\rm qfh},\mathbb{Z}/n).$$

This implies the degeneration of the spectral sequence $I$ at $E_1$, giving the isomorphism

$$\mathrm{Ext}^*_{\rm qfh}(\tilde{\mathcal{F}}_{\rm qfh},\mathbb{Z}/n) \cong \mathrm{Ext}^*_{\rm qfh}((\tilde{\mathcal{F}}_*)_{\rm qfh},\mathbb{Z}/n).$$

By the homotopy invariance of the functors (2.3), the homology presheaf

$$H_q((\tilde{\mathcal{F}}_*)_{\rm qfh})$$

forms a homotopy invariant presheaf on $Sch/k$; the transfers for the qfh-sheaves $(\tilde{\mathcal{F}}_q)_{\rm qfh}$ give transfers for $H_q((\tilde{\mathcal{F}}_*)_{\rm qfh})$. This allows one to use the comparison with h-cohomology (theorem 9.1) and the rigidity theorem 8.3 to compare the spectral sequence $II$ with the spectral sequence

$$III_2^{p,q} = \mathrm{Ext}^p_{Ab}(H_q(\mathcal{F}_*(\mathrm{Spec}\ k),\mathbb{Z}/n) \Longrightarrow \mathrm{Ext}^{p-q}_{Ab}(\mathcal{F}_*(\mathrm{Spec}\ k),\mathbb{Z}/n),$$

giving the isomorphism

$$\mathrm{Ext}^*_{Ab}(\mathcal{F}_*(\mathrm{Spec}\ k),\mathbb{Z}/n) \cong \mathrm{Ext}^*_{\rm qfh}((\tilde{\mathcal{F}}_*)_{\rm qfh},\mathbb{Z}/n)$$

and completing the proof.

## 11. Suslin homology and Bloch's higher Chow groups

To complete the discussion, it is necessary to relate Suslin homology and Bloch's higher Chow groups.

For $X$ in $\mathbf{Sch}/k$, and $Y$ normal, let $Z_0(X)(Y)$ denote the free abelian group on the subvarieties $W$ of $X \times Y$ which are *quasi-finite* over $Y$. As in §4, sending a smooth $Y$ to $Z_0(X)(Y)$ defines a contravariant functor on $\mathbf{Sm}/k$. Taking $Y$ to be the cosimplicial scheme $\Delta^*$, and taking the associated homological complex defines the complex $Z_*(X)$, with

$$Z_n(X) = Z_0(X)(\Delta^n).$$

Suppose $X$ has pure dimension $d$ over $k$. As a subvariety $W$ of $X \times \Delta^p$ which is quasi-finite over $\Delta^p$ obviously has the proper intersection with all faces, there is the inclusion of complexes

$$Z_*(X) \to \mathcal{Z}^d(X, *). \tag{11.1}$$

Suslin [48] has shown

**Theorem 11.1.** *Suppose that $X$ is affine. Then the inclusion* (11.1) *is a quasi-isomorphism.*

Let $\mathrm{CH}^q(X, p; \mathbb{Z}/n)$ denote the homology

$$\mathrm{CH}^q(X, p; \mathbb{Z}/n) = H_p(\mathcal{Z}^q(X, *) \otimes \mathbb{Z}/n) = H_p((\mathcal{Z}^q(X, *) \otimes^L \mathbb{Z}/n).$$

There is an analog of theorem 7.1:

**Theorem 11.2.** *There is a unique extension of the functor $Z_0(X)[1/p]$ on $\mathbf{Sm}/k$ to a qfh-sheaf $z_0(X)$ on $\mathbf{Sch}/k$*

Applying theorem 10.1 gives

**Proposition 11.3.** *Let $n$ be prime to char$(k)$ and suppose $X$ is affine of dimension $d$ over $k$. There is a natural isomorphism*

$$CH^d(X, *; \mathbb{Z}/n)^\vee = H_*^{sing}(z_0(X), \mathbb{Z}/n)^\vee \cong Ext^*_{qfh}(z_0(X), \mathbb{Z}/n).$$

Here $M^\vee$ is the dual

$$M^\vee := \mathrm{Hom}_{\mathbb{Z}/n}(M, \mathbb{Z}/n)$$

of a $\mathbb{Z}/n$-module $M$.

The qfh-sheaves $z_0(X)$ are contravariantly functorial in $X$ for open immersions, and covariantly functorial for proper maps. If $Y$ is a closed subscheme of $X$ with complement $U$, the following sequence

$$0 \to z_0(Y) \to z_0(X) \to z_0(U) \tag{11.2}$$

is easily seen to be exact. Letting $\tilde{z_0}(X)_\mathrm{h}$ denote the h-sheaf associated to $z_0(X)$, there is the

**Lemma 11.4.** *The sequence*

$$0 \to \tilde{z_0}(Y)_h \to \tilde{z_0}(X)_h \to \tilde{z_0}(U)_h \to 0 \tag{11.3}$$

*induced from the sequence* (11.2) *is an exact sequence of h-sheaves on* $\mathbf{Sch}/k$.

Remark 11.5. As mentioned in §4, the Suslin homology of the functor $C_0(X)$ is an algebraic version of homology; the Suslin homology of the functor $Z_0(X)$ can similarly be viewed as an algebraic version of *Borel-Moore homology*. In fact, recall from theorem 7.1 that the functor $C_0(X)[1/p]$ extends to the qfh-sheaf $c_0(X)$ on $\mathbf{Sch}/k$. The h-sheaf $\tilde{c_0}(-)_{\mathrm{h}}$ associated to the qfh-sheaf $c_0(-)$ satisfies a Mayer-Vietoris property, giving the exact sequence

$$0 \to \tilde{c_0}(U \cap V)_{\mathrm{h}} \to \tilde{c_0}(U)_{\mathrm{h}} \oplus \tilde{c_0}(V)_{\mathrm{h}} \to \tilde{c_0}(U \cup V)_{\mathrm{h}} \to 0 \tag{11.4}$$

By theorem 10.1, the sequences (11.3) and (11.4) give rise to a localization sequence for the mod $n$ Suslin homology $H_*^{\mathrm{sing}}(Z_0(-), \mathbb{Z}/n)$ and a Mayer-Vietoris sequence for the mod $n$ Suslin homology $H_*^{sing}(-, \mathbb{Z}/n)$ (assuming $n$ prime to the characteristic). In [50], modifications of the sequences (11.3) and (11.4) give rise to a localization sequence for $H_*^{\mathrm{sing}}(Z_0(-), \mathbb{Z})$ of $Z_0(-)$ and a Mayer-Vietoris sequence for $H_*^{sing}(-, \mathbb{Z})$ (assuming characterstic zero).

If $X$ is proper over $k$ we have

$$z_0(X) = c_0(X).$$

By theorem 9.1, formula (5.2) and corollary 10.2, we have the isomorphisms

$$\mathrm{Ext}^*_{\mathrm{h}}(\tilde{z_0}(X)_{\mathrm{h}}, \mathbb{Z}/n) \cong H^*_{\text{ét}}(X, \mathbb{Z}/n) \tag{11.5}$$

for $X$ proper. Using the long exact Ext-sequence arising from the exact sequence of lemma 11.4, and the corresponding long exact Gysin sequence for étale cohomology with compact supports, one extends the isomorphism (11.6) to the isomorphism

$$\mathrm{Ext}^*_{\mathrm{h}}(\tilde{z_0}(X)_{\mathrm{h}}, \mathbb{Z}/n) \cong H^*_c(X, \mathbb{Z}/n) \tag{11.6}$$

to arbitrary $X$; here $H_c^*$ is the étale cohomology with compact suports.

With the help of (11.6), one arrives at the following comparsion of the mod-$n$ higher Chow groups, and étale cohomology for affine $X$:

**Theorem 11.6.** *Let $X$ be an affine variety over an algebraically closed field $k$, and let $n > 0$ be an integer prime to char$(k)$. Assume that $q > d := \dim(X)$. Then there is a natural isomorphism*

$$CH^q(X, p; \mathbb{Z}/n) \cong H_c^{2(d-q)+p}(X, \mathbb{Z}/n(d-q))^{\vee}.$$

The proof uses the homotopy property to replace $X$ with $X \times \mathbb{A}^{q-d}$, which reduces to the case $q = d$; one then applies proposition 11.3 and the isomorphism (11.6). Using Poincaré duality, theorem 8.3 implies

**Corollary 11.7.** *Let $X$ be a smooth affine variety over an algebraically closed field $k$, and let $n > 0$ be an integer prime to char$(k)$. Assume that $q \geq d := \dim(X)$. Then there is a natural isomorphism*

$$CH^q(X, p; \mathbb{Z}/n) \cong H^{2d-p}_{\acute{e}t}(X, \mathbb{Z}/n(q)).$$

The results 11.2 and 11.2 extend to quasi-projective $X$ (resp. smooth quasi-projective $X$) if $\mathrm{char}(k) = 0$, using the localization property for the higher Chow groups [5] and the analogous Gysin sequence for étale cohomology with compact supports.

## 12. Epilogue

We have not described the beautiful constructions and results in the fundamental papers [50], [52] and [20], which form the foundations for many of the constructions we have described. We omitted as well mention of the works on Lawson homology and related topics [17], [18], and [16], which formed a large part of the motivation for many of the constructions of [50] and [20]. We have also not described the various constructions of triangulated motivic categories and motivic complexes, in [23],[28], [34], [36], [53] and [33]. Presumably, the categorical constructions all lead to equivalent categories and the complexes all yield equivalent cohomologies, but this is at present not known.

More recently, Suslin and Voevodsky [51] have reduced the Quillen-Lichtenbaum conjectures for motivic cohomology to the *Bloch-Kato conjecture* which asserts that the *Galois symbol* on Milnor $K$-theory of a field $F$ defines an isomorphism

$$K_q^M(F)/n \to H^q_{\text{ét}}(F, \mu_n^{\otimes q})$$

for all $n$ prime to the characteristic of $F$. Relying on results of Rost [43], [44], Voevodsky has given a proof of the mod 2 Bloch-Kato conjecture (see [54]), which thus completely solves the mod 2 Quillen-Lichtenbaum conjecture for motivic cohomology. Using the Bloch-Lichtenbaum spectral sequence [6] relating $K$-theory and motivic cohomology, Weibel [55] has shown how to compute the mod 2 $K$-theory of $\mathbb{Z}$; this has recently been extended by Kahn [30] to a computation of the mod 2 $K$-theory of all number rings, verifying that mod 2 algebraic $K$-theory agrees with mod 2 étale $K$-theory for all number rings. There is a whole host of unanswered questions in the area of motivic cohomology and algebraic $K$-theory; the new viewpoint offered by Voevodsky, Suslin and others will surely lead to further breakthroughs.

## References

1. M. Artin, **Grothendieck topologies**, Seminar notes, Harvard Univ. Dept of Math., 1962.
2. M. Atiyah, **K-theory**, Benjamin 1965.
3. M. Atiyah and F. Hirzebruch, *Vector bundles and homogeneous spaces*, Am. Math Soc. Symp. in Pure Math. **III**(1961) 7-38.
4. S. Bloch, *Algebraic cycles and higher $K$-theory*, Adv. in Math. **61** No. 3(1986) 267-304.
5. S. Bloch, *The moving lemma for the higher Chow groups*, Alg. Geom. **3**(1994) 537-568.
6. S. Bloch and S. Lichtenbaum, *A spectral sequence for motivic cohomology*, preprint (1995).
7. A. Borel and J.-P. Serre, *Le théorem de Riemann-Roch*, Bull. Soc. Math. France **86**(1958) 97-136.
8. C. Chevalley, *Anneaux de Chow et Applications*, Sém. C. Chevalley, **2** Paris, 1958.
9. W.-L. Chow, *On the equivalence classes of cycles in an algebraic variety*, Ann. of Math. **64**(1956) 450-479.
10. Deligne *et al.*, *Cohomologie Etale*, Lecture Notes in Math. **569**, Springer Verlag 1977.
11. A. Dold and R. Thom, *Quasifaserungen und unendliche symmetrische Produkte*, Ann. of Math. **67**(1956) 230-281. .
12. A. Duady and J.-L. Verdier, *Séminaire de Géométrie Analytique de l'Ecole Normal Sup.* Astérisque **36-7**, 1976.
13. W. Dwyer and E. Friedlander, *Algebraic and étale K-theory*, Trans. Amer. Math. Soc. **292**(1985) 247-280.
14. E. Friedlander, *Etale K-theory I: Connections with Etale Cohomology and Algebraic Vector Bundles*, Invent. Math. **60**(1980) 109-150..
15. E. Friedlander, *Etale K-theory II: Connections with algebraic K-theory*, Ann. Ec. Norm. Sup. 4ᵉ sér. **15** no. 2(1982) 231-256.

16. E. Friedlander and O. Gabber, *Cycle spaces and intersection theory*, in **Topological methods in modern mathematics**(1993) 325-370.
17. E. Friedlander and B. Lawson, *Duality relating spaces of algebraic cocycles and cycles*, preprint (1994).
18. E. Friedlander and B. Lawson, *Moving algebraic cycles of bounded degree*, preprint (1994).
19. E. Friedlander and B. Mazur, *Correspondence homomorphisms for singular varieties*, preprint (1994).
20. E. Friedlander and V. Voevodsky, *Bivariant cycle cohomology*, preprint (1995).
21. W. Fulton, **Intersection Theory**, Springer-Verlag, New York 1984.
22. O. Gabber, *K-theory of henselian local rings and henselian pairs*, in **Algebraic K-theory, Commutative Algebra, and Algebraic Geometry**, Cont. Math. **126**(1992) 59-70.
23. D. Grayson, *Weight filtrations via commuting automorphisms*, preprint (1993).
24. A. Grothendieck, *Classes de faisceaux et théorème de Riemann-Roch*, Exposé 0, **SGA 6**, Lecture Notes in Math. 225(1971) 297-364.
25. A. Grothendieck, *La théorie des classes des Chern*, Bull. Soc. Math. France **86**(1958) 137-154.
26. Grothendieck *et al.*, *Théorie de Topos et Cohomologie Etale des Schémas*, Lecture Notes in Math. **269**, Springer Verlag 1972.
27. H. Gillet and R. Thomason, *On the K-theory of strict hensel local rings and a theorem of Suslin*, JPAA **34**(1984) 241-254.
28. M. Hanamura, *Mixed motives and algebraic cycles, I, II*, preprints (1995).
29. H. deJong, *Resolution of singularities in characteristic* $p > 0$, preprint (1995).
30. B. Kahn, *The Quillen-Lichtenbaum conjecture at the prime 2* preprint (1997).
31. M. Karoubi and O. Villamayor, *$K$-théorie algébrique et $K$-théorie topologique*, Math. Scand. **28**(1971) 265-307.
32. M. Levine, *Bloch's higher Chow groups revisted*, in **K-theory, Strasbourg, 1992**, Asterisque **226**(1994)235-320.
33. M. Levine, Motivic cohomology and algebraic cycles, preprint(1995).
34. S. Lichtenbaum, *Values of zeta-functions at non-negative integers*, in **Number Theory**, Lecture Notes in Math. 1068(1984) 127-138.
35. J.S. Milne, **Etale Cohomology**, Princeton Math. Ser. **33**, Princeton Univ. Press 1980.
36. F. Morel, *Théorie de l'homotopie et motifs, I*, preprint(1995).
37. D. Mumford, *Rational equivalence of zero-cycles on surfaces*, J. Math. Kyoto Univ. **9** (1968), 195-204.
38. D. Quillen, *Cohomology of groups*, International Congress of Mathematics, Nice 1970.
39. D. Quillen, *Higher algebraic K-theory I*, in **Algebraic K-theory I: Higher K-theories**, Lecture Notes in Math. **341**, Springer Verlag(1973) 85-147.
40. J. Roberts, *Chow's moving lemma*, in **Algebraic Geometry**, Oslo 1979, F. Oort editor.
41. A.A. Roitman, *Rational equivalence of zero-cycles*, Math. of the USSR Sbornik **18**(1972) 571-588.
42. A.A. Roitman, *The torsion of the group of* 0*-cycles modulo rational equivalence*, Ann. of Math. **111**(1979) 415-569.
43. M. Rost, *On the spinor norm and* $A_0(X, \mathcal{K}_1)$ *for quadrics*, preprint (1988).
44. M. Rost, *Some new results on the Chow groups of quadrics*, preprint (1990).
45. F. Severi, *Problèmes résolus et problèmes nouveaux dans la théories des systémes d'équivalence*, Proc. Intern. Cong. Math., **3**(1954), 529-541.
46. A. Suslin, *On the K-theory of algebraically closed fields*, Invent. Math. **73**(1983) 241-245.
47. A. Suslin, *On the K-theory of local fields*, JPAA **34**(1984) 301-318.
48. A. Suslin, *Higher Chow groups of affine varieties and étale cohomology*, preprint (1994).
49. A. Suslin and V. Voevodsky, *Singular homology of abstract varieties*, Inv. Math. **123** Fasc. 1(1996) 61-94.
50. A. Suslin and V. Voevodsky, *Relative cycles and Chow sheaves*, preprint (1994).
51. A. Suslin and V. Voevodsky, *Bloch-Kato conjecture and motivic cohomology with finite coefficients*, preprint(1995).
52. V. Voevodsky, *Homology of schemes II*, preprint (1994).
53. V. Voevodsky, *Triangulated categories of motives over a field*, preprint (1995).
54. V. Voevodsky, *The Milnor conjecture*, preprint (1997).
55. C. Weibel, *The two-torsion in the K-theory of the integers*, preprint (1996).

56. A. Weil, **Foundations of Algebraic Geometry**, AMS Colloquium publ. **29**, AMS 1962.

Dept. of Math., Northeastern University, Boston MA 02115, USA
*E-mail address*: marc@neu.edu