Math 1137, Summer 2003

Homework 4: 1,4,8-12,16,26,27,60,65 p.108

Exercise: 1 p.108
None of the following are functions from $\mathbb{R}$ to $\mathbb{R}$.

a) $f(x) = \frac{1}{x}$. The problem here is that f is not defined for 0. A function must be defined for every element in its domain.
b) $f(x) = \sqrt{x}$ is not defined on negative numbers $x < 0$.
c) $f(x) = \pm\sqrt{x^2+1}$. This f associates two real numbers to any x. The expression $\sqrt{x^2+1}$ is perfectly well-defined for all $x \in \mathbb{R}$ but a function can only associate one output value to any input.

Exercise: 4 p.108
Remember that the range means all possible images (outputs) of a function.

a) f assigns to each nonnegative integer its last digit. Domain=$\mathbb{N}$ and Range=$\{0,1,2,3,4,5,6,7,8,9\}$.
b) f assigns the next largest integer to a positive integer. Domain=$\mathbb{N}^* = \{1,2,3,\ldots\}$ and Range=$\{2,3,4,\ldots\}$.
c) f assigns to a bit string the number of one bits in the string. Domain=bitstrings=$\{(b_0,b_1,b_2,\ldots) \mid b_i \in \{0,1\} \text{ for all } i \in \mathbb{N}\}$ and Range=$\mathbb{N}$. (We can describe the set of bitstrings in a few different ways: 1) as the set of functions $h : \mathbb{N} \to \{0,1\}$; 2)as the infinte Cartesian product $\{0,1\} \times \{0,1\} \times \cdot$ which can be written succintly as $\{0,1\}^{\mathbb{N}}$.
d) f assigns to a bit string the number of bits in a string. (I assume this simply means that all bits are zero after a certain point.) Domain is bitstrings and Range=$\mathbb{N}$.

Exercise: 8 p.109

a) $\lfloor 1.1 \rfloor = 1$
b) $\lceil 1.1 \rceil = 2$
c) $\lfloor -0.1 \rfloor = -1$
d) $\lceil -0.1 \rceil = 0$
e) $\lceil 2.99 \rceil = 3$
f) $\lceil -2.99 \rceil = -2$
g) $\lfloor \frac{1}{2} + \lceil \frac{1}{2} \rceil \rfloor = \lfloor \frac{1}{2} + 1 \rfloor = \lfloor \frac{3}{2} \rfloor = 1$
h) $\lceil \lfloor \frac{1}{2} \rfloor + \lceil \frac{1}{2} \rceil + \frac{1}{2} \rceil = \lceil 0 + 1 + \frac{1}{2} \rceil = \lceil \frac{3}{2} \rceil = 2$

Exercise: 9 p.109

a) $\lfloor\rfloor\ \lceil\rceil$
b) $\lceil \frac{3}{4} \rceil = 1$
c) $\lfloor \frac{7}{8} \rfloor$
d) $\lceil -\frac{3}{4} \rceil = 0$
e) $\lfloor -\frac{7}{8} \rfloor = -1$
f) $\lceil 3 \rceil = 3$
g) $\lfloor -1 \rfloor = -1$
h) $\lfloor \frac{1}{2} + \lceil \frac{3}{2} \rceil \rfloor = \lfloor \frac{1}{2} + 2 \rceil \rfloor = 2$
i) $\lfloor \frac{1}{2} \cdot \lfloor \frac{5}{2} \rfloor \rfloor = \lfloor 1 \rfloor = 1$

Exercise: 10 p.109
One-to-one menas that the f sends different elements to different images. Mathematically speaking, f is one-to-one when $f(x) = f(y) \Rightarrow x = y$.

a) this is one-to-one.
b) not one-to-one since $f(a) = b = f(b)$.
c) not one-to-one since $f(a) = d = f(d)$.

Exercise: 11 p.109
Onto means that every element in the target space ($\{a, b, c, d\}$) gets hit as the image of some element in the domain. Only the function in a) is onto. In b) the function misses a and the same thing happens in c).

Exercise: 12,13 p.109
Each of the following functions is considered $f : \mathbb{Z} \to \mathbb{Z}$.

a) $f(n) = n-1$. To see algebraically that this is one-to-one, consider the equation $f(n) = f(m)$, i.e. $n-1 = m-1$. Adding 1 to both sides, we get $n = m$. This is the mathematical definition of a function being one-to-one. To see that it is onto, we consider the equation $f(n) = y$. The function will be onto if such an equation has a solution in $\mathbb{Z}$ for every possible y in the range. Here the solution is just $n = y + 1$. Since y is an integer, then n is an integer and everything works. f is one-to-one and onto.

b) $f(n) = n^2 + 1$. One might be able to see directly that this is not one-to-one since $f(-n) = f(n)$ so for example $f(-2) = f(2) = 5$. (If we tried to solve $f(n) = f(m)$, we would get the equation $n^2 + 1 = m^2 + 1$ which is equivalent to $n^2 - m^2 = 0$ and hence $(n - m)(n + m) = 0$. The only way this last equation can be 0 is if $n - m = 0$ (i.e. $n = m$) or $n + m = 0$ (i.e. $n = -m$).)

f is not onto either since for example f never hits 3. $f(n) = 3 \Leftrightarrow n^2 = 2$ but $n^2 = 2$ has no solutions in the integers.

c) $f(n) = n^3$. f is one-to-one and we need to find some way to be convinced of this. $f(n) = f(m) \Leftrightarrow n^3 = m^3 \Leftrightarrow (\frac{n}{m})^3 = 1$. The only real number r that satisfies $r^3 = 1$ is $r = 1$. Hence, we deduce that $\frac{n}{m} = 1 \Leftrightarrow n = m$ so f is one-to-one. On the other hand, f is not onto since for example there is no integer n such that $n^3 = 2$.

d) $f(n)\lceil \frac{n}{2} \rceil$. This certainly is not one-to-one since for example $f(1) = \lceil \frac{1}{2} \rceil = 1 = f(2)$. But $f(n)$ is surjective. To show this, we consider $f(n) = y$ where y is some integer. $\lceil \frac{n}{2} \rceil$ is the unique integer y that satifies

$$y - 1 < \frac{n}{2} \leq y$$

this is equivalent to $2y - 2 < n \leq 2y$. Hence $n = 2y$ or $n = 2y - 1$ will produce $f(n) = y$.

Exercise: 16 p. 109
All functions in the exercise are from $\mathbb{N}$ to $\mathbb{N}$.

a) one-to-one but not onto: $f(x) = x^2$. It's one-to-one in $\mathbb{N}$ but it doesn't hit every element in the target.

b) onto but not one-to-one: $f(x) = \lfloor x/2 \rfloor$ will do.

c) both one-to-one and onto but not the identity function:

$$f(x) = \begin{cases} x + 1 & \text{if } x \text{ is even} \\ x - 1 & \text{if } x \text{ is odd} \end{cases}$$

For the integers $0, 1, 2, 3, 4, 5, 6, \ldots$, this function puts out $1, 0, 3, 2, 5, 4, 7, \ldots$.

d) neither one-to-one or onto: $f(x) = 1$ is a simple example, or $f(x) = (x - 3)^2$. (It isn't onto but also $f(1) = 4 = f(5)$.

Exercise: 26 p.109
Let $f : B \to C$ and $g : A \to B$ be two functions. We begin by assuming that f and $f \circ g$ are one-to-one. (Remember that $f \circ g$ means that we apply g first and then f. Consider two elemenst n, m in A. We suppose that $g(n) = g(m)$. If we apply the function f to both sides of this equation we get $f(g(n)) = f(g(m))$ which is equivalent to $(f \circ g)(n) = (f \circ g)(m)$. But since $f \circ g$ is one-to-one, we get from the mathematical defintion that $n = m$. Hence $g(n) = g(m) \Rightarrow n = m$ so g itself is one-to-one. (Note, we didn't even use the fact that f is one-to-one.

Exercise: 27 p.109
We consider two functions as in the previous exercise but this time we assume that f and $f \circ g$ are onto. In this case, however, g does not have to be onto. Here is the simplest example that illustrates this phenomenon. Take $A = \{a\}$, $B = \{0, 1\}$ and $C = \{x\}$. Define the functions f and g as $g(a) = 0$, $f(0) = x$ and $f(1) = x$. Then $f \circ g$ is the function defined simply by $(f \circ g)(a) = x$ and it's easy to see that $f \circ g$ and f are both onto but g is not since it doesn't hit the element 1 in B.

Exercise: 60 p.110
(I will skip this on the answer sheet.)

Exercise: 65 p.111

a) We prove that $\lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$. For all real numbers x, $\lfloor x \rfloor$ is an integer. Furthermore, by definition of the ceiling function for all integers n, $\lceil n \rceil = n$. The claim follows from these two facts.

b) It is not true that $\lfloor 2x \rfloor = 2\lfloor x \rfloor$ for all real numbers x. The fraction $\frac{1}{2}$ serves as a counter example: the left hand side of the proposed formula gives $\lfloor 1 \rfloor = 1$ while the right hand side gives $2\lfloor \frac{1}{2} \rfloor = 0$.

c) We prove that for all real numbers x and y, $\lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil$. By definition, $\lceil x \rceil = m$ means m is the unique integer such that $m - 1 < x \leq m$ and $\lceil y \rceil = n$ means n is the unique integer such that $n - 1 < x \leq n$. By adding the inequalities, we can conlude that

$$m + n - 2 < x + y \leq m + n$$

Now $\lceil x + y \rceil$ is the smallest integer greater or equal to $x + y$ but the above inequality implies that $m + n - 2 < \lceil x + y \rceil \leq m + n$ so $\lceil x + y \rceil = m + n$ or $m + n - 1$. The claim follows.

d) It is certainly not true that $\lceil xy \rceil = \lceil x \rceil \lceil y \rceil$. Take $x = -1$ and $y = -1$. This pair serves as an eay counter example. Perhaps a more interesting example would be to take $x = \frac{1}{2}$ and $y = 3$. Then $\lceil xy \rceil = 2$ but $\lceil x \rceil \lceil y \rceil = 3$.

e) We show that $\left\lceil \frac{x}{2} \right\rceil = \left\lfloor \frac{x+1}{2} \right\rfloor$ is not true for all real numbers x. A counter example is $x = \frac{1}{2}$. $\left\lceil \frac{1}{4} \right\rceil = 1$ but the right hand side is $\left\lceil \frac{3}{4} \right\rceil = 0$.

In fact we can prove a stronger result. First, by definition $\left\lceil \frac{x}{2} \right\rceil = n$ means that n is the unique integer satisfying $n - 1 < \frac{x}{2} \leq n$ that is to say $2n - 2 < x \leq 2n$. Also by definition $\left\lfloor \frac{x+1}{2} \right\rfloor = m$ means that m is the unique integer satisfying $m \leq \frac{x+1}{2} < m + 1$ that is to say $2m - 1 \leq x < 2m + 1$. If we want that $n = m$ then the only values of x that make this work are $x \in [2n - 1, 2n]$ in fancy set notation, this means that the proposed equality holds only for x in the set of real numbers

$$\bigcup_{n=-\infty}^{\infty} [2n - 1, 2n] = \cdots [-3, -2] \cup [-1, 0] \cup [1, 2] \cup [3, 4] \cup [5, 6] \cdots$$

(In particular, the equality holds for all integers, though not for all real numbers.)