

## Cohomology of tails and stable cohomology over Koszul quiver algebras

**Roberto Martínez Villa**

Instituto de Matemáticas  
UNAM, Apdo Postal 61-B  
Morelia, Mich., Mexico, 58089  
mvilla@matmor.unam.mx

**Alex Martsinkovsky**

Mathematics Department  
Northeastern University  
Boston, MA 02115, USA  
alexmart@neu.edu

**Abstract.** The main result of this announcement shows that, under Koszul duality between quiver algebras, cohomology of tails is naturally isomorphic to stable cohomology. As an application, we have a new conceptual proof of the Serre duality for the category of tails over a generalized Artin - Schelter regular Koszul algebras. In particular, the Serre duality formula turns out to be equivalent to the Auslander - Reiten formula for selfinjective artin algebras.

### Introduction

The purpose of this short note is twofold. First, it serves as an announcement of some of the results contained in [5]. Secondly, it provides a quick and, hopefully, transparent introduction to the ideas and methods used in the quoted paper. For that second purpose, a special attempt has been made to present the results with a minimum of technical details. For the same reason, we have slightly rearranged the order of exposition and included examples, which are either not present in the original paper or have been completely rewritten. The proofs have been explained at a level of general ideas and the most basic tools. For complete details and further results the reader is referred to the quoted paper, which will appear elsewhere.

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## 1 Notation

Let  $Q$  be a finite quiver and  $\mathbb{k}$  a field. The path algebra of  $Q$  over  $\mathbb{k}$  will be denoted  $\mathbb{k}Q$ . This algebra is nonnegatively graded by path length: vertices are considered as paths of length zero, arrows as paths of length one, and the length of any other path is defined inductively.

**Definition 1.1** A quiver algebra  $\Lambda$  is the quotient algebra  $\Lambda := \mathbb{k}Q/I$ , where  $I$  is a two-sided homogeneous ideal contained in the square of the ideal generated by the arrows.

Notice that a quiver algebra may or may not be artin, but the finiteness of  $Q$  implies that in the vector space decomposition  $\Lambda = \coprod_{i \geq 0} \Lambda_i$  each  $\Lambda_i$  is a finite-dimensional  $\mathbb{k}$ -vector space. The graded two-sided ideal  $J := \coprod_{i \geq 1} \Lambda_i$  is called the graded Jacobson radical of  $\Lambda$ .

In this paper we shall deal exclusively with  $\mathbb{Z}$ -graded left  $\Lambda$ -modules. The class of all such modules with degree zero homomorphisms between them forms an abelian category, denoted  $\text{Gr Mod } \Lambda$ . In this category:

- there are enough projectives,
- finitely generated modules have projective covers,
- the Krull - Remak - Schmidt Theorem holds, and
- the principal projectives are in one-to-one correspondence with the vertices of  $Q$ .

Given an integer  $k$ , we can shift a graded module  $M$  to obtain a new graded module  $M[k]$  with  $M[k]_j := M_{k+j}$ . For each integer  $k$ , the corresponding shift operation gives rise to an autoequivalence of  $\text{Gr Mod } \Lambda$ .

The quotient module  $\Lambda/J$  is a direct sum of (graded) simples. The definition above shows that those simple summands are in one-to-one correspondence with the vertices of  $Q$ . It is easy to see that any graded simple is isomorphic to one of those summands and that  $\Lambda$  is basic. Furthermore, a projective cover of the simple module corresponding to a vertex of  $Q$  is isomorphic to the principal projective corresponding to that vertex.

**Example 1.2** Let  $Q$  be the quiver with one vertex and one arrow. The quiver algebra  $\mathbb{k}Q$  is isomorphic to  $\mathbb{k}[X]$ . The simple module  $\mathbb{k}$  is the unique graded simple. On the other hand, Euclid's argument on the infinitude of primes shows that  $\mathbb{k}[X]$  has infinitely many nongraded simples.

## 2 Koszul quiver algebras and Koszul duality

Let  $\Lambda$  be a quiver algebra and  $M \in \text{Gr Mod } \Lambda$  and

$$\mathbf{P} \dots \rightarrow P_n \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

a projective resolution of  $M$ . We shall say that  $\mathbf{P}$  is of type  $FP_n$  if the projectives  $P_0, \dots, P_n$  are finitely generated, and that  $\mathbf{P}$  is of type  $FP_\infty$  if all  $P_i$  are finitely generated. Recall that  $M$  is said to be of type  $FP_n$  (resp.,  $FP_\infty$ ) if it has a projective resolution of type  $FP_n$  (resp.,  $FP_\infty$ ). Schanuel's Lemma shows that  $M$  is of type  $FP_\infty$  if and only if it is of type  $FP_n$  for each  $n \geq 0$ .

**Remark 2.1** It is clear that if  $\mathbf{P}$  is of type  $FP_\infty$ , then  $M$  has a minimal resolution.

**Definition 2.2** Suppose  $M \in \text{Gr Mod } \Lambda$ .

- a) We say that  $M$  is Koszul if it has a projective resolution  $\mathbf{P}$  of type  $FP_\infty$  such that each nonzero  $P_i$  is generated in degree  $i$ . (In particular,  $M$  has to be generated in degree zero).
- b) We say that  $\Lambda$  is Koszul if all graded simple  $\Lambda$ -modules are Koszul.

**Remark 2.3** As a minimal resolution is isomorphic to a direct summand of any other resolution, the previous remark shows that the projective resolution  $\mathbf{P}$  whose existence is postulated in part a) is necessarily minimal.

**Examples 2.4** Let  $Q$  be a quiver with one vertex and  $n$  arrows. The corresponding path algebra  $\mathbb{k}Q$  is isomorphic to the free algebra  $\mathbb{k}\langle X_1, \dots, X_n \rangle$  on  $n$  letters. Thus  $\mathbb{k}Q$ , and therefore any graded quotient algebra of it, is graded local. For any quiver algebra  $\Lambda := \mathbb{k}Q/I$  there will be only one graded simple, denoted (in abused notation)  $\mathbb{k}$ . Thus to establish the Koszul property we only have to check the minimal resolution of  $\mathbb{k}$ . Here are some of the typical choices for  $I$ :

1.  $I := \{0\}$ . Then  $\Lambda = \mathbb{k}\langle X_1, \dots, X_n \rangle$  and the minimal resolution of  $\mathbb{k}$  is

$$0 \longrightarrow \Lambda^n[-1] \xrightarrow{[X_1, \dots, X_n]} \Lambda \longrightarrow \mathbb{k} \longrightarrow 0,$$

showing that  $\mathbb{k}\langle X_1, \dots, X_n \rangle$  is Koszul.

2.  $I := J^2$ . In this case  $\Lambda = \mathbb{k}Q/J^2$  is a radical square zero algebra on  $n$  letters. Thus the matrices in the minimal projective resolution of  $\mathbb{k}$  (and, in fact, of any finite module) must be linear. This shows that  $\Lambda$  is Koszul.
3.  $I := (X_i X_j - X_j X_i)_{i,j=1, \dots, n}$ . In this case  $\Lambda = \mathbb{k}[X_1, \dots, X_n]$  is the polynomial algebra on  $n$  letters. It is well-known that the Koszul complex on  $X_1, \dots, X_n$  gives a projective resolution of  $\mathbb{k}$ . As the matrix entries of each map in the Koszul complex are linear,  $\mathbb{k}[X_1, \dots, X_n]$  is Koszul.
4.  $I := (X_i X_j + X_j X_i, X_i^2)_{i,j=1, \dots, n}$ . In this case,  $\Lambda = \wedge^\bullet(\mathbb{k}^n)$  is the exterior algebra on  $n$  letters. This algebra is also Koszul, as can be deduced from general theory (see Th. 2.5, (v)). For the reader who does not like general theory we offer this exercise: write down an explicit minimal projective resolution of  $\mathbb{k}$  and verify that  $\Lambda$  is Koszul.<sup>1</sup>

Before we state the next result, we briefly recall that the Yoneda product  $gf \in \text{Ext}^{k+l}(L, N)$  of the classes  $f \in \text{Ext}^k(L, M)$  and  $g \in \text{Ext}^l(M, N)$  is defined as the class of the composition of a lifting of  $g$  with a lifting of  $f$  (to the respective shifted projective resolutions). This makes  $\text{Ext}^*(A, B)$  a left graded module over  $\text{Ext}^*(B, B)$ .

The next theorem, extracted from ([2] and [3]), collects basic facts about Koszul algebras and Koszul modules.

**Theorem 2.5** *Let  $\Lambda := \mathbb{k}Q/I$  be a Koszul algebra and  $\Gamma := \prod_{k \geq 0} \text{Ext}_\Lambda^k(\Lambda_0, \Lambda_0)$  its Yoneda algebra. Let  $F : \text{Gr Mod } \Lambda \rightarrow \text{Gr Mod } \Gamma$  be the functor given by  $F(-) := \prod_{k \geq 0} \text{Ext}_\Lambda^k(-, \Lambda_0)$ . Then:*

- (i) *If  $M$  is a finitely generated graded semisimple  $\Lambda$ -module, then  $F(M)$  is a finitely generated graded projective  $\Gamma$ -module generated in degree zero.*
- (ii) *If  $M$  is a finitely generated graded projective  $\Lambda$ -module, then  $F(M)$  is a finitely generated graded semisimple  $\Gamma$ -module generated in degree zero.*
- (iii)  *$\Lambda$  is quadratic, i.e. the ideal  $I$  is generated by  $\mathbb{k}$ -linear combinations of paths of length two.*

<sup>1</sup>We recommend to skip this exercise in the first reading.

- (iv) The Yoneda algebra  $\Gamma = \prod_{k \geq 0} \text{Ext}_{\Lambda}^k(\Lambda_0, \Lambda_0)$  is Koszul and  $\Lambda$  is the Yoneda algebra of  $\Gamma$ .
- (v) The algebra  $\Gamma$  is isomorphic to  $\mathbb{k}Q^{op}/L$ , where  $Q^{op}$  is the opposite quiver of  $Q$  and  $L$  is the ideal generated by its degree two part  $L_2$  which is obtained as follows. Let  $V := \mathbb{k}Q_2$ ,  $V^o := \mathbb{k}Q_2^{op}$  and  $\langle -, - \rangle : V \times V^o \rightarrow \mathbb{k}$  be the bilinear form defined on the basis elements by

$$\langle \alpha\beta, \beta'\alpha' \rangle := \begin{cases} 1 & \text{if } \alpha = \alpha' \text{ and } \beta = \beta' \\ 0 & \text{otherwise.} \end{cases}$$

Then  $L_2 := I_2^\perp$ , where  $I_2 := I \cap \mathbb{k}Q_2$ .

- (vi) The opposite algebra of  $\Lambda$  is Koszul.
- (vii) If  $M$  is a Koszul  $\Lambda$ -module, then both  $J^k M[k]$  and  $\Omega^k M[k]$  are Koszul for each  $k \geq 1$ .
- (viii) Let  $K_\Lambda$  and  $K_\Gamma$  denote the categories of Koszul modules over  $\Lambda$  and, respectively,  $\Gamma$  with degree zero maps. Then the functor  $F : \text{Gr Mod } \Lambda \rightarrow \text{Gr Mod } \Gamma$  defined above induces a duality, denoted by the same letter,  $F : K_\Lambda \rightarrow K_\Gamma$  such that, for any Koszul  $\Lambda$ -module  $M$ , there are natural isomorphisms  $F(J_\Lambda^k M[k]) \simeq \Omega^k F(M)[k]$  and  $F(\Omega^k M[k]) \simeq J_\Gamma^k F(M)[k]$  for each  $k \geq 1$ . The functor  $G : K_\Gamma \rightarrow K_\Lambda$ , given by  $G(-) := \prod_{k \geq 0} \text{Ext}_\Gamma^k(-, \Gamma_0)$ , is a quasi-inverse of  $F$ .

**Remark 2.6** The term “natural” used in the last part of the above theorem presupposes that  $\Omega$  is a functor. The construction of  $\Omega$  used in the quoted reference shows that this is indeed the case.

The next result ([4], Prop. 2.4) shows that, under mild assumptions, graded modules of finite projective dimension can be approximated by Koszul modules.

**Theorem 2.7** Let  $\Lambda := \mathbb{k}Q/I$  be a Koszul algebra and  $M$  a graded  $\Lambda$ -module of finite projective dimension. Assume that  $M$  is of type  $FP_\infty$ . Then there exists an integer  $k$  such that the shift  $M_{\geq k}[k]$  of the truncated module  $M_{\geq k}$  is Koszul.

### 3 Tails and their cohomology

A classical result of Serre asserts that the category of coherent sheaves on a projective space is equivalent to the category of finitely generated graded modules over the corresponding polynomial algebra modulo the subcategory of modules of finite length. The latter is a special case of a general technique, also introduced by Serre under the name of classes of abelian groups and nowadays known as quotient categories. We briefly recall this construct in our setting.

Let  $\Gamma$  be a graded quiver algebra with graded radical  $J$  and  $\text{Fl } \Gamma$  the full subcategory of  $\text{Gr Mod } \Gamma$  of  $J$ -torsion modules. Recall that a module is  $J$ -torsion if any element of it can be annihilated by a suitable power of  $J$ . It is obvious that  $\text{Fl } \Gamma$  is a Serre subcategory. The corresponding quotient category  $\text{Gr Mod } \Gamma / \text{Fl } \Gamma$ , which we are about to describe, is denoted  $\text{Tails } \Gamma$ .

- The objects of  $\text{Tails } \Gamma$  are just graded  $\Gamma$ -modules.
- The morphisms in  $\text{Tails } \Gamma$  are defined by the formula

$$\text{Hom}_{\text{Tails } \Gamma}(M, N) := \varinjlim \text{Hom}_{\text{Gr Mod } \Gamma}(M', N/N'),$$

where  $M'$  and  $N'$  are, respectively, submodules of  $M$  and  $N$  such that  $M/M'$  and  $N/N'$  are in  $\text{Fl}\Gamma$ , and  $(M', N') \leq (M'', N'')$  if and only if  $M'' \subseteq M'$  and  $N' \subseteq N''$ .

Thus a morphism in  $\text{Tails}\Gamma$  is represented by the slanted arrow in the following diagram with exact columns:

$$\begin{array}{ccc}
 0 & & 0 \\
 \downarrow & & \downarrow \\
 M' & & N' \\
 \downarrow & \searrow & \downarrow \\
 M & & N \\
 \downarrow & & \downarrow \\
 M/M' & & N/N' \\
 \downarrow & & \downarrow \\
 0 & & 0
 \end{array}$$

and with the southwest and northeast entries in  $\text{Fl}\Gamma$ . A graded module  $M$  viewed as an object of  $\text{Tails}\Gamma$  will be denoted  $\widetilde{M}$ .

It is a well-known result that the category  $\text{Tails}\Gamma$  (in fact, any quotient category) is abelian with enough injectives and that the canonical functor  $\pi : \text{Gr Mod}\Gamma \rightarrow \text{Tails}\Gamma$  is exact. As the next result shows ([4], Prop. 4.5), under mild assumption, the homological algebra in  $\text{Tails}\Gamma$  admits a simple description.

**Theorem 3.1** *Let  $\Gamma$  be a graded quiver algebra such that all graded simples are of type  $FP_\infty$ ,  $M$  a graded  $\Gamma$ -module of type  $FP_\infty$ , and  $N$  any graded  $\Gamma$ -module. Then  $\text{Ext}_{\text{Tails}\Gamma}^n(\widetilde{M}, \widetilde{N})$  is naturally isomorphic to  $\varinjlim \text{Ext}_{\text{Gr Mod}\Gamma}^n(M_{\geq k}, N)$  for any integer  $n \geq 0$ . If, in addition,  $M$  is generated in a single degree, then there is a natural isomorphism*

$$\text{Ext}_{\text{Tails}\Gamma}^n(\widetilde{M}, \widetilde{N}) \cong \varinjlim \text{Ext}_{\text{Gr Mod}\Gamma}^n(J^k M, N)$$

#### 4 Stable cohomology

Now we want to review (the graded version of) stable cohomology, a construction going back to R.-O. Buchweitz [1]. Let  $\Lambda$  be any graded ring and  $M, N \in \text{Gr Mod}\Lambda$ . Stable cohomology  $\underline{\text{Ext}}_{\text{Gr Mod}\Lambda}^p(M, N)$  is defined by the formula

$$\underline{\text{Ext}}_{\text{Gr Mod}\Lambda}^p(M, N) := \varinjlim_{k \geq \max\{-p, 0\}} \underline{\text{Hom}}_{\text{Gr Mod}\Lambda}(\Omega^{k+p} M, \Omega^k N).$$

In this formula the underlined  $\text{Hom}$  stands for morphisms in  $\text{Gr Mod}\Lambda$  modulo projectives. More precisely, two degree zero morphisms are in the same class modulo projectives if their difference is a composition of two homogeneous maps composed at a graded projective. This underlined  $\text{Hom}$ , viewed as a collection of morphisms, gives rise to a category, denoted  $\underline{\text{Gr Mod}}\Lambda$ , whose objects are graded  $\Lambda$ -modules. As was observed by Eckmann - Hilton, this is an example of an abstract homotopy category, with the maps factoring through projectives as the class of null-homotopic maps. One of the advantages of this category is that the operation  $\Omega$  becomes a functor. Consequently, it gives rise to a well-defined directed system in the right-hand side of the above formula, whose limit is taken as the definition of stable cohomology.

It is immediate from that definition that stable cohomology admits a downward dimension shift in the contravariant argument and an upward dimension shift in

the covariant argument. Moreover, as attaching equal amounts of extra  $\Omega$ 's to both variables does not change the limit, the new cohomology admits *arbitrary* dimension shifts in both directions.

**Example 4.1** Assume that  $\Lambda$  is noetherian and selfinjective. In this case  $\Omega$  is an autoequivalence of the full subcategory of  $\underline{\text{Gr Mod}}\Lambda$  consisting of finitely generated modules. For such modules  $M$  and  $N$ , this immediately identifies the degree zero part of stable cohomology:

$$\underline{\text{Ext}}_{\text{Gr Mod } \Lambda}^0(M, N) \cong \underline{\text{Hom}}_{\text{Gr Mod } \Lambda}(M, N).$$

Using dimension shifts we can identify the remaining cohomology groups:

$$\underline{\text{Ext}}_{\text{Gr Mod } \Lambda}^i(M, N) \cong \underline{\text{Hom}}_{\text{Gr Mod } \Lambda}(\Omega^i M, N)$$

and

$$\underline{\text{Ext}}_{\text{Gr Mod } \Lambda}^{-i}(M, N) \cong \underline{\text{Hom}}_{\text{Gr Mod } \Lambda}(M, \Omega^i N),$$

where in both formulas  $i \geq 1$ . Furthermore, regardless of the properties of  $\Lambda$ , the canonical surjection  $\text{Hom}_{\text{Gr Mod } \Lambda}(\Omega M, N) \rightarrow \underline{\text{Hom}}_{\text{Gr Mod } \Lambda}(\Omega M, N)$  factors through the canonical surjection  $\text{Hom}_{\text{Gr Mod } \Lambda}(\Omega M, N) \rightarrow \text{Ext}_{\text{Gr Mod } \Lambda}^1(M, N)$ , giving rise to the canonical natural surjection

$$\text{Ext}_{\text{Gr Mod } \Lambda}^1(M, N) \rightarrow \underline{\text{Hom}}_{\text{Gr Mod } \Lambda}(\Omega M, N).$$

As  $\Lambda$  is selfinjective, it is easy to check that this map is in fact an isomorphism. Thus we have a natural isomorphism

$$\underline{\text{Ext}}_{\text{Gr Mod } \Lambda}^i(M, N) \cong \text{Ext}_{\text{Gr Mod } \Lambda}^i(M, N)$$

for each  $i \geq 1$ . Since  $M$  is finitely presented, there is a natural isomorphism  $\underline{\text{Hom}}_{\text{Gr Mod } \Lambda}(M, N) \cong \text{Tor}_1^{\text{Gr Mod } \Lambda}(\text{Tr } M, N)$ . As a result, we have a natural isomorphism

$$\underline{\text{Ext}}_{\text{Gr Mod } \Lambda}^{-i}(M, N) \cong \text{Tor}_{i+1}^{\text{Gr Mod } \Lambda}(\text{Tr } M, N)$$

for each  $i \geq 0$ .

## 5 Koszul duality, cohomology of tails, and stable cohomology

We can now state the main result of the paper.

**Theorem 5.1** *Let  $\Gamma$  be a Koszul quiver algebra with Yoneda algebra  $\Lambda$ ,  $M$  and  $N$  Koszul  $\Gamma$ -modules, and  $F : K_\Gamma \rightarrow K_\Lambda$  the Koszul duality between the categories of Koszul modules. Then there is a natural isomorphism*

$$\text{Ext}_{\text{Tails } \Gamma}^n(\widetilde{M}, \widetilde{N}[p]) \cong \underline{\text{Ext}}_{\text{Gr Mod } \Lambda}^{n+p}(F(N)[p], F(M))$$

for any integer  $p$  and any integer  $n$ .

In colloquial terms, Koszul duality interchanges cohomology of tails and stable cohomology. Notice however the nontrivial change in the cohomological degrees. As the theorem shows, this is the result of the degree disparity in the input variables (recall that Koszul modules are, by definition, generated in degree zero). The main ingredient of the proof of the above result is Th. 2.5, (viii). It shows that Koszul duality interchanges  $J$  and  $\Omega$ . But the two cohomology theories in questions are defined in a similar manner, using a limiting process, except one uses  $J$  and the other  $\Omega$ .

## 6 Noncommutative Serre duality

The purpose of this section is to provide, using Koszul duality, a new approach to noncommutative Serre duality, whose existence was established earlier by the first author ([4], Th. 4.9) in a more general setting. We believe that the approach outlined here is more transparent and conceptual. We begin by recalling

**Definition 6.1** A (graded) quiver algebra  $\Gamma$  is said to be generalized Artin - Schelter regular of dimension  $n$  if:

1. All graded simples are of type  $FP_\infty$  and of projective dimension  $n$ .
2. If  $S$  is a graded simple, then  $\text{Ext}_\Gamma^k(S, \Gamma) = 0$  for  $0 \leq k \leq n - 1$ .
3. The functor  $\text{Ext}_\Gamma^n(-, \Gamma)$  induces a bijection between the isomorphism classes of graded simple  $\Gamma$ - and graded simple  $\Gamma^{op}$ - modules.

The importance of this class of algebras is explained by the following result ([4], Th. 3.1).

**Theorem 6.2** *Let  $\Lambda$  be an indecomposable finite-dimensional Koszul quiver algebra of Loewy length  $n$ . Then  $\Lambda$  is selfinjective if and only if its Yoneda algebra is generalized Artin - Schelter regular of dimension  $n$ .*

Let  $\Lambda$  be a selfinjective quiver algebra of Loewy length  $n$ . Being basic, it is Frobenius. Choose a Nakayama automorphism  $\nu : \Lambda \rightarrow \Lambda$  and let  $\mathcal{N}() := D(*)[-n]$  be the graded Nakayama functor. Here, for any graded  $\Lambda$ -module  $M$ , the modules  $D(M) := \text{Hom}_{\mathbb{k}}(M, \mathbb{k})$  and  $M^* := \text{Hom}_\Lambda(M, \Lambda)$  inherit their graded structures from their arguments.

**Lemma 6.3**  *$\mathcal{N}(M) \cong^\nu M$  for any locally finite module  $M$ , where  ${}^\nu M$  is the twist of  $M$  by the Nakayama automorphism  $\nu$  (i.e.,  $M = {}^\nu M$  as sets and  $\lambda \cdot m := \nu(\lambda)m$  for  $m \in M$ ). In particular, the restriction of the Nakayama functor to the subcategory of locally finite modules is an autoequivalence.*

Let  $\Gamma$  be a generalized Artin - Schelter regular Koszul quiver algebra of dimension  $n$  with Yoneda algebra  $\Lambda$ . We can now state the promised noncommutative Serre duality theorem.

**Theorem 6.4** *There exists an algebra automorphism  $\sigma : \Gamma \rightarrow \Gamma$  such that, for any  $\Gamma$ -modules  $M$  and  $N$  of type  $FP_\infty$  and any  $i$  with  $0 \leq i \leq n - 1$ , there is a natural isomorphism*

$$D(\text{Ext}_{\text{Tails } \Gamma}^{n-1-i}(\widetilde{N}, {}^\sigma \widetilde{M}[-n])) \cong \text{Ext}_{\text{Tails } \Gamma}^i(\widetilde{M}, \widetilde{N})$$

We want to sketch the proof of the theorem. By Koszul duality,  $\Gamma$  is the Yoneda algebra of  $\Lambda$ . Thus the inverse of the Nakayama automorphism  $\nu^{-1} : \Lambda \rightarrow \Lambda$  induces an automorphism  $\sigma : \Gamma \rightarrow \Gamma$  such that  ${}^{\nu^{-1}} F(M) \cong F(\sigma M)$  for any Koszul  $\Gamma$ -module  $M$ . In the quotient category  $\text{Tails } \Gamma$  any two (locally finite) modules that differ in only finitely many degrees are (canonically and naturally) isomorphic. Thus the modules  $M$  and  $N$  can be truncated. As  $\Gamma$  is of finite global dimension, Th. 2.7 shows that we may assume that  $M$  and  $N$  are shifts of Koszul modules. Any truncation of (a shift of) a Koszul module is again a shift of a Koszul module. This follows from the fact that a Koszul module is generated in a single degree, thus making truncation equivalent to multiplying by a power of the radical  $J$ , and from Th. 2.5, (vii). Thus we may assume that  $M$  and  $N$  are shifts of Koszul modules generated in a single degree. Therefore, shifting both modules to degree zero, we may assume that both modules are Koszul. After applying Th. 5.1 and taking into

account that  $F(\sigma M) \cong \nu^{-1} F(M)$ , the problem is now reduced to establishing the following natural isomorphism:

$$D(\underline{\text{Ext}}_{\text{Gr Mod } \Lambda}^{-i-1}(\nu^{-1} F(M)[-n], F(N))) \cong \underline{\text{Ext}}_{\text{Gr Mod } \Lambda}^i(F(N), F(M)).$$

It turns out that this assertion follows from a more general result, in which the Koszul assumption is no longer needed. The precise statement is captured in the next lemma.

**Lemma 6.5** *Let  $\Lambda$  be a graded Frobenius algebra of Loewy length  $n$ , and  $A$  and  $B$  finitely generated graded  $\Lambda$ -modules. Then, for each integer  $i$ , there is a natural isomorphism*

$$D(\underline{\text{Ext}}_{\text{Gr Mod } \Lambda}^{-i-1}(\nu^{-1} A[-n], B)) \cong \underline{\text{Ext}}_{\text{Gr Mod } \Lambda}^i(B, A) \quad (6.1)$$

As Example 4.1 shows, the left-hand side of the above formula is naturally isomorphic to  $D(\text{Tor}_{i+1}^{\text{Gr Mod } \Lambda}(\Omega \text{Tr}^{\nu^{-1}} A, B[n]))$ . From this point, only standard tools are needed to reach the right-hand side. Those tools are: the duality between the functors  $\text{Tor}$  and  $\text{Ext}$ , the duality  $D$  and its (stably) functorial relations  $D\Omega \cong \Omega^{-1}D$  and  $D \text{Tr} \cong \Omega^2 \mathcal{N}[n]$ , and the fact that the Nakayama equivalence comes from the Nakayama automorphism (Lemma 6.3).

**Remark 6.6** One may call (6.1) the Serre duality for Frobenius algebras.

**Remark 6.7** It was pointed out to us by Idun Reiten, that (6.1) is equivalent to (the graded version of) the Auslander - Reiten formula:

$$D(\text{Ext}_{\text{Gr Mod } \Lambda}^1(M, D \text{Tr } N)) \cong \underline{\text{Hom}}_{\text{Gr Mod } \Lambda}(N, M)$$

To see that, use a dimension shift on the left-hand side of (6.1) and rewrite it as  $D(\underline{\text{Ext}}_{\text{Gr Mod } \Lambda}^1(\nu^{-1} A[-n], \Omega^{i+2} B))$ . By Example 4.1, over a selfinjective algebra, stable cohomology is isomorphic to the usual cohomology in positive degrees. This makes the left-hand side into  $D(\text{Ext}_{\text{Gr Mod } \Lambda}^1(\nu^{-1} A[-n], \Omega^{i+2} B))$ . Applying the Nakayama automorphism to both arguments and using Lemma 6.3, we have  $D(\text{Ext}_{\text{Gr Mod } \Lambda}^1(A[-n], \mathcal{N}\Omega^{i+2} B))$ . As  $\Lambda$  is selfinjective,  $\mathcal{N}\Omega^2$  is stably isomorphic to, and thus can be replaced by,  $D \text{Tr}[-n]$  in the covariant argument of  $\text{Ext}_{\text{Gr Mod } \Lambda}^1$ . We now have  $D(\text{Ext}_{\text{Gr Mod } \Lambda}^1(A[-n], D \text{Tr } \Omega^i B)[-n])$ . Shifting both arguments by  $n$  and using the Auslander - Reiten formula, we have  $\underline{\text{Hom}}_{\text{Gr Mod } \Lambda}(\Omega^i B, A)$ . Using Example 4.1 again, this is  $\underline{\text{Ext}}_{\text{Gr Mod } \Lambda}^i(B, A)$ .

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