Math 1137, Summer 2003

Homework 14: 1-3,6,19-21,25,27,42,43 p.409

Exercise: 1 p.409

a) 2,12,72,432,2592...
b) 2,4,16,256,65536...
c) 1,2,5,11,26...
d) 1,1,$2 \cdot 1 + 2^2 \cdot 1 = 6$,$3 \cdot 6 + 3^2 \cdot 1 = 27$, $4 \cdot 27 + 4^2 \cdot 6 = 204$...
e) 1,2,0, $0 + 1 = 1$, $1 + 2 = 3$, $3 + 0 = 3$, $3 + 1 = 4$, $4 + 3 = 7$...

Exercise: 2 p.409

a) -1,2,-4,8,-16,32,...
b) 2,-1,-3,-2,1,3,...
c) 1,3,27,2187,14 348 907,617 673 396 283 947,...
d) -1,0,1,3,13,74,...
e) 1,1,2,2,1,1,...

Exercise: 3 p.409
Let $a_n = 2^n + 5 \cdot 3^n$ for $n \geq 0$.
a) $a_0 = 6$, $a_1 = 17$, $a_2 = 49$, $a_3 = 143$, $a_4 = 421$.
b) $5a_1 - 6a_0 = 85 - 36 = 49 = a_2$; $5a_2 - 6a_1 = 245 - 102 = 143 = a_3$; $5a_3 - 6a_2 = 715 - 294 = 421 = a_4$.
c) To prove that $a_n = 5a_{n-1} - 6a_{n-2}$ for all $n \geq 2$, we don't really need to use induction though it would be a perfectly valid apporach. Instead, just plug the definition of a_n into the RHS and check that it indeed is equal to a_n.

$$\begin{aligned} 5a_{n-1} - 6a_{n-2} &= 5(2^{n-1} + 5 \cdot 3^{n-1}) - 6(2^{n-2} + 5 \cdot 3^{n-2}) \\ &= 5 \cdot 2^{n-1} - 6 \cdot 2^{n-2} + 25 \cdot 3^{n-1} - 30 \cdot 3^{n-2} \\ &= 10 \cdot 2^{n-2} - 6 \cdot 2^{n-2} + 75 \cdot 3^{n-2} - 30 \cdot 3^{n-2} \\ &= 4 \cdot 2^{n-2} + 45 \cdot 3^{n-2} \\ &= 2^n + 5 \cdot 3^n \\ &= a_n \end{aligned}$$

Exercise: 6 p.409

a) $a_n = 3$. A recurrence relation: $a_{n+1} = a_n$ for $n \geq 0$.
b) $a_n = 2n$. A recurrence relation: $a_{n+1} = a_n + 2$ for $n \geq 0$.
c) $a_n = 2n + 3$. A recurrence relation: $a_{n+1} = a_n + 2$ for $n \geq 0$. (Same relation as b) but different initial condition.
d) $a_n = 5^n$. A recurrence relation: $a_{n+1} = 5a_n$ for $n \geq 0$.
e) $a_n = n^2$. A recurrence relation: $a_{n+1} = a_n + 2n + 1$ for $n \geq 0$ or $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$ for $n \geq 3$.
f) $a_n = n^2$. A recurrence relation: $a_{n+1} = a_n + 2n + 2$ for $n \geq 0$ or $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$ for $n \geq 3$. (Note the second relation proposed is the same as that in g).)
g) $a_n = n + (-1)^n$. A recurrence relation: $a_n = a_{n-2} + 2$ for $n \geq 2$.
h) $a_n = n!$. A recurrence relation: $a_n = na_{n-1}$ for $n \geq 1$.

Exercise: 19 p.410
A stamp vending machine accepts only dolar coins, \$1 bills and \$5 bills.
a) Let a_n represent the number of ways to deposit n dollars into the veding machine, where the order in which we deposit them matters. Think about it this way. What was the last money type added:

- if it was a dollar coin, there are a_{n-1} ways to order the $n-1$ dollars put in before.
- if it was a dollar bill, there are also a_{n-1} ways to order the $n-1$ dollars put in before.
- if it was a five dollar bill, there are a_{n-5} ways to order the $n-5$ dollars put in before.

Thus the recurrence relation that a_n satisfies is $a_n = 2a_{n-1} + a_{n-5}$. This holds for $n \geq 6$ since a_n only makes sense for $n \geq 1$.

b) We need to provide initial conditions for how to put in 1 through 5 dollars. $a_1 = 2$, $a_2 = 4$, $a_3 = 8$, $a_4 = 16$, $a_5 = 33$.

c) Using these initial conditions and the recurrence relation we get the following outputs to the function: 2,4,8,16,33,68,140,288,592,1217... Thus $a_{10} = 1217$.

Exercise: 20 p.410
Using the same explanation as in the previous exercise, using the explanation in the text, the recurrence relation that describes how many ways to pay n pesos if order matters is:

$$a_n = a_{n-1} + a_{n-2} + 2a_{n-5} + 2a_{n-10} + a_{n-20} + a_{n-50} + a_{n-100}$$

Exercise: 21 p.410
To solve this problem (think through this), it suffices to define $a_0 = 1$ and $a_n = 0$ if $n < 0$. We then get the following outputs for the sequence a_n:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
a_n	1	1	2	3	5	10	17	31	54	95	171	302	539	955	1694	3011	5343	9494

Exercise: 25 p.410
We consider bit strings of length n that do not contain 3 consecutive 0s. Call this function a_n.

a) We work to get a recurrence relation for a_n. Think of a bit string of length n and what it can look like on the end. We break the situations into three mutually exclusive possibilities: the string ends with **1, *10, or 100.

If the last digit is a 1 then the previous $n-1$ bits can be any bitstring that don't have 3 consecutive 0s. Thus, strings of length n that end with a 1 contribute a_{n-1} to the total of a_n.

If the end is 10, we don't need to worry about what comes before the 1, so we could have any bitstring of length $n-2$. This contributes a_{n-2} possibilities..

Similarly, tot he above two cases, if the string terminates with 100, then the previous $n-3$ bits are just a bitstring of length $n-3$ without 3 consecutive 0s. Again contributes a_{n-3}.

Thus a recurrence relation is $a_n = a_{n-1} + a_{n-2} + a_{n-3}$.

b) We only need the first three initial conditions. $a_1 = 2$, $a_2 = 4$ and $a_3 = 7$.

c) To find out how many bitstrings of length 7 have no consecutive 0s means we want to find a_7. The first seven terms in the sequence are:

$$2, 4, 7, 13, 24, 44, 81$$

Thus $a_7 = 81$.

Exercise: 27 p.410
We consider the ways to climb n stairs where the person can climb one or two stairs at a time. Call this function a_n.

a) We work to get a recurrence relation for a_n. This is easier than #25. Either the last step one took was a single stair or a double stair. If it was a single stair, there are a_{n-1} wasy to climb the ones before. If it was a double stair, there are a_{n-2} ways to climb the previous $n-2$. Thus a recurrence relation is $a_n = a_{n-1} + a_{n-2}$.

b) We only need the first two initial conditions. $a_1 = 1$ and $a_2 = 2$.

c) The first eight terms in the sequence are:

$$1, 2, 3, 5, 8, 13, 21, 44.$$

Thus $a_8 = 44$.

Exercise: 42 p.411

Let b_n be the number of ways to cover a $2 \times n$ chess board with 1×2 pieces.

a) If the top right corner has a horizontal tile, then another horizontal tile is below it and the previous $2 \times (n-2)$ board can be covered in b_{n-2} ways. If the top right corner is covered by a vertical tile, then the previous $2 \times (n-1)$ board can be covered in b_{n-1} ways. Thus the recurrence relation is $b_n = b_{n-1} + b_{n-2}$.

b) the initial conditions are $b_1 = 1$ and $b_2 = 2$.

c) Going through the terms of the sequence, we get $b_{17} = 2584$.

Exercise: 43 p.411

Let b_n be the number of ways cover a walkway of length n with red, green or gray tiles where no two red tiles are adjacent and two tiles of the same color are considered indistinguishable. This last condition means that a green tile followed by a green tile is just one situation.

a) We must look at how we can create these walkways by breaking into distinct possibilities what can happen at the end. If the last tile is either green or gray, the previous $n-1$ tiles can be covered in b_{n-1} ways. Together, these provide $2b_{n-1}$ ways.

   If the last tile is red then the condition that no two red tiles are adjacent means that the $(n-1)$-st tile must be either gray or green. In both of these cases, we can cover the previous $n-2$ tiles in b_{n-2}.

   In total, the recurrence relation is $b_n = 2(b_{n-1} + b_{n-2})$.

b) The initial conditions are $a_1 = 3$ and $a_2 = 8$.

c) The first seven terms of the sequence are: 3,8,22,60,164,448,1224.