

## Math 1137, Summer 2003

### Homework 9: 1,4,7,17,27,28 p.223

#### Exercise: 1 p.223

We want to prove that the product of any three consecutive integers is divisible by 6. We can begin by calling the first of the three consecutive integers  $n$  and the next two will then be  $n + 1$  and  $n + 2$ . Hence, we are suppose to prove that for all  $n \in \mathbb{Z}$ ,  $6 \mid n(n+1)(n+2)$ .

We can prove this by first showing that 2 divides  $n(n+1)(n+2)$  and then that 3 does. To show that 2 divides, note that if  $n$  is even then of course  $2 \mid n(n+1)(n+2)$  (by Theorem 1 (2) p.154). If  $n$  is odd then  $n + 1$  is even and thus  $2 \mid n + 1$  and hence  $2 \mid n(n+1)(n+2)$  (by the same theorem).

To show that 3 divides  $n(n+1)(n+2)$ , we do a proof with three cases. 1) Suppose  $n \equiv 0 \pmod{3}$ . Then  $3 \mid n$  and hence  $3 \mid n(n+1)(n+2)$ . 2) Suppose  $n \equiv 1 \pmod{3}$ . Then  $3 \mid n + 2$  and hence  $3 \mid n(n+1)(n+2)$ . 3) Suppose finally that  $n \equiv 2 \pmod{3}$ . Then  $3 \mid n + 1$  and hence  $3 \mid n(n+1)(n+2)$ .

Thus, we've shown that both 2 and 3 divide  $n(n+1)(n+2)$ . Therefore, the least common multiple of 2 and 3 also divides  $n(n+1)(n+2)$  and the least common multiple is 6. Thus for all  $n \in \mathbb{Z}$  we conclude that  $6 \mid n(n+1)(n+2)$ .

#### Exercise: 4 p. 223

The exercise is to show that there are no solutions in integers to the equation  $2x^2 + 5y^2 = 14$ . Following the method of example 4 on page 216 of the text, we see that we only need to check a few possible solutions. For  $|x| \geq 3$ ,  $2x^2 > 14$  and hence we know that any possible solution has  $|x| \leq 2$ . Furthermore, if  $|y| \geq 2$ ,  $5y^2 > 14$  and hence we know that any possible solution has  $|y| \leq 1$ . Also, because we are squaring both  $x$  and  $y$ , then if  $(x, y)$  is a solution, then the four possible combinations of signs with  $(\pm x, \pm y)$  also occur as solutions. Thus, we only need to check the pairs  $(x, y)$  with  $0 \leq x \leq 2$  and  $0 \leq y \leq 1$ .

We check one at a time:  $(0, 0) \rightarrow LHS = 0$  doesn't work;  $(0, 1) \rightarrow LHS = 5$  doesn't work;  $(1, 0) \rightarrow LHS = 2$  doesn't work;  $(1, 1) \rightarrow LHS = 7$  doesn't work;  $(2, 0) \rightarrow LHS = 8$  doesn't work;  $(2, 1) \rightarrow LHS = 13$  doesn't work.

This covers all the cases and we've proved that  $2x^2 + 5y^2 = 14$  has no integers solutions.

#### Exercise: 7 p. 223

Consider the equation  $3x^2 - 8y = 1$ . We wish to show it doesn't have integer solutions. Notice we can't use the same technique as the previous problem since we can't cut down to a finite number of possible solutions.

Here's one clever way to solve this problem. Consider the equation modulo 4. The equation becomes  $3x^2 \equiv 1 \pmod{4}$  (because  $8 \equiv 0$ ). However, notice the possible squares of integers modulo 4:

$$\begin{aligned}0^2 &\equiv 0 \\1^2 &\equiv 1 \\2^2 &\equiv 4 \equiv 0 \\3^2 &\equiv 9 \equiv 1\end{aligned}$$

Thus, no matter what  $x$  is as an integer, we know that  $x^2$  is congruent to 0 or 1 modulo 4. Hence  $3x^2 \equiv 0$  or  $3$  but never 1. Hence, since our equation has no solution modulo 4, it cannot have solutions in integers.

#### Exercise: 17 p. 224

Prove that  $\sqrt[3]{3}$  is not a rational number, i.e. is not a fraction. Let's prove this by contradiction.

Suppose that  $\sqrt[3]{3}$  is equal to a fraction  $\frac{p}{q}$  written in reduce form, i.e. the greatest common divisor between  $p$  and  $q$  is 1, i.e. that  $p$  and  $q$  are relatively prime.

$$\sqrt[3]{3} = \frac{p}{q} \iff p^3 = 3q^3$$

From this last expression, we see that by unique factorization of integers, 3 must divide  $p^3$ . Again by the unique factorization of integers, if  $3 \nmid p$  then  $3 \nmid p^3$ . Since we know that  $3 \mid p^3$ , then using the contrapositive of this statement,  $3 \mid p$ . We can then write  $p = 3p'$  and then

$$p^3 = 3q^3 \iff 27(p')^3 = 3q^3 \iff 9(p')^3 = q^3$$

Now, using an identical argument as in the above paragraph, we can conclude that  $3 \mid q$ . However, this leads to a contradiction since we assumed that  $\gcd(p, q) = 1$  and we just showed that  $3 \mid \gcd(p, q)$ . Thus,  $\sqrt[3]{3}$  cannot be a rational number.

**Exercise:** 27 p. 224

A constructive proof means that we explicitly show how to solve the proposed problem. Let's show that for every positive integer  $n$  there's an integer divisible by more than  $n$  primes.

Let's first recall the result that there exists an infinite number of primes. Let's now number them in order of appearance so the set of primes is  $\{p_1, p_2, p_3, \dots\}$ :

$$p_1 = 2, \quad p_2 = 3, \quad p_3 = 5, \quad p_4 = 7, \quad p_5 = 11, \dots$$

Given any integer  $n$ , we can construct the integer  $m = p_1 p_2 p_3 \cdots p_{n+1}$  as the product of the first  $n + 1$  prime numbers. By construction, the integer  $m$  is divisible by more than  $n$  primes.

**Exercise:** 28 p. 224

Let's consider the statement: "For every prime number  $n$ ,  $n + 2$  is prime." This statement is clearly false and to see that it is false it suffices to find a counter example. The first counter-example is with  $n = 2$  and  $n + 2 = 4$  (4 is composite).

Of course, one might doctor the statement to say that every odd prime has this property but that is also false since we can find the counter example  $n = 7$  which give  $n + 2 = 9 = 3^2$ .