Math 1137, Summer 2003

Homework 12: 1,7,8,10,18,19,25,28,39,41,42 p.310

Exercise: 1 p.310
18 math majors and 325 computer science majors at college.

a) If we pick two representatives, one from each major, this is the scenario of the product rule so the total number of ways to pick this pair is $18 \times 325 = 5850$.

b) If we pick one representative from either major (assuming that no one is double majoring), then this is the scenario of the addition rule. There are $18 + 325 = 343$ ways to pick such a rep.

Exercise: 7 p.310
The number of choices for three different initials. For each initial, one could foreseeably have any of 26 letters. Thus, the answer is $26 \times 26 \times 26 = 26^3 = 17,576$.

Exercise: 8 p.310
If we assume that in the group of three initials, none of the letters are repeated, we have 26 choices for the first letter, then 25 choices for the second and 24 choices for the third. Hence the answer is $26 \times 25 \times 24 = 15,600$.

Exercise: 10 p.310
Bit strings of length 8. 2 choices for each bit, no other conditions. Ans: $2^8 = 256$.

Exercise: 18 p.310
We consider positive integers less than 1000 and count how many satisfy certain properties.

a) divisible by 7: the answer is $\lfloor \frac{1000}{7} \rfloor = 142$.

b) divisible by 7 but not by 11. We must subtract from those divisible by 7 integers divisible by 7 and 11, i.e. divisible by 77. Answer: $\lfloor \frac{1000}{7} \rfloor - \lfloor \frac{1000}{77} \rfloor = 130$.

c) divisible by 7 and 11. Answer: $\lfloor \frac{1000}{77} \rfloor = 12$

d) divisible by 7 or 11. This follows the inclusion exclusion pronciple ($|A \cup B| = |A| + |B| - |A \cap B|$ where A is the set of integers less than 1000 divisible by 7 and B the integers less than 1000 divisible by 11.). Answer: $\lfloor \frac{1000}{7} \rfloor + \lfloor \frac{1000}{11} \rfloor - \lfloor \frac{1000}{77} \rfloor = 142 + 90 - 12 = 220$.

e) divisible by exactly one of 7 or 11. In this case, we need to calculate $|A \cup B| - |A \cap B| = 220 - \lfloor \frac{1000}{77} \rfloor = 208$.

f) divisible by neither 7 nor 11. In this case, we want to calculate $|\overline{A \cup B}| = |U| - |A \cup B|$ where $U = \{n \in \mathbb{N}^* | n \leq 1000\}$. Answer: $1000 - 220 = 780$.

g) have distinct digits. We have to be more careful with this one for we don't consider 01 as a proper representation of an integer. Hence 7 has distinct digits, though if we wrote 007, it wouldn't look like it had distinct digits. Let's break it into three parts depending on whether the number has one, two or three digits. (Notice that 1000 doesn't have distinct digits so we don't need to count it.) 1) All one digit numbers have distinct digits but they're are only 9 to chose from. 2) For two digit integers, we have nine choices for the first digit (since it can't be a 0) and again 9 choice for the second since we've eliminated one but 0 because available again. So we have a total of 81 with 2 digits. 3) With three digits, we have 9 choices for the first, 9 choices again for the second and 8 for the third. Thus $9 \times 9 \times 8 = 641$. Adding these up we get a total of 731 positive integers ledd than 1000 that have distinct digits.

h) have distinct digits and are even. (This just means the last digit must be even.) Break into three cases again. 1) with one digit we only have 4 possibilities. 2) with two digits we break into two subparts. If the first digit is even we have 4 choices for the first digit and 4 for the second, while is the first digit is odd we have 5 choices for the first and five choices for the secodn digit. In total, we have $4 \times 4 + 5 \times 5 = 41$. 3) with three digits, we repeat the same reasoning as in (2) but pick first the first digit, then the third and then we have no constraints on the second one. We end up with a total of $8 \times 41 = 328$. Adding all together, we get the answer of: 373.

Exercise: 19 p.311
We consider the positive integers between 100 and 999. How many have the following property:

a) Are divisible by 7. The safest way to get the number of multiples of 7 between 100 and 999 is by looking for the smallest bigger than 100, the largest less than 999 and counting how many lie between there. That's a little long. One way to be sure is to calculate: $\lfloor \frac{999}{7} \rfloor - \lfloor \frac{99}{7} \rfloor = 128$.

b) That are odd: $\lfloor\frac{999}{2}\rfloor - \lfloor\frac{99}{2}\rfloor = 450$.
c) Have the same three decimals: only 9 digits to choose from. Only 9.
d) Are NOT divisible by 4. Well, let's first count those which are divisible by 4. $\lfloor\frac{999}{4}\rfloor - \lfloor\frac{99}{4}\rfloor = 225$. Not we have to take $900 - 225 = 675$.
e) Are divisible by 3 or 4. We will need to use the inclusion exclusion principle here so that we don't double count numbers which are divisible by both 3 and 4.

$$\left(\lfloor\frac{999}{3}\rfloor - \lfloor\frac{99}{3}\rfloor\right) + \left(\lfloor\frac{999}{4}\rfloor - \lfloor\frac{99}{4}\rfloor\right) - \left(\lfloor\frac{999}{12}\rfloor - \lfloor\frac{99}{12}\rfloor\right) = 450$$

f) Are not divisible by either 3 or 4: Just take $900 - (e) = 450$.
g) Are divisible by three but not by 4: Think set theory here. If you call A the set of numbers between 100 and 999 that are divisible by 3 and B the set of numbers divisible by 4 then the numbers divisible by 3 but not by 4 is precisely the set $A - B$. T count this precisely, remove from A its intersection with B. $|A - B| = |A| - |A \cap B|$ The number is $\left(\lfloor\frac{999}{3}\rfloor - \lfloor\frac{99}{3}\rfloor\right) - \left(\lfloor\frac{999}{12}\rfloor - \lfloor\frac{99}{12}\rfloor\right) = 225$.
h) Are divisible by 3 and 4: $\lfloor\frac{999}{12}\rfloor - \lfloor\frac{99}{12}\rfloor = 75$.

Exercise: 25 p.311
License plates made with three digits followed by three letters: $10^3 \times 26^3 = 17,576,000$.

License plates made with three letters followed by three digits: $17,576,000$.

Either one of the two possibilites: Add them up since we're not double counting: 35,152,000.

Exercise: 28 p.311
Strings of eight English letters...
a) letter can be repeated: $26^8 = 208\,827\,064\,576$.
b) no letters can be repeated: $26 \times 25 \times 24 \times 23 \times 22 \times 21 \times 20 \times 19 = 62\,990\,928\,000$.
c) start with an X and letters can be repeated: $26^7 = 8\,031\,810\,176$.
d) start with an X and letters can't be repeated: $25 \times 24 \times 23 \times 22 \times 21 \times 20 \times 19 = 2\,422\,728\,000$.
e) start and end with an X and letters can be repeated: $26^6 = 308\,915\,776$.
f) that start with the letters BO and leters can be repeated: $26^6 = 308\,915\,776$.
g) that start and end with the letters BO and leters can be repeated: $26^4 = 496\,976$.
h) that start or end with the letters BO and leters can be repeated: (inclusion-exclusion principle) $26^6 + 26^6 - 26^4 = 617\,374\,576$.

Exercise: 39 p.312

a) The bride must sit next to the groom. Since there are only six seats, there are only 5 (pairs of seats) where the bride and groom can sit. Also, whether the bride sits to the left or to the right of the groom affords us 2 choices. Then for all the remaining people, there are 4! ways to seat them. Hence: $5 \times 2 \times 4! = 240$.
b) If the bride must not sit next to the groom, this is the opposite scenario as the one previously described so we take the total number of seatings and subtract part a). That is $6! - 240 = 480$.
c) If the bride is positioned somewhere next to the groom, then we can break the problem down as follows. If the groom is at the far right, the bride has 5 places to sit. If he is the next seat in, the bride has 4 places to sit... Hence, the bride and groom have 5+4+3+2+1=15 ways to sit. Then the remaining people just fill up the 4 slots. Thus the answer to the problem is $15 \times 4! = 360$.

Exercise: 41 p. 312
Bit strings of length 10 beginning with three 0's: $2^7 = 128$

Bit strings of length 10 ending with two 0's: $2^8 = 256$

We can't just add these to get the answer to the problem since we would then be counting some stuff twice: the strings beginning with three 0's and ending with two 0's: $2^5 = 32$. Hence our answer is $128 + 256 - 32 = 352$

Exercise: 42 p.312
Bit strings of length 10 containing five consecutive 0's or five consecutive 1's. First of all notice that if we count the number of bit strings containing five consecutive 0's then this number will also be the same as if it contained five consecutive 1's.

Containing five consecutive 0': The difficulty with this exercise is that we must beware of double counting things. I would suggest finding how many have exactly five consecutive 0's, then how many have six consecutive 0's, exactly 7, etc.

- exactly five 0's: If the 00000 start the string, then we will have $000001, -, -, -, -$, so $2^4 = 16$ choices; if the 00000 are in the middle (not on the end), we have something that looks like $-, 1000001, -, -$, so $2^3 = 8$ choices, and if the 00000 is on the end, then the scenario will look like $-, -, -, -, 100000$ and so 16 choices again. In total $(16 + 8 + 8 + 8 + 8 + 16) = 64$ choices.
- exactly six 0's: Use the same reasoning to get that in this case there are $(8 + 4 + 4 + 4 + 8) = 28$ choices.
- exactly seven 0's: $(4 + 2 + 2 + 4) = 12$ choices.
- exactly eight 0's: $(2 + 1 + 2) = 5$ choices.
- exactly nine 0's: $(1 + 1) = 2$ choices.
- exactly ten 0's: 1 choice.

In total, for five consecutive 0's, there are $64 + 28 + 12 + 5 + 2 + 1 = 112$.

Now for five consecutive 1's we also have 112 such bit strings. However, in order to answer our problem, we can't simply add these two together since we would be double counting two situations (0000011111 and 1111100000). Thus the answer to our problem is $112 + 112 - 2 = 222$.