

Math 1137, Summer 2003

Homework 13: 2,3,5,7,10,11,19,22,23,31,38 p.324; 4,8,12,19,21 p.333

Exercise: 2 p.324

Permutations on a set of 7 elements. In the way they've stated the problem, it means you consider all the elements (not just r of them), so the answer is $P(7,7) = 7! = 5040$.

Exercise: 3 p.324

Permutations on a set of 7 elements where the list must end in a . This means we only have six slots to fill and 6 letters to choose from. $P(6,6) = 6! = 720$.

Exercise: 5 p.324

a) 120; b) 720; c) 8; d) 6720; e) $8! = 40320$; f) $P(10,9) = 10! = 3,628,800$.

Exercise: 7 p.324

$P(9,5) = 15,120$.

Exercise: 10 p.325

In this problem order matters. We print all six names so the answer is $P(6,6) = 6! = 720$.

Exercise: 11 p.325

Bit strings of length 10.

- with exactly four 1s: $C(10,4) = 210$
- with at most four 1s: $C(10,0) + C(10,1) + C(10,2) + C(10,3) + C(10,4) = 386$.
- with at least three 1s: $C(10,3) + C(10,4) + C(10,5) + C(10,6) + C(10,7) + C(10,8) + C(10,9) + C(10,10) = 2^{10} - (C(10,0) + C(10,1) + C(10,2)) = 1024 - (1 + 10 + 45) = 968$.
- with an equal number of 0s and 1s: $C(10,5) = 252$

Exercise: 19 p.325

A coin is flipped 10 times. We count how many possible outcomes are there

- in total: $2^{10} = 1024$.
- that contain exactly two heads. We must simply select on which flips (2 out of 10) the two heads come up. $C(10,2) = 45$.
- that contain at most three tails. At least three tails means no tails, or exactly 1 tail or exactly 2 tails or exactly 3 tails. Answer: $C(10,0) + C(10,1) + C(10,2) + C(10,3) = 176$.
- contain the same number of heads and tails: $C(10,5) = 252$.

Exercise: 22 p.325

We consider permutations of the string ABCDEFGH. How many strings:

- contain the string ED. There are 9 ways to place the string ED in ten possible slots and then there are $8!$ ways to choose the rest of the letters: $9 \times 8!$
- contain the string CDE. With the same reasoning, we get: $8 \times 7!$.
- contain the strings BA and FGH. We first need to figure out in how many ways we can place the substrings BA and FGH. Since once we know where B goes we know where A goes and since once we know where F goes we know where GH go, then choosing positions for BA and FGH is like ordering two out of seven slots. Thus $P(7,2)$ ways to place BA and FGH. Now to place the rest of the letters it's merely $5!$. In total, $P(7,2) \times 5! = 5040$.
- contain the strings, AB, DE and GH. Reasoning in the same way as above, there are $P(7,3)$ ways to place AB, DE and GH and then $4!$ ways to place the remaining 4 letters. $P(7,3) \times 4! = 5040$.
- contain the strings CAB and BED. Clearly it must contain the string CABED. There are six ways to place this and $5!$ ways to place the rest. Hence: $6 \times 5! = 720$.
- contain the strings BCA and ABF. Since no letters are repeated in a permutation, this is impossible. Answer: 0.

Exercise: 23 p.325

We place eight men and five women in a line so that no women stand next to each other. The hint is to first position the men (there are $8!$ ways to do that) and then think about where the women can go. Any woman can go between any two men or on the ends. Hence, there are 9 places the women can go and order matters so picking five of those nine places gives $P(9,5)$ options. Answer: $8! \times P(9,5)$.

Exercise: 31 p.326

English has 21 consonants and 5 vowels. How many strings of six letters contain

- exactly one vowel: we have to choose where the vowel goes: $C(6,1)$ ways. There are then 5 ways to choose the vowel and 21^5 ways to choose the consonants. Answer: $C(6,1) \times 5 \times 21^5 = 122,523,030$
- exactly two vowels: we have to choose where the vowels go: $C(6,2)$ ways. There are then 5^2 ways to choose the vowels and 21^4 ways to choose the consonants. Answer: $C(6,2) \times 5^2 \times 21^4 = 72,930,375$
- at least one vowel: We can get this by subtracting no vowels away from all possibilities: $26^6 - 21^6 = 223,149,655$.
- at least two vowels: We can get this by subtracting no vowels case and the exactly one vowel case away from all possibilities: $26^6 - 21^6 - C(6,1)5 \times 21^5 = 100,626,625$.

Exercise: 38 p.326

This exercise is actually quite easy. Just break the problem down and use the multiplication rule. Answer: $C(45,3) \times C(57,4) \times C(69,5)$.

Exercise: 4 p.333

This exercise use the binomial theorem. The coefficient of x^5y^8 in $(x+y)^{13}$ is by the binomial theorem $C(13,8) = 1287$.

Exercise: 8 p.333

The term containing x^8y^9 in $(3x+2y)^{17}$ is $C(17,9)(3x)^8(2y)^9$ so the coefficient to x^8y^9 is: $C(17,9)3^82^9 = 81662929920$.

Exercise: 12 p.333

The eleventh row for Pascal's triangle is:

$$1 \quad 11 \quad 55 \quad 165 \quad 330 \quad 462 \quad 462 \quad 330 \quad 165 \quad 55 \quad 11 \quad 1$$

Exercise: 19 p.333

The formula for the binomial coefficient is $\binom{n}{r} = \frac{n!}{r!(n-r)!}$. We take the usual right hand side first:

$$\begin{aligned} \binom{n-1}{r} + \binom{n-1}{r-1} &= \frac{(n-1)!}{r!(n-1-r)!} + \frac{(n-1)!}{(r-1)!(n-1-r+1)!} \\ &= \frac{(n-1)!}{r!(n-1-r)!} + \frac{(n-1)!}{(r-1)!(n-r)!} \\ &= \frac{(n-1)!(n-r)}{r!(n-r)!} + \frac{(n-1)!r}{r!(n-r)!} \\ &= \frac{n(n-1)!}{r!(n-r)!} = \frac{n!}{r!(n-r)!} \\ &= \binom{n}{r} \end{aligned}$$

Exercise: 21 p.333

We want to prove in two different ways that $k\binom{n}{k} = n\binom{n-1}{k-1}$.

- With a combinatorial proof: Notice that $k\binom{n}{k}$ counts the number of ways to select a subset of k elements from a set of size n and then to pick some element from this chosen subset. On the other hand, this election process amounts to picking a number first (n ways) and then, to create a subset of size k that will contain the element we first picked, choose $k-1$ elements from the remaining $n-1$. Thus, the second process (which ultimately is equivalent to the first), can be done in $n\binom{n-1}{k-1}$ ways. This proves they are equal.

b) algebraically:

$$k \binom{n}{k} = \frac{k \cdot n!}{k!(n-k)!} = \frac{n!}{(k-1)!(n-k)!} = \frac{n \cdot (n-1)!}{(k-1)!(n-1-(k-1))!} = n \binom{n-1}{k-1}$$