Math 1137, Summer 2003

Homework 11: 1,2,4,5,8,12,14 p.270

Exercise: 1 p.270

a) $f(n+1) = f(n) + 2$ and $f(0) = 1$. We have $f(1) = f(0) + 2 = 3$, $f(2) = f(1) + 2 = 5$, $f(3) = f(2) + 2 = 7$, $f(4) = f(3) + 2 = 9$.

b) $f(n+1) = 3f(n)$ and $f(0) = 1$. We have $f(1) = 3f(0) = 3$, $f(2) = 3f(1) = 9$, $f(3) = 3f(2) = 27$, $f(4) = 3f(3) = 81$.

c) $f(n+1) = 2^{f(n)}$ and $f(0) = 1$. We have $f(1) = 2^f(0) = 2$, $f(2) = 2^f(1) = 2^2 = 4$, $f(3) = 2^f(2) = 2^4 = 16$, $f(4) = 2^f(3) = 2^16 = 65536$.

d) $f(n+1) = f(n)^2 + f(n) + 1$ and $f(0) = 1$. We have $f(1) = f(0)^2 + f(0) + 1 = 3$, $f(2) = f(1)^2 + f(1) + 1 = 3^2+3+1 = 13$, $f(3) = f(2)^2+f(2)+1 = 13^2+13+1 = 183$, and $f(4) = f(3)^2+f(3)+1 = 183^2+183+1 = 33673$.

Exercise: 2 p.270

a) $f(n+1) = -2f(n)$ and $f(0) = 3$. We have $f(1) = -2f(0) = -6$, $f(2) = -2f(1) = 12$, $f(3) = -2f(2) = -24$, $f(4) = -2f(3) = 48$, $f(5) = -2f(4) = -96$.

b) $f(n+1) = 3f(n) + 7$ and $f(0) = 3$. We have: $f(1) = 3f(0) + 7 = 16$, $f(2) = 3f(1) + 7 = 55$, $f(3) = 3f(2) + 7 = 172$, $f(4) = 3f(3) + 7 = 523$, $f(5) = 3f(4) + 7 = 1576$.

c) $f(n+1) = f(n)^2 - 2f(n) - 2$ and $f(0) = 3$. We have: $f(1) = f(0)^2 - 2f(0) - 2 = 1$, $f(2) = f(1)^2 - 2f(1) - 2 = -3$, $f(3) = f(2)^2 - 2f(2) - 2 = 13$, $f(4) = f(3)^2 - 2f(3) - 2 = 141$, $f(5) = f(4)^2 - 2f(4) - 2 = 19597$.

d) $f(n+1) = 3^{f(n)/3}$ and $f(0) = 3$. We have: $f(1) = 3^{f(0)/3} = 3$, $f(2) = 3^{f(1)/3} = 3$, $f(3) = 3^{f(2)/3} = 3$, $f(4) = 3^{f(3)/3} = 3$, $f(5) = 3^{f(4)/3} = 3$.

Exercise: 4 p.271

a) $f(n+1) = f(n) - f(n-1)$ with $f(0) = f(1) = 1$. We have $f(2) = 1 - 1 = 0$, $f(3) = 0 - 1 = -1$, $f(4) = -1 - 0 = -1$ and $f(5) = -1 + 1 = 0$.

b) $f(n+1) = f(n)f(n-1)$ with $f(0) = f(1) = 1$. We have $f(2) = 1$, $f(3) = 1$, $f(4) = 1$ and $f(5) = 1$.

c) $f(n+1) = f(n)^2 + f(n-1)^3$ with $f(0) = f(1) = 1$. We have $f(2) = 2$, $f(3) = 2^2+1^3 = 5$, $f(4) = 5^2+2^3 = 33$, $f(5) = 33^2 + 5^3 = 1214$.

d) $f(n+1) = f(n)/f(n-1)$ with $f(0) = f(1) = 1$. We have $f(2) = 1$, $f(3) = 1$, $f(4) = 1$ and $f(5) = 1$.

Exercise: 5 p.271

a) $f(0) = 0$ and $f(n) = 2f(n-2)$ for $n \geq 1$. This doesn't give a valid definition for a function because for example, it doesn't output a value for $f(1)$.

b) $f(0) = 1$ and $f(n) = f(n-1) - 1$ for $n \geq 1$. This is a valid definition. All values can by obtained from previous ones. A formula is $f(n) = 1 - n$.

c) $f(0) = 2$, $f(1) = 3$ and $f(n) = f(n-1) - 1$ for $n \geq 2$. This is also a valid definition since it really only starts using the recursive formulation starting at $n = 2$. A formula for the function is

$$f(n) = \begin{cases} 2 & \text{if } n = 0 \\ 3 - n & \text{if } n > 0 \end{cases}$$

d) $f(0) = 1$, $f(1) = 2$ and $f(n) = 2f(n-2)$ for $n \geq 2$. This is also a valid definition since the recursive formula goes back to $n - 2$ and we're given two initial conditions. After a little work, one can check that a formula for this function is

$$f(n) = 2^{\lceil \frac{n}{2} \rceil}$$

e) $f(0) = 1$, $f(n) = 3f(n-1)$ if n is odd and $n \geq 1$ or $f(n) = 9f(n-2)$ if n is even and $n \geq 2$. This also gives a valid definition since when the definition for n being odd gives a value for $f(1)$. Then after that both definitions, whether n is odd or even, are well-defined. Despite the complicated recursive defintion, a formula for f is easy: $f(n) = 3^n$.

Exercise: 8 p.271

For each of these problems, since the exercise says to start at $n = 1$, we get the initial term simply by plugging in $n = 1$ into a_n. However, to get the recursive definition, we have to think a little more to see what works.

a) Let $a_n = 4n - 2$. We have $a_1 = 2$. Each time we increase the index, we add 4 to the previous term in the sequence. Hence $a_{n+1} = a_n + 4$. Another way to see this is that

$$a_{n+1} = 4(n+1) - 2 = 4n + 4 - 2 = (4n - 2) + 4 = a_n + 4$$

b) Let $a_n = 1 + (-1)^n$. We have $a_1 = 0$. Note that the first few terms of the sequence a_n are $0, 2, 0, 2, 0, 2, \ldots$ Using this, you might notice right away that $a_{n+1} = 2 - a_n$. Another way to see this is by noticing that:

$$\frac{a_{n+1} - 1}{a_n - 1} = \frac{(-1)^{n+1}}{(-1)^n} = -1 \Longrightarrow a_{n+1} - 1 = 1 - a_n \Longrightarrow a_{n+1} = 2 - a_n$$

c) Let $a_n = n(n+1)$. We have $a_1 = 2$. The terms of the sequence go as follows: $2, 6, 12, 20, \ldots$ So you might notice that for $n > 1$, we have $a_n = a_{n-1} + 2n$. You could also notice that $a_{n+1} = (n+1)(n+2) = \frac{n+2}{n} n(n+1) = \left(1 + \frac{2}{n}\right) a_n$.
You might also notice that the difference between terms is $a_{n+1} - a_n$ and creates a sequence $4, 6, 8, 10, 12, \ldots$ Then, taking differences again, we get $(a_{n+2} - a_{n+1}) - (a_{n+1} - a_n) = a_{n+2} - 2a_{n+1} + a_n$ and this turns out to always be 2. Thus we get the relation $a_{n+2} = 2a_{n+1} - a_n + 2$ for $n \geq 1$. Using this latter form, we need to provide the first two initial conditions, $a_1 = 2$ and $a_2 = 6$.

d) Let $a_n = n^2$. We can give Two kinds of recursive definitions. First and easiest, is given by $a_1 = 1$ and $a_{n+1} = a_n + 2n + 1$. The second is similar to the last option in the above formulation: $a_{n+2} = 2a_{n+1} - a_n + 2$ with $a_1 = 1$ and $a_2 = 4$.

Exercise: 12 p.271

Let f_n be the n'th Fibonacci number as defined in class. ($f_0 = 0$, $f_1 = 1$, and for $n \geq 0$, $f_{n+2} = f_{n+1} + f_n$.) We want to prove that

$$f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1} \tag{1}$$

This is a typical problem for which we should use a proof by induction. (It asks us to prove that some formula is valid for all positive integers n and the terms involved, the Fibonacci numbers, are defined recursively.)

Basis Step: The first positive integer is $n = 1$. The left hand side of the formula (1) is just $f_1 = 1$. The right hand side is $f_1 f_2 = 1 \times 1 = 1$. Check.

Induction Step: Let's suppose that formula (1) holds for n. We now try to prove that it also must hold for $n + 1$.

$$\begin{aligned} f_1^2 + f_2^2 + \cdots + f_n^2 + f_{n+1}^2 &= f_n f_{n+1} + f_{n+1}^2 \qquad \text{by induction hypothesis} \\ &= f_{n+1}(f_n + f_{n+1}) \\ &= f_{n+1} f_{n+2} \qquad \text{by definition of Fibonacci numbers} \end{aligned}$$

Thus the formula holds for $n + 1$. By induction, formula (1) is true for all positive integers.

Exercise: 14 p.271

The problem asks us to consider the Fibonacci sequence f_n and prove that for all $n \in \mathbb{N}^*$ we have $f_{n-1} f_{n+1} - f_n^2 = (-1)^n$. (Even though this is a starred problem, it really isn't that hard.) Let's prove it by induction.

Basis Step: We check the formula for $n = 1$. $f_0 f_2 - f_1^2 = -1 = (-1)^1$ so it works.

Induction Step: We assume that $f_{n-1} f_{n+1} - f_n^2 = (-1)^n$ for some n. We remind you that the definition of Fibonacci gives us in particular $f_{n+1} = f_n + f_{n-1}$ which is equivalent to $f_{n+1} - f_n = f_{n-1}$. The next stage after the induction assumption is:

$$\begin{aligned} f_n f_{n+2} - f_{n+1}^2 &= f_n(f_{n+1} + f_n) - f_{n+1}^2 \\ &= f_n^2 + f_n f_{n+1} - f_{n+1}^2 \\ &= f_n^2 - f_{n+1}(f_{n+1} - f_n) \\ &= f_n^2 - f_{n+1} f_{n-1} \\ &= -(f_{n-1} f_{n+2} - f_n^2) \\ &= -(-1)^n = (-1)^{n+1} \end{aligned}$$

This last calculation proves the formula for all positive integers by induction.