MTH U481 : SUMMER 1, 2005: PRACTICE PROBLEMS FOR MIDTERM

1). A biased coin has probability $p = 0.6$ of coming up Heads. The coin is tossed 5 times. Assume that the tosses are independent.

a). What is the probability that the first two tosses are Heads, and the last three are Tails?

b). What is the probability that the first two tosses are Heads?

c). What is the probability that exactly two of the tosses come up Heads, in any order?

2). Based on past experience, the time taken by a randomly chosen student to complete a test in the course "Probability 1" varies between 35 minutes and 65 minutes. Let T denote the time taken for the test, and suppose that the pdf for T is

$$f(t) = \begin{cases} c(65 - t) & \text{if } 35 \leq t \leq 65 \\ 0 & \text{else} \end{cases}$$

a) Find the value of the constant c.

b) Find the probability that the time taken is greater than 50 minutes.

c) Find the cdf of T (your answer should present different expressions for the cdf in three different intervals).

3). Two fair dice are thrown at random. Define the discrete random variable Y as follows: if the numbers on the dice are equal, then $Y = 1$; if the numbers on the dice are different, then $Y = 0$. Find the pmf of Y.

4). The pdf for a continuous random variable T is given as follows:

$$f_T(t) = \begin{cases} t & \text{if } 0 \leq t \leq 1, \\ 1 & \text{if } 1 \leq t \leq 1.5, \\ 0 & \text{otherwise.} \end{cases}$$

a) Sketch the graph of $f_T(t)$.

b) Calculate $P(0.5 \leq T \leq 1.25)$.

5). The pdf for a continuous random variable Y is

$$f_Y(y) = \begin{cases} c\,(2y - y^2) & \text{if } 0 \le y \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

where c is a constant.
a). Find the value of c.
b). Calculate $P(0 \le Y \le 1)$.
c). Find the cdf $F_Y(y)$.

6). An urn contains 4 chips, numbered $1, 2, 3, 4$. Two chips are chosen at random, without replacement. Let X be the smaller of the two numbers on the chips; if the two numbers are equal, then X is equal to this number.
a). Find the pmf of X.
b). Find the expected value of X.
c). Find the variance of X.
d). Find the standard deviation of X.

7). A continuous random variable X has the following pdf:

$$f_X(x) = \begin{cases} k(1 + x^2) & \text{for } -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of k.
(b) Compute E[X].
(c) Compute VAR[X].

1). A moderately skilled darts player throws a dart at the interval $[0, 1]$. The probability density function (pdf) for the position X where her dart lands is

$$f(x) = 6x - 6x^2, \qquad 0 \le x \le 1$$

a). Sketch the graph of the pdf, and calculate the mean and the variance of X.
Answer: E[X] = 0.5; VAR[X] = 0.05

b). What is the probability that the dart lands in the interval $[0.3, 0.7]$?
Answer: 0.568

c). If she throws two darts, and the positions where they land are independent, find the probability that both of them land in the interval $[0.3, 0.7]$.
Answer: 0.323

2). A continuous random variable X has the following cdf:

$$F_X(x) = P(X \le x) = \begin{cases} 0 & \text{if } x \le -1 \\ c(1 + x)^2 & \text{if } -1 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$$

a) Find the constant c, and sketch the graph of F_X. [BE CAREFUL!! this is the cdf, NOT the pdf!! Remember that the cdf is continuous]
Answer: c = 0.25.

b) Find $P(X \leq 1/2)$.
Answer: 9/16

c) Find $P(|X| \geq 1/2)$.
Answer: 1/2

d) Calculate the pdf f_X (your answer should present different expressions for the pdf in three different intervals).
Answer:

$$f_X(x) = \begin{cases} 0 & \text{if } x \leq -1 \\ (1+x)/2 & \text{if } -1 \leq x \leq 1 \\ 0 & \text{if } x > 1 \end{cases}$$

3). A discrete random variable X can take on the three values $\{1, 3, 4\}$. It is known that $P(X = 1) = 0.2$ and $P(X = 3) = 0.5$. Fnd the pmf of X and use it to calculate E[X] and VAR[X].
Answer: E[X] = 2.9; VAR[X] = 1.09

4). The voltage X across a resistor is a random variable, and it has a normal distribution with mean $\mu = 10$ Volts and standard deviation $\sigma = 2.5$ Volts. Find the probability that the voltage is between 7 Volts and 12 Volts.
Answer: $\Phi(0.8) - \Phi(-1.2) = 0.673$

5). A biased coin has probability $p = 0.3$ of coming up Heads. The coin is tossed 1000 times by a patient student of probability. Let X be the number of Heads that appear. Find the mean and the standard deviation of X.
Answer: E[X] = 300; STD[X] = 14.5.

6). A reliable type of aircraft engine is known to fail with a probability $p = 10^{-5}$ after one year of continuous operation. There are five thousand of these engines in use on aircraft. Use the Poisson approximation to compute the probability that two of these engines would fail independently after one year of continuous operation.
Answer: $\lambda = 0.05$, P(X=2) = 0.0012.

7). Two resistors R_1 and R_2 are placed in series, so their combined resistance is $R = R_1 + R_2$. Suppose that R_1 and R_2 are both random variables, where R_1 is normal with mean 50 and standard deviation 5, and R_2 is normal with mean 80 and standard deviation 10. Find the mean and standard deviation of R.
Answer: E[R] = 130; STD[R] = 11.18.

8). An urn contains 4 chips, numbered $1, 2, 3, 4$. A chip is chosen at random. Let X_1 be the number on this chip. This chip is then replaced, and a second chip is chosen at random from the same urn. Let X_2 be the number on this chip.
a). Find the expected value of X_1.
Answer: 2.5
b). Find the expected value of X_2.
Answer: 2.5
c). Find the expected value of the sum $X_1 + X_2$.
Answer: 5
d). Find the expected value of $\min(X_1, X_2)$.
Answer: 15/8
e). Find the covariance COV$[X, Y]$ (Hint: you don't need to do any calculations).
Answer: 0

9). A certain make of car has a lifetime given by a normal distribution with mean 100,000 miles and standard deviation 20,000 miles.
a). Find the probability that a randomly selected car has lifetime less than 70,000 miles.
Answer: $1 - \Phi(1.5) = 0.0668$
b). What percentage of cars will have a lifetime greater than 115,000 miles?
Answer: $1 - \Phi(0.75) = 0.2266$

10). A rare disease occurs only in 1 person out of every 10,000. If a population of 1000 people is tested, what is the probability that one or more people have the disease?
Hint: use the Poisson distribution to estimate the answer.
Answer: $\lambda = 0.1$; prob = 0.095

11). A point is chosen randomly and uniformly from the triangle with corners at $(0,0)$, $(1,0)$ and $(1,2)$. Let X and Y be the coordinates of the point chosen.
a). Find $f(x,y)$, the joint pdf of X and Y. Be sure to describe the region where $f(x,y)$ is non-zero, and it's value in that region.
Answer:

$$f(x,y) = \begin{cases} 1 & \text{for } (x,y) \text{ in triangle} \\ 0 & \text{else} \end{cases}$$

b). Calculate $P(X \leq Y)$.
Answer: 1/2
c). Calculate the marginal pdf's f_X and f_Y for X and Y. Again be sure to describe the intervals where they are non-zero.
Answer:

$$f_X(x) = \begin{cases} 2x & \text{for } 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

$$f_Y(y) = \begin{cases} 1 - y/2 & \text{for } 0 \leq y \leq 2 \\ 0 & \text{else} \end{cases}$$

d). Are X and Y independent? Explain.
Answer: No, since $f_X(x) f_Y(y) \neq f(x,y)$
e). Calculate the mean and variance of X.
Answer: E[X] = 2/3; VAR[X] = 1/18
f). Calculate the mean and variance of Y.
Answer: E[Y] = 2/3; VAR[Y] = 2/9