Math 1137, Summer 2003

Homework 10: 3,6,14,19,20,24,25 p.236; 1-3,5-7,10,11,22,37,47,51,62,65 p.253

Exercise: 3 p.236

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a) $a_n = 2^n + 1$. Starting with $n = 0$ the terms in the sequence are $1, 3, 5, 9, \ldots$
b) $a_n = (n+1)^{n+1}$. Starting with $n = 0$ the terms in the sequence are $1^1 = 1, 2^2 = 4, 3^3 = 27, 4^4 = 256, \ldots$
c) $a_n = 1 + (-1)^n$. Starting with $n = 0$ the terms in the sequence are $2, 0, 2, 0, \ldots$
d) $a_n = -(-2)^n$. Starting with $n = 0$ the terms in the sequence are $-1, 2, -4, 8, \ldots$

Exercise: 6 p.236

a) $10, 7, 4, 1, -2, -5, -8, -11, -14, -17$
b) $1, 3, 6, 10, 15, 21, 28, 36, 45, 55$
c) $3^n - 2^n$ starting at $n = 0$: $0, 1, 5, 19, 65, 211, 665, 2059, 6305, 19171$
d) $\lfloor \sqrt{n} \rfloor$ starting at $n = 0$: $0, 1, 1, 1, 2, 2, 2, 2, 2, 3$
e) $1, 2, 3, 5, 8, 13, 21, 34, 55, 89$
f) Starting at $n = 1$: $(1)_2 = 1, (11)_2 = 3, (111)_2 = 7, (1111)_2 = 15, (11111)_2 = 31, (111111)_2 = 63, (1111111)_2 = 127, (11111111)_2 = 255, (111111111)_2 = 511, (1111111111)_2 = 1023$
g) 1,2,2,4,8,11,33,37,148,153
h) a_n is the largest integer k such that $k! =\leq n$. (The factorials starting at $k = 1$ are $1! = 1$, $2! = 2$, $3! = 1 \times 2 \times 3 = 6$, $4! = 1 \times 2 \times 3 \times 4 = 24$) Starting at $n = 1$, the terms of a_n are: $1, 2, 2, 2, 2, 3, 3, 3, 3, 3$

Exercise: 14 p.237
Let $S = \{1, 3, 5, 7\}$.
a) $\displaystyle\sum_{j \in S} j = 1 + 3 + 5 + 7 = 16$
b) $\displaystyle\sum_{j \in S} j^2 = 1 + 9 + 25 + 49 = 84$
c) $\displaystyle\sum_{j \in S} \frac{1}{j} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} = \frac{178}{105}$
d) $\displaystyle\sum_{j \in S} 1 = 1 + 1 + 1 + 1 = 4$

Exercise: 19 p.237
We start with any sequence of real numbers (a_n) and we consider the sum $\sum_{j=1}^{n} (a_j - a_{j-1})$. This simplifies easily as follows:

$$\begin{aligned}
\sum_{j=1}^{n} (a_j - a_{j-1}) &= (a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_n - a_{n-1}) \\
&= (-a_0 + a_1) + (-a_1 + a_2) + (-a_2 + a_3) + \cdots + (-a_{n-1} + a_n) \\
&= -a_0 + (a_1 - a_1) + (a_2 - a_2) + \cdots + (a_{n-1} - a_{n-1}) + a_n \\
&= a_n - a_0
\end{aligned}$$

We can use this trick when we deal with "telescoping" sums: sums that are precisely in the form $\sum_{j=1}^{n} (a_j - a_{j-1})$. Telescoping sums may not be as easy to recognize right away as one might think.

Exercise: 20 p.237
This exercise tells us to "use the identity $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$". This nothing more fancy that a normal addition of

fractions:

$$\frac{1}{k} - \frac{1}{k+1} = \frac{k+1}{k(k+1)} - \frac{k}{k(k+1)} = \frac{1}{k(k+1)}$$

Consequently, we have telescoping going on the following summation:

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1} = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n-1} - \frac{1}{n} = 1 - \frac{1}{n} = \frac{n-1}{n}$$

Exercise: 24 p.237

This exercise asks us to evaluate the summation:

$$\sum_{k=99}^{200} k^3$$

and it also tells us to look at Table 2 in which we find the formula

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

However, the summation we are supposed ot calculate doesn't start at 1 but at 99. Here's how we can get the value of the summation:

$$\begin{aligned}\sum_{k=99}^{200} k^3 &= \sum_{k=1}^{200} k^3 - \sum_{k=1}^{98} k^3 \\ &= \left(\frac{200 \times 201}{2}\right)^2 - \left(\frac{98 \times 99}{2}\right)^2 \\ &= (100 \times 201)^2 - (49 \times 99)^2 \\ &= 380477799\end{aligned}$$

Exercise: 25 p.237

We want to calculate $\sum_{k=1}^{m} \lfloor \sqrt{k} \rfloor$. This problem is more difficult than all the above problems in this homework set but let's take a crack at it anyway. Sometimes we can get an idea of what a formula may look like if we write down a bunch of terms in the summation:

$$\sum_{k=1}^{m} \lfloor \sqrt{k} \rfloor = 1+1+1+2+2+2+2+2+3+3+3+3+3+3+3+4+\ldots+\lfloor \sqrt{m} \rfloor$$

The first remark to make is that in the summation it looks as though 1 gets repeated 3 times, 2 gets repeat 5 times, 3 gets repeated 7 times... A hypothesis for this could be that the integer i gets repeated $2i+1$ times. Let's prove this first. Let $n = \lfloor \sqrt{k} \rfloor$. It is the unique integer that satisfies $n \leq \sqrt{k} < n+1$. We can rephrase our question as how many integers k have $\lfloor \sqrt{k} \rfloor = n$? All these numbers are positive integers and squaring is an increasing function on positive real numbers so we can square each term in this inequality and not have to worry about reversing any signs. Thus $n^2 \leq k < (n+1)^2$. Thus there are $(n+1)^2 - n^2$ integers k such that $\lfloor \sqrt{k} \rfloor = n$, but $(n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n+1$. This proves our hypothesis.

We can now get a little closer to calculating a formula for our summation. Let N^2 be the largest square integer less than m. (For example 16 is the largest square less than 21, or 49 is the largest square less than 50.) Then we split our summation into two parts:

$$\begin{aligned}\sum_{k=1}^{m}\lfloor\sqrt{k}\rfloor &= \sum_{k=1}^{N^2-1}\lfloor\sqrt{k}\rfloor + \sum_{k=N^2}^{m}\lfloor\sqrt{k}\rfloor \\ &= \sum_{k=1}^{N-1} k(2k+1) + \sum_{k=N^2}^{m} N \\ &= \sum_{k=1}^{N-1} 2k^2 + k + (m - N^2 + 1)N \\ &= 2\sum_{k=1}^{N-1} k^2 + \sum_{k=1}^{N-1} k + (m - N^2 + 1)N\end{aligned}$$

In order to put this into a nice form, we need to use the formulas listed in table 2 on page 76, namely $\sum_{k=1}^{m} k = \frac{m(m+1)}{2}$ and $\sum_{k=1}^{m} k^2 = \frac{m(m+1)(2m+1)}{6}$. Thus, our formula becomes:

$$\begin{aligned}\sum_{k=1}^{m}\lfloor\sqrt{k}\rfloor &= 2\frac{(N-1)N(2N-1)}{6} + \frac{(N-1)N}{2} + (m - N^2 + 1)N \\ &= \frac{(N-1)N(4N-2)}{6} + \frac{3(N-1)N}{6} + (m - N^2 + 1)N \\ &= \frac{(N-1)N(4N+1)}{6} + (m - N^2 + 1)N\end{aligned}$$

where we remind that $N = \lfloor\sqrt{m}\rfloor$.

Exercise: 1 p.253
This exercise asks us to guess at a formula for the sum of the first n even positive integers. The first few value of n give us:

$$\begin{aligned} n=1: &\quad 2 = 2 \\ n=2: &\quad 2+4 = 6 \\ n=3: &\quad 2+4+6 = 12 \\ n=4: &\quad 2+4+6+8 = 20\end{aligned}$$

So here's where the guess work begins. If we notice that 2=1*2, 6=2*3, 12=3*4 and 20=4*5 then we could make a guess that the formula for the first n even positive integers is

$$\sum_{i=1}^{n} 2i = n(n+1) \tag{1}$$

Exercise: 2 p.253
Now we use induction to prove the formula we found in the previous exercise.

Basis Step: Already done for $n = 1$.

Induction Step: Assume that Formula (1) is correct for n. We look at the left hand side of the formula but where we stop at $n+1$ instead of n:

$$\begin{aligned}\sum_{i=1}^{n+1} 2i &= \left(\sum_{i=1}^{n} 2i\right) + 2(n+1) \\ &= n(n+1) + 2(n+1) \qquad \text{by the induction hypothesis} \\ &= (n+1)(n+2) \qquad \text{by factorization}\end{aligned}$$

This last line is formula 1 with $n+1$ instead of n. Hence by induction we've proved formula (1) for all $n \in \mathbb{N}^*$.

Exercise: 3 p.253
We want to prove the following formula by induction:

$$3 + 3 \cdot 5^1 + s \cdot 5^2 + \cdots + 3 \cdot 5^n = \frac{3}{4}(5^{n+1} - 1) \tag{2}$$

Basis Step: For $n = 0$, the left is side is just 3. The right hand side is $\frac{3}{4}(5 - 1) = 3$. Formula (2) works for $n = 0$.

Induction Step: Now we assume that formula (2) is correct for n and we want to prove that this forces the formula to be correct for $n + 1$. The left hand side of the formula for $n + 1$ is:

$$\begin{aligned}
3 + 3 \cdot 5^1 + s \cdot 5^2 + \cdots + 3 \cdot 5^n + 3 \cdot 5^{n+1} &= \frac{3}{4}(5^{n+1} - 1) + 3 \cdot 5^{n+1} \qquad \text{by hypothesis} \\
&= 3\Big(\frac{5^{n+1} - 1}{4} + 5^{n+1}\Big) \\
&= 3\frac{5^{n+1} - 1 + 4 \cdot 5^{n+1}}{4} \\
&= \frac{3}{4}(5 \cdot 5^{n+1} - 1) \\
&= \frac{3}{4}(5^{n+2} - 1)
\end{aligned}$$

This last expression is the right hand side of the formula but with $n+1$. Hence, by induction the formula is correct for all values of $n \in \mathbb{N}$.

Exercise: 5 p.253
This exercise asks us to find and prove a formula for the summation:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \sum_{i=1}^{n} \frac{1}{2^i}$$

Let's take a look at a few situations:

$$\begin{aligned}
\frac{1}{2} &= \frac{1}{2} \\
\frac{1}{2} + \frac{1}{4} &= \frac{3}{4} \\
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} &= \frac{3}{4} + \frac{1}{8} = \frac{7}{8} \\
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} &= \frac{7}{8} + \frac{1}{16} = \frac{15}{16}
\end{aligned}$$

By this stage, it looks like we can guess at the possible formula:

$$\sum_{i=1}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n} \tag{3}$$

Let's prove this formula by induction on n.

Basis Step: For $n = 1$, the left hand side of (3) is just $\frac{1}{2}$ while the right hand side is $1 - \frac{1}{2} = \frac{1}{2}$. Formula works for $n = 1$.

Induction Step: Now we assume that formula (3) is correct for n and we want to prove that this forces the formula to be correct for $n + 1$.

$$\begin{aligned}
\sum_{i=1}^{n+1} \frac{1}{2^i} &= \sum_{i=1}^{n} \frac{1}{2^i} + \frac{1}{2^{n+1}} \\
&= 1 - \frac{1}{2^n} + \frac{1}{2^{n+1}} \\
&= 1 - \Big(\frac{1}{2^n} - \frac{1}{2^{n+1}}\Big) \\
&= 1 - \Big(\frac{2}{2^{n+1}} - \frac{1}{2^{n+1}}\Big) \\
&= 1 - \frac{1}{2^{n+1}}
\end{aligned}$$

Thus, we've proved the induction step as well and we've proved formula (3) by induction.

Exercise: 6 p.253
This exercise asks us to guess at a formula for:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=\sum_{i=1}^{n}\frac{1}{i(i+1)}$$

The first few values of n provide us with the following list:

$$\begin{aligned}
n=1: &\quad \frac{1}{2}=\frac{1}{2}\\
n=2: &\quad \frac{1}{2}+\frac{1}{6}=\frac{2}{3}\\
n=3: &\quad \frac{1}{2}+\frac{1}{6}+\frac{1}{12}=\frac{2}{3}+\frac{1}{12}=\frac{3}{4}\\
n=4: &\quad \frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}=\frac{4}{5}
\end{aligned}$$

The very process of simplifying the fractions seems to indicate that our formula should be:

$$\sum_{i=1}^{n}\frac{1}{i(i+1)}=\frac{n}{n+1} \tag{4}$$

Exercise: 7 p.253
This exercise asks us to prove formula (4) by induction. Here we go:

Basis Step: Already done for $n=1$.

Induction Step: Assume that Formula (4) holds for n. We wish to prove that it also holds for $n+1$. The left hand side of the formula gives:

$$\begin{aligned}
\sum_{i=1}^{n+1}\frac{1}{i(i+1)} &= \sum_{i=1}^{n}\frac{1}{i(i+1)}+\frac{1}{(n+1)(n+2)}\\
&=\frac{n}{n+1}+\frac{1}{(n+1)(n+2)} \qquad \text{by induction hypothesis}\\
&=\frac{n(n+2)+1}{(n+1)(n+2)}\\
&=\frac{n^2+2n+1}{(n+1)(n+2)}=\frac{(n+1)^2}{(n+1)(n+2)}\\
&=\frac{n+1}{n+2}
\end{aligned}$$

This is the proper form of the fomula for $n+1$ so by induction, we have prove formula (4) for all $n\in\mathbb{N}^*$.

Exercise: 10 p.253
We want to prove the following formula by induction:

$$1\cdot 1!+2\cdot 2!+\cdots+n\cdot n!=(n+1)!-1 \tag{5}$$

Basis Step: For $n=1$, the left is side is just 1. The right hand side is $2!-1=2-1=1$. Formula (5) works for $n=1$.

Induction Step: Now we assume that formula (5) is correct for n and we want to prove that this forces the formula to be correct for $n+1$. The left hand side of the formula for $n+1$ is:

$$\begin{aligned}
1\cdot 1!+2\cdot 2!+\cdots+n\cdot n!+(n+1)\cdot(n+1)! &= (n+1)!-1+(n+1)\cdot(n+1)! \qquad \text{by hypothesis}\\
&=(n+1)!\Big[1+n+1\Big]-1\\
&=(n+2)\cdot(n+1)!-1\\
&=(n+2)!-1
\end{aligned}$$

This last expression is the right hand side of the formula but with $n+1$. Hence, by induction the formula is correct for all values of $n \in \mathbb{N}^*$.

Exercise: 11 p.253
Assume $h > -1$. We want to show by mathematical induction that $1 + nh \leq (1+h)^n$ for all integers $n \geq 0$. This is called Bernoulli's inequality.

Basis Step: For $n = 0$, the left is side is just 1. The right hand side is $(1+h)^0 = 1$ and hence Bernoulli's inequality holds for $n = 0$.

Induction Step: We assume the inequality is true for n and attempt to prove it for $n+1$. What I think gives the quickest induction proof is to start from the right hand side of the inequality:

$$\begin{aligned}
(1+h)^{n+1} &= (1+h)^n(1+h) && \text{to make the expression } (1+h)^n \text{ appear} \\
&\geq (1+nh)(1+h) && \text{by induction hypothesis and because } 1+h>0 \\
&\geq 1+nh+h+nh^2 \\
&\geq 1+(n+1)h+nh^2
\end{aligned}$$

This last expression is not quite the left hand side of Bernoulli's inequality. However, since $n \geq 0$ and h^2 is always nonnegative, then nh^2 is nonnegative and $1 + (n+1)h + nh^2 \geq 1 + (n+1)h$. Hence we deduce that $1 + (n+1)h \leq (1+h)^{n+1}$.

By induction, Bernoulli's inequality holds for all $n \in \mathbb{N}$.

Exercise: 22 p.254
This exercise asks us to show that for all nonnegative integers n, 6 divides $n^3 - n$. Let's do the steps.

Basis Step: For $n = 0$, $n^3 - n = 0$ and everything divides 0 so 6 divides 0. The statement holds trivially for $n = 0$.

Induction Step: We assume that given some n, 6 divides $n^3 - n$. We can express this differently by saying that $\frac{1}{6}(n^3 - n)$ is an integer. Now let's try to prove that the same will hold for $n+1$.

We expand

$$(n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + 1 - n - 1 = n^3 + 3n^2 + 3n - n = (n^3 - n) + 3(n^2 + n) = (n^3 - n) + 3n(n+1)$$

Dividing this last expression by 6 we get:

$$\frac{(n+1)^3 - (n+1)}{6} = \frac{n^3 - n}{6} + \frac{3n(n+1)}{6} = \frac{n^3 - n}{6} + \frac{n(n+1)}{2}$$

To prove that 6 divides $(n+1)^3 - (n+1)$, we need to show that this expression is an integer. We haven't used the induction hypothesis yet but we see $\frac{1}{6}(n^3 - n)$ in the above expression and the induction hypothesis tells us that this is an integer. However, at first glance, we don't know if $\frac{1}{2}n(n+1)$ is an integer. However, for any integer, either n is even and $n+1$ is odd or n is odd and $n+1$ is even. Hence, for all integers n, $n(n+1)$ is an even integer and hence $\frac{1}{2}n(n+1)$ is an integer. Thus 6 divides $(n+1)^3 - (n+1)$ and by induction 6 divides $n^3 - n$ for all integers $n \in \mathbb{N}$.

Exercise: 37 p.254
The exercise asks us to show that

$$\sum_{\{a_1,\ldots,a_k\} \subseteq \{1,2,\ldots,n\}} \frac{1}{a_1 a_2 \cdots a_k} = n \tag{6}$$

where the sum is taken over all nonempty subsets of $\{1, 2, \ldots, n\}$.

The hardest part of this exercise is to determine what the question is really asking. Let's show a simple example with say $n = 3$. We must first obtain all the non-empty subsets of $\{1, 2, 3\}$. They are:

$$\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$$

Thus the summation tells us to add together the following fractions:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 3} + \frac{1}{2\cdot 3} + \frac{1}{1\cdot 2\cdot 3} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \frac{1}{6} = 3$$

The formula works for $n = 3$. Let's go ahead an follow the induction steps:

Basis Step: For $n=1$, the only non-empty subset of $\{1\}$ is itself. Hence, the left hand side of formula (6) is just 1 which is of course equal to 1.

Induction Step: We assume formula (6) is true for n and now we try to show that it is true for $n+1$. Now the summation involves all the non-empty subsets of $\{1,2,\ldots,n+1\}$. The main difficulty here is that there are now many more subsets than in the summation for the formula with n. We can divide the non-empty subsets of $\{1,2,\ldots,n,n+1\}$ into three categories:

$$\begin{cases} 1)\ U & \text{where } U \text{ is any subset of } \{1,2,ldots,n\} \\ 2)\ U\cup\{n+1\} & \text{where } U \text{ is any subset of } \{1,2,ldots,n\} \\ 3)\ \{n+1\} & \text{(this singleton set is not covered in the above two cases)} \end{cases}$$

Grouping the non-empty subsets of $\{1,2,\ldots,n+1\}$ in this way we have:

$$\begin{aligned} \sum_{\{a_1,\ldots,a_k\}\subseteq\{1,2,\ldots,n+1\}} \frac{1}{a_1a_2\cdots a_k} &= \sum_{\text{①}}\frac{1}{a_1a_2\cdots a_k}+\sum_{\text{②}}\frac{1}{a_1a_2\cdots a_k}+\sum_{\text{③}}\frac{1}{a_1a_2\cdots a_k} \\ &= n+\frac{1}{n+1}\cdot n+\frac{1}{n+1} \quad \text{both ① and ② change by induction hypothesis} \\ &= \frac{n(n+1)+n+1}{n+1}=\frac{(n+1)^2}{n+1} \\ &= n+1 \end{aligned}$$

Thus, by induction, formula (6) holds for all $n\in\mathbb{N}^*$.

Exercise: 47 p.255

Let $A_1,A_2,\ldots,A_n$ be n subsets of a universal set U. We want to prove a multiple version of deMorgan's law:

$$\overline{\bigcup_{k=1}^{n} A_k}=\bigcap_{k=1}^{n}\overline{A_k} \tag{7}$$

We'll prove this by induction:

Basis Step: We can begin with $n=1$ in which case both the left hand side and right hand side of equation (7) are $\overline{A_1}$.

Induction Step: Let's assume that the formula is true for n and prove that it must be true for $n+1$.

$$\begin{aligned} \overline{\bigcup_{k=1}^{n+1} A_k} &= \overline{\Big(\bigcup_{k=1}^{n} A_k\Big)\cup A_{n+1}} \\ &= \overline{\bigcup_{k=1}^{n} A_k}\cap\overline{A_{n+1}} && \text{by deMorgan's law on two sets} \\ &= \bigcap_{k=1}^{n}\overline{A_k}\cap\overline{A_{n+1}} && \text{by the induction hypothesis} \\ &= \bigcap_{k=1}^{n+1}\overline{A_k} \end{aligned}$$

Thus, we've proved the induction step and formula (7) is proved by induction.

Exercise: 51 p. 255

We need to find a hole in the argument provided in this exercise. After all, the result of their argument would prove that all horses have the same color. We know this is not true. On the other hand, we should not doubt the technique of proofs by induction because the method has been proven mathematically. The problem lies somewhere in the argument provided in the text.

Well, here it is. The key issue is that $P(1)\to P(2)$ is not true. The statement $P(1)$="Every subset of horses of cardinality 1 is such that all the horses therein have the same color." is true. However, the book's proof that $P(n)\to P(n+1)$ is true works only when sets of size $n+1$ can be subdivided into **overlapping** subsets of size n. However, this only starts at $n=2$.

Hence, $P(1) \to P(2)$ is false and the induction does not hold.

Exercise: 62 p.256
We wish to prove that $21 \mid 4^{n+1} + 5^{2n-1}$ for all $n \in \mathbb{N}^*$.

Basis Step: We begin with $n = 1$ in which case $4^{n+1} + 5^{2n-1} = 16 + 5 = 21$ and of course 21 divides 21. So the relationship holds for $n = 1$.

Induction Step: Now, we suppose that $21 \mid 4^{n+1} + 5^{2n-1}$ for some n and we try to show that the same relatioship holds for $n + 1$. Here's a quick way to do this. For $n + 1$, we have

$$\begin{aligned}
4^{(n+1)+1} + 5^{2(n+1)-1} &= 4^{n+1+1} + 5^{(2n-1)+2} \\
&= 4 \times 4^{n+1} + 25 \times 5^{(2n-1)} \\
&= 4 \times 4^{n+1} + (4 + 21) \times 5^{(2n-1)} \\
&= 4 \times 4^{n+1} + 4 \times 5^{(2n-1)} + 21 \times 5^{(2n-1)} \\
&= 4\big(4^{n+1} + 5^{2n-1}\big) + 21 \times 5^{2n-1}
\end{aligned}$$

Now, by the induction hypothesis, $21 \mid 4^{n+1} + 5^{2n-1}$ so $21 \mid 4\big(4^{n+1} + 5^{2n-1}\big)$. Also by definition of divisibility 21 divides $21 \times 5^{2n-1}$. Thus

$$\begin{aligned}
&21 \mid 4\big(4^{n+1} + 5^{2n-1}\big) + 21 \times 5^{2n-1} \qquad \text{so} \\
&21 \mid 4^{(n+1)+1} + 5^{2(n+1)-1}
\end{aligned}$$

Thus, we've proved by induction that $21 \mid 4^{n+1} + 5^{2n-1}$ for all $n \in \mathbb{N}^*$.

Exercise: 65 p.256
Consider the equation in integers:

$$x^2 + y^2 + z^2 + w^2 = 2xyzw$$

Suppose that the 4-tuple (x, y, z, w) is a solution for this equation. If we consider the parity (odd or even) of numbers, it's clear that the right hand side of the equation is even and thus either none, two or all four of the integers x, y, z, w must be odd. (Squaring a number preserves odd or even and in order to get an even number, you must add up an even number of odd numbers.) We first want to show that none of the integers x, y, z, w must be odd.

Suppose that only exactly two of them are odd. (Without loss of generality, suppose that x and y are odd while z and w are even.) Consider the equation mod 4. Since z and w are even, $8 \mid 2xyzw$ and hence the right hand size is congruent to 0 modulo 4. Also, since z and w are even, then $z^2 \equiv 0$ and $w^2 \equiv 0$. We also know that since x and y are odd, $x^2 \equiv 1$ and $y^2 \equiv 1$. Thus, no matter what x, y, z, w are in this situation, reducing the equation modulo 4 gives: $2 \equiv 0$ which is a contradiction. Hence, we've eliminated the possibility that exactly two of the integers are odd.

Suppose now that all four of the integers are odd. This time, let's look at the equation modulo 8. We know (from work in class) that $x^2 \equiv 1$, $y^2 \equiv 1$, $z^2 \equiv 1$ and $w^2 \equiv 1$ all modulo 8, $x^2 + y^2 + z^2 + w^2 \equiv 4 \pmod 8$. Now writing $x = 2x' + 1$, $y = 2y' + 1$ and the same for z and w, we can expand out the right hand side of the equation to get:

$$\begin{aligned}
2xyzw = 32x'y'z'w' + 16x'y'z' + 16x'y'w' + 8x'y' + 16x'z'w' + 8x'z' + 8x'w' + 4x' \\
+ 16y'z'w' + 8y'z' + 8y'w' + 4y' + 8z'w' + 4z' + 4w' + 2
\end{aligned}$$

This looks very ugly but the point of this is that modulo 8, this line becomes

$$2xyzw \equiv 4(x' + y' + z' + w') + 2 \pmod 8$$

Now, we really don't now anything about the parity of x', y', z' or w'. However, there are two possibilities for the $x' + y' + z' + w'$: whether it is odd or even. If the sum is odd (i.e. equal to $2n + 1$ for some integer n), then $2xyzw \equiv 8n + 4 + 2 \equiv 6$. If the sum is even (i.e. equal to $2n$ for some integer n), then $2xyzw \equiv 8n + 2 \equiv 2$. Either way, we cannot have $x^2 + y^2 + z^2 + w^2 = 2xyzw$ with all x, y, z, w odd, since the left hand side is congruent to 4 mod 8 while the right hand side is congruent to 2 or 6 mod 8.

SO, we've finally proved that if (x, y, z, w) is a solution to the equation, all of them have to be even.

Now since all x, y, z, w are even, we can write $x = 2x'$, $y = 2y'$, $z = 2z'$, $w = 2w'$ for integers x', y', z', w'. We get in particular,

$$x^2+y^2+z^2+w^2 = 2xyzw \Longleftrightarrow 4(x')^2+4(y')^2+4(z')^2+4(w')^2 = 32x'y'zw \Longleftrightarrow (x')^2+(y')^2+(z')^2+(w')^2 = 8x'y'z'w'$$

At this stage we, get to a point where our reasoning can repeat. We know that if n is odd, then $n^2 \equiv 1 \pmod 8$ while it's easy to check that if n is even then $n^2 \equiv 0$ or 4 mod 8. We know from the above reasoning that an even number of the x', y', z', w' can be odd. If exactly two of them are odd, then $(x')^2+(y')^2+(z')^2+(w')^2 \equiv 2$ or 6, the right hand side is congruent to 0. A contradiction. If all four of them are odd, then $(x')^2+(y')^2+(z')^2+(w')^2 \equiv 4$ while again the right hand side is congruent to 0. Another contradiction. Thus, all x', y', z', w' are even, which means that all x, y, z, w are divisible by 4.

We can then divide x', y', z', w' by 2 again and reduce to another equation, but the same reasoning as in the above paragraph will hold so that we'll conclude that x, y, z, w are divisible by 8.

This process of reasoning can go on and on, so that x, y, z, w are divisble by any power of 2. This leads to a contradiction, since for every integer m, there exists a power of 2 greater than m and hence there exist no integers x, y, z, w with the required property. This line of reasoning is what's called infinite descent and proves that $x^2 + y^2 + z^2 + w^2 = 2xyzw$ has no solutions.