Math 1137, Summer 2003

Homework 5: 1,2,4,14,16,24 p.85 and 1,3,11,12,15,19,21,24,25,41,42 p.94

Exercise: 1 p.85

a) $\{x \,|\, x \text{ is a real number such that } x^2 = 1\} = \{-1, 1\}$
b) $\{x \,|\, x \text{ is a positive integer less than } 12\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$
c) $\{x \,|\, x \text{ is a square of an integer and } x < 100\} = \{0, 1, 4, 9, 16, 25, 36, 49, 64, 81\}$.
d) $\{x \,|\, x \text{ is an integer such that } x^2 = 21\} = \emptyset$

Exercise: 2 p.85

a) $\{0, 3, 6, 9, 12\} = \{x \in \mathbb{N} \,|\, 3 \text{ divides } x \text{ and } x \leq 12\}$
b) $\{-3, -2, -1, 0, 1, 2, 3\} = \{x \in \mathbb{Z} \,|\, |x| \leq 3\}$. Don't get confused between the bar for the set builder notation and the bar for the absolute value.
c) $\{m, n, o, p\} = \{x \,|\, x \text{ is a letter in the English alphabet between } m \text{ and } p\}$

Exercise: 4 p.85
$A = \{2, 4, 6\}$, $B = \{2, 6\}$, $C = \{4, 6\}$ and $D = \{4, 6, 8\}$. Subset list:

$$B \subset A, \qquad C \subset A, \qquad C \subset D$$

and no others.

Exercise: 14 p.85

a) The cardinality of $\emptyset$ is 0.
b) The cardinality of $\{\emptyset\}$ is 1.
c) The cardinality of $\{\emptyset, \{\emptyset\}\}$ is 2.
d) The cardinality of $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ is 3.

Exercise: 16 p.45
Consider the power set $P(A)$ of any finite set A. If someone gives us $P(A)$, we can recover the original set A by taking the largest element (set) in $P(A)$. Furthermore, it's not hard to see that the maximal element in $P(A)$ is unique. (If someone claimed there was more than one maximal set in $P(A)$, we would take the union of those allegedly maximal sets and get one that is bigger. Contradiction.) Thus if $P(A) = P(B)$ then their maximal elements are unique and equal and hence $A = B$.

Exercise: 24 p.85
Let $A = \{a, b, c\}$, $B = \{x, y\}$ and $C = \{0, 1\}$. (Remember in the following order in the set doesn't matter but order within the triplets does.)
a) $A \times B \times C = \{(a,x,0), (a,x,1), (a,y,0), (a,y,1), (b,x,0), (b,x,1), (b,y,0), (b,y,1), (c,x,0), (c,x,1), (c,y,0), (c,y,1)\}$
b) $C \times B \times A = \{(0,x,a), (0,x,b), (0,x,c), (0,y,a), (0,y,b), (0,y,c), (1,x,a), (1,x,b), (1,x,c), (1,y,a), (1,y,b), (1,y,c)\}$
c) $C \times A \times B = \{(0,a,x), (0,a,y), (0,b,x), (0,b,y), (0,c,x), (0,c,y), (1,a,x), (1,a,y), (1,b,x), (1,b,y), (1,c,x), (1,c,y)\}$
d) $B \times B \times B = \{(x,x,x), (x,x,y), (x,y,x), (x,y,y), (y,x,x), (y,x,y), (y,y,x), (y,y,y)\}$

Exercise: 1 p.94

a) $A \cap B$ is the set of students who live within one mile of school and walk to class.
b) $A \cup B$ is the set of students who live within one mile of school or walk to class.
c) $A - B$ is the set of students who live within one mile of school but don't walk to class.
d) $B - A$ is the set of students who walk to class but don't live within one mile of school.

Exercise: 3 p.95
Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$.

a) $A \cup B = \{0,1,2,3,4,5,6\}$
b) $A \cap B = \{3\}$
c) $A - B = \{1,2,4,5\}$
d) $B - A = \{0,6\}$

Exercise: 11 p.95
We want to show that $\overline{A \cap B \cap C} = \bar{A} \cup \bar{B} \cup \bar{C}$.

a) By showing that one is inside the other. If $x \in \overline{A \cap B \cap C}$, then x doesn't lie in the common intersection. Hence, $x \notin A$ or $x \notin B$ or x$\notin C$. But this last statement is the same as saying $x \in \bar{A} \cup \bar{B} \cup \bar{C}$. This argument show that $\overline{A \cap B \cap C} \subseteq \bar{A} \cup \bar{B} \cup \bar{C}$.
To prove that $\bar{A} \cup \bar{B} \cup \bar{C} \subseteq \overline{A \cap B \cap C}$ we start with the same assumption. Suppose $x \in \bar{A} \cup \bar{B} \cup \bar{C}$. Then $x \notin A$ or $x \notin B$ or x$\notin C$ which means that it can't lie in the common intersection. Hence $x \in \overline{A \cap B \cap C}$. Thus $\overline{A \cap B \cap C} \supseteq \bar{A} \cup \bar{B} \cup \bar{C}$ and we conclude that since the two sets are subsets of each other, they must be equal.

b) Membership table:

A	B	C	$\overline{A \cap B \cap C}$	$\bar{A}$	$\bar{B}$	$\bar{C}$	$\bar{A} \cup \bar{B} \cup \bar{C}$
1	1	1	0	0	0	0	0
1	1	0	1	0	0	1	1
1	0	1	1	0	1	0	1
1	0	0	1	0	1	1	1
0	1	1	1	1	0	0	1
0	1	0	1	1	0	1	1
0	0	1	1	1	1	0	1
0	0	0	1	1	1	1	1

Columns 4 and 8 are identical so the corresponding sets are the same.

Exercise: 12 p.95

a) By definition, $A \cap B = \{x \,|\, x \in A \text{ and } x \in B\}$. However, by simplification $x \in A$ and $x \in B \Longrightarrow x \in A$. Thus if $x \in A \cap B$ then $x \in A$. In other words, $(A \cap B) \subseteq A$.
b) By definition, $A \cup B = \{x \,|\, x \in A \text{ or } x \in B\}$. However, by addition $x \in A \Longrightarrow x \in A$ or $x \in B$. Thus if $x \in A$ then $x \in A \cup B$. In other words, $A \subseteq A \cup B$.
c) By definition, $A - B = \{x \,|\, x \in A \text{ but } x \notin B\}$. However, by simplification $x \in A$ and $x \notin B \Longrightarrow x \in A$. Thus if $x \in A - B$ then $x \in A$. In other words, $(A - B) \subseteq A$.
d) Consider the proposition $x \in A \cap (B - A)$. Using logic, this means that $(x \in A) \wedge (x \in B) \wedge (x \notin A)$ which is equivalent to

$$(x \in A) \wedge (x \in B) \wedge \neg(x \in A) \Leftrightarrow \Big((x \in A) \wedge \neg(x \in A)\Big) \wedge (x \in B) \Leftrightarrow \mathbf{F} \wedge (x \in B) \Leftrightarrow \mathbf{F}$$

We understand this result as there is no x in the proposed set so $A \cap (B - A) = \emptyset$.
e) Consider the proposition $x \in A \cup (B - A)$. In logic, this translates to

$$\begin{aligned}(x \in A) \vee \Big((x \in B) \wedge \neg(x \in A)\Big) &\Longleftrightarrow \Big((x \in A) \vee (x \in B)\Big) \wedge \Big((x \in A) \vee \neg(x \in A)\Big) \\ &\Longleftrightarrow \Big((x \in A) \vee (x \in B)\Big) \wedge U \\ &\Longleftrightarrow (x \in A) \vee (x \in B)\end{aligned}$$

Thus $x \in A \cup (B - A)$ is equivalent to $x \in A \cup B$ and hence $A \cup (B - A) = A \cup B$.

Exercise: 15 p.95

$$\begin{aligned}A - B &= \{x \,|\, x \in A \text{ and } x \notin B\} \\ &= \{x \,|\, x \in A \text{ and } x \in \bar{B}\} \\ &= A \cap \bar{B}\end{aligned}$$

Exercise: 19 p.95
Let $A = \{0, 2, 4, 6, 8, 10\}$, $B = \{0, 1, 2, 3, 4, 5, 6\}$ and $C = \{4, 5, 6, 7, 8, 9, 10\}$.

a) $A \cap B \cap C = \{4, 6\}$.
b) $A \cup B \cup C = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.
c) $(A \cup B) \cap C = \{0, 1, 2, 3, 4, 5, 6, 8, 10\} \cap C = \{4, 5, 6, 8, 10\}$.
d) $(A \cap B) \cup C = \{0, 2, 4, 6\} \cup C = \{0, 2, 4, 5, 6, 7, 8, 9, 10\}$.

Exercise: 21 p.95

a) $A \cup B = A$ means that B contributes nothing new to A so we know that $B \subseteq A$.
b) $A \cap B = A$ this time means that $A \subseteq B$.
c) $A - B = A$ means that B doesn't take anything away from A so we conclude that $A \cap B = \emptyset$. (It does not mean that $A = \emptyset$.)
d) $A \cap B = B \cap A$ is true for every pair of sets A and B so this tells us nothing.
e) $A - B = B - A$: If $x \in A - B$ then in particular $x \in A$ but if $x \in B - A$ then $x \notin A$. Hence this condition $A - B = B - A$ implies that $A - B = \emptyset = B - A$ and hence that $A = B$.

Exercise: 24 p.95
$A = \{1, 3, 5\}$ and $B = \{1, 2, 3\}$. Then $A \oplus B = \{2, 5\}$.

Exercise: 25 p.95
The symmetric difference between the set of math majors at school and the set of computer science majors at school is the set of people who are either math majors or science majors but are not both.

Exercise: 41 p.95
Let $U = \{1, 2, \ldots, 10\}$ and we consider how to decode binary strings of length 10 in subsets of U. I'll call the inverse encoding process b^{-1}.

a) $b^{-1}(11110\,01111) = \{1, 2, 3, 4, 7, 8, 9, 10\}$
b) $b^{-1}(01011\,11000) = \{2, 4, 5, 6, 7\}$
c) $b^{-1}(10000\,00001) = \{1, 10\}$

Exercise: 42 p.95
For any universal set U whose subsets we encode into binary strings as described in class,

a) $b^{-1}(000\ldots0) = \emptyset$
b) $b^{-1}(111\ldots1) = U$