

# ABSOLUTE, RELATIVE, AND TATE COHOMOLOGY OF MODULES OF FINITE GORENSTEIN DIMENSION

LUCHEZAR L. AVRAMOV AND ALEX MARTSINKOVSKY

*To Idun Reiten on her 60th birthday*

## INTRODUCTION

We study finite modules  $M$  over a noetherian (on both sides) ring  $R$ .

Auslander and Bridger [2] singled out the class of  $R$ -modules of finite Gorenstein dimension (or G-dimension) as a common generalization of modules of finite projective dimension over arbitrary rings, and arbitrary modules over commutative Gorenstein rings. Projective modules always have G-dimension 0. Over a Gorenstein ring the modules of G-dimension 0 are precisely the maximal Cohen-Macaulay modules; their central position in the category of finite  $R$ -modules is emphasized by the maximal Cohen-Macaulay approximations of Auslander and Buchweitz [3].

Classically, G-dimension is studied by means of functors  $\text{Ext}_R^n(M, \_)$ . Our approach is based on the interaction of this *absolute cohomology* theory with two other theories that are naturally defined on the category of modules of finite G-dimension.

The *relative cohomology* functors  $\text{Ext}_G^n(M, \_)$  treat modules of G-dimension zero as projectives. Their vanishing defines a numerical invariant, the relative dimension  $\text{reldim}_G M$ , which refines the classical projective dimension:  $\text{reldim}_G M \leq \text{proj dim}_R M$ , with equality when the projective dimension is finite.

At the other extreme, the *Tate cohomology* functors  $\widehat{\text{Ext}}_R^n(M, \_)$  capture only those properties of  $M$  that are shared by all its syzygy modules. This theory is rigid, in the sense that vanishing of any one of these functors implies the vanishing of all of them; vanishing characterizes modules of finite projective dimension.

Relative and Tate functors come equipped with natural transformations,

$$\varepsilon_G^n: \text{Ext}_G^n \rightarrow \text{Ext}_R^n \quad \text{and} \quad \varepsilon_R^n: \text{Ext}_R^n \rightarrow \widehat{\text{Ext}}_R^n.$$

Our main result shows that, in all degrees, they provide remarkably tight connections between the three theories: There exists an exact sequence of functors

$$0 \longrightarrow \text{Ext}_G^1 \xrightarrow{\varepsilon_G^1} \text{Ext}_R^1 \longrightarrow \cdots \longrightarrow \text{Ext}_G^n \xrightarrow{\varepsilon_G^n} \text{Ext}_R^n \xrightarrow{\varepsilon_R^n} \widehat{\text{Ext}}_R^n \xrightarrow{\delta_R^n} \text{Ext}_G^{n+1} \longrightarrow \cdots$$

This is new even for maximal Cohen-Macaulay modules over Gorenstein local rings.

In Section 1 we summarize some basic facts about complexes of modules, used throughout the paper, and introduce an intrinsic notion of minimality for complexes. Sections 2 and 3 provide a concise treatment of the classical aspects of finite G-dimension developed in [2]. Unlike the module-theoretic point of view of

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the original and of the recent systematic expositions of Christensen [13] and Mašek [29], our approach is based on explicit constructions of complexes.

The next two sections contain elementary, self-contained constructions of the relative and Tate theories. We emphasize the parallelism of their properties and provide essentially parallel arguments, some of which may be new.

The relative cohomology in Section 4 is ‘relative homological algebra’ in the sense of Eilenberg and Moore [16], and MacLane [25], with respect to the class of modules of G-dimension 0. These sources are mostly concerned with existential aspects of the theory, while we focus on hands-on constructions of proper resolutions. As an application we recover properties of modules of finite G-dimension, due to Auslander and Bridger [6]. Also, we establish the equality of relative and Gorenstein dimensions, proved over commutative Gorenstein rings by Enochs and Jenda [19].

Tate cohomology, introduced through complete resolutions, has been the subject of several recent expositions, in particular by Buchweitz [11] and Cornick and Kropholler [14]. However, we could not find in the literature an adequate treatment of two essential points: functoriality in the contravariant argument and naturality of comparison maps. To tackle them, in Section 5 we incorporate into the notion of complete resolution a comparison morphism to a projective resolution. Such a device might prove useful in other contexts as well.

For the special class of G-perfect modules, defined in [2], relative and Tate cohomology groups have functorial comparisons with appropriate absolute *homology* groups: this is the topic of Section 6. In Section 7 we derive the exact sequence above, involving the comparison morphisms. In both sections the arguments heavily rely on explicit constructions developed earlier in the paper.

In Section 8 we produce and characterize minimal resolutions over commutative local rings. We prove that in all theories ‘minimal’ constructions yield minimal resolutions, and that our general concept of minimality specializes to a variety of notions, introduced earlier in the Gorenstein case in more or less *ad hoc* ways.

In the final two sections we test the techniques developed in the paper. For finite modules over a commutative local ring we consider relative and Tate versions, both of Betti numbers and of Bass numbers. Exploring the relation of these invariants with their absolute prototypes and with Auslander’s ‘delta invariants’, we uncover patterns that may be rather surprising from an absolutist perspective.

## 1. COMPLEXES. MINIMALITY

Let  $R$  be an *associative ring* and  $\mathcal{M} = \mathcal{M}(R)$  the *category of left  $R$ -modules*.

Right  $R$ -modules are treated as left modules over the opposite ring,  $R^o$ . For any left or right module  $M$ , the dual  $\text{Hom}_R(M, R)$  is equipped with the canonical action on the other side, and  $(\ )^*$  denotes either of the dualization functors

$$\text{Hom}_R(\ , R): \mathcal{M}(R)^{\text{op}} \longrightarrow \mathcal{M}(R^o) \quad \text{or} \quad \text{Hom}_{R^o}(\ , R): \mathcal{M}(R^o)^{\text{op}} \longrightarrow \mathcal{M}(R).$$

Complexes have differentials of degree  $-1$ , and are often displayed as sequences

$$\mathbf{B} = \cdots \longrightarrow B_{n+1} \xrightarrow{\partial_{n+1}^{\mathbf{B}}} B_n \xrightarrow{\partial_n^{\mathbf{B}}} B_{n-1} \longrightarrow \cdots$$

The *dual complex* of  $\mathbf{B}$  is the complex of  $R^o$ -modules with  $(B_{-n})^*$  in degree  $n$ :

$$\mathbf{B}^* = \cdots \longrightarrow (B_{-n-1})^* \xrightarrow{(\partial_{-n}^{\mathbf{B}})^*} (B_{-n})^* \xrightarrow{(\partial_{-n+1}^{\mathbf{B}})^*} (B_{-n+1})^* \longrightarrow \cdots$$

We identify a module  $M$  and the complex with  $M$  in degree zero and 0 elsewhere.

Let  $i$  be an integer. The  $i$ th *syzygy module* of  $\mathbf{B}$  is  $\Omega^i \mathbf{B} = \text{Coker } \partial_{i+1}^{\mathbf{B}}$ . The  $i$ th *shift* of  $\mathbf{B}$  is the complex  $\Sigma^i \mathbf{B}$  with  $n$ th component equal to  $B_{n-i}$  and  $\partial_n^{\Sigma^i \mathbf{B}} = (-1)^i \partial_{n-i}^{\mathbf{B}}$ . We write  $\mathbf{B}_{<i}$  or  $\mathbf{B}_{\leq i-1}$  for the subcomplex of  $\mathbf{B}$  with  $n$ th component equal to  $B_n$  for  $n < i$  and to 0 for  $n \geq i$ , and set  $\mathbf{B}_{\geq i} = \mathbf{B}_{>i-1} = \mathbf{B}/\mathbf{B}_{<i}$ .

A *homomorphism*  $\beta: \mathbf{B} \rightarrow \mathbf{C}$  of degree  $i$  is a sequence of  $R$ -linear maps  $\beta_n: B_n \rightarrow C_{n+i}$  for  $n \in \mathbb{Z}$ . All homomorphisms of degree  $i$  form an abelian group, denoted  $\text{Hom}_R(\mathbf{B}, \mathbf{C})_i$ , that we identify with  $\prod_{n \in \mathbb{Z}} \text{Hom}_R(B_n, C_{n+i})$ . It appears as the  $i$ th component of a complex  $\text{Hom}_R(\mathbf{B}, \mathbf{C})$  of abelian groups, with differential

$$\partial(\beta_n) = (\partial_{n+i}^{\mathbf{C}} \beta_n - (-1)^i \beta_{n-1} \partial_n^{\mathbf{B}}) \quad \text{for } \beta = (\beta_n) \in \text{Hom}_R(\mathbf{B}, \mathbf{C})_i.$$

For any  $i \in \mathbb{Z}$ , the cycles in  $\text{Hom}_R(\mathbf{B}, \mathbf{C})_i$  are the *chain maps*  $\mathbf{B} \rightarrow \mathbf{C}$  of degree  $i$ . Chain maps  $\beta, \beta': \mathbf{B} \rightarrow \mathbf{C}$  are in the same homology class, denoted  $\beta \sim \beta'$ , if and only if they are *homotopic*. As usual, we set  $\text{H}^i \text{Hom}_R(\mathbf{B}, \mathbf{C}) = \text{H}_{-i} \text{Hom}_R(\mathbf{B}, \mathbf{C})$ .

A *morphism* of complexes is a chain map of degree 0. A *quasiisomorphism* is a morphism  $\beta: \mathbf{B} \rightarrow \mathbf{C}$  with  $\text{H}_n(\beta): \text{H}_n(\mathbf{B}) \rightarrow \text{H}_n(\mathbf{C})$  bijective for all  $n$ ; if such a  $\beta$  is surjective, then each cycle of  $\mathbf{C}$  is the image of some cycle of  $\mathbf{B}$ . Since homology classes in  $\text{H}_0 \text{Hom}_R(\mathbf{B}, \mathbf{C})$  represent homotopy classes of morphisms, the *Lifting Lemma* and *Extension Lemma* below are obtained by unraveling definitions.

**1.1.** Let  $\beta: \mathbf{B} \rightarrow \mathbf{C}$  be a morphism of complexes.

- (1) Let  $\mathbf{P}$  be a complex such that  $\text{Hom}_R(\mathbf{P}, \beta): \text{Hom}_R(\mathbf{P}, \mathbf{B}) \rightarrow \text{Hom}_R(\mathbf{P}, \mathbf{C})$  is a quasiisomorphism. For each morphism  $\gamma: \mathbf{P} \rightarrow \mathbf{C}$  there is a morphism  $\alpha: \mathbf{P} \rightarrow \mathbf{B}$  with  $\gamma \sim \beta \alpha$  (even  $\gamma = \beta \alpha$  if  $\text{Hom}_R(\mathbf{P}, \beta)$  is surjective). If  $\gamma': \mathbf{P} \rightarrow \mathbf{C}$  and  $\alpha': \mathbf{P} \rightarrow \mathbf{B}$  satisfy  $\gamma' \sim \gamma$  and  $\gamma' \sim \beta \alpha'$ , then  $\alpha' \sim \alpha$ .
- (2) Let  $\mathbf{I}$  be a complex such that  $\text{Hom}_R(\beta, \mathbf{I}): \text{Hom}_R(\mathbf{C}, \mathbf{I}) \rightarrow \text{Hom}_R(\mathbf{B}, \mathbf{I})$  is a quasiisomorphism. For each morphism  $\alpha: \mathbf{B} \rightarrow \mathbf{I}$  there exists a morphism  $\gamma: \mathbf{P} \rightarrow \mathbf{B}$  with  $\gamma \beta \sim \alpha$  (even  $\gamma \beta = \alpha$  if  $\text{Hom}_R(\beta, \mathbf{I})$  is surjective). If  $\alpha': \mathbf{B} \rightarrow \mathbf{I}$  and  $\gamma': \mathbf{C} \rightarrow \mathbf{I}$  satisfy  $\alpha' \sim \alpha$  and  $\gamma' \beta \sim \alpha'$ , then  $\gamma' \sim \gamma$ .

This is often applied by means of a standard *Comparison Lemma*. A complex  $\mathbf{A}$  is *bounded below* (respectively, *above*) if  $A_i = 0$  for all  $i \ll 0$  (respectively,  $i \gg 0$ ).

**1.2.** Let  $\beta: \mathbf{B} \rightarrow \mathbf{C}$  be a quasiisomorphism of complexes of  $R$ -modules.

- (1) If  $\mathbf{P}$  is a bounded below complex of projective  $R$ -modules, then  $\text{Hom}_R(\mathbf{P}, \beta)$  is a quasiisomorphism; it is surjective when  $\beta$  is surjective.
- (2) If  $\mathbf{I}$  is a bounded above complex of injective  $R$ -modules, then  $\text{Hom}_R(\beta, \mathbf{I})$  is a quasiisomorphism; it is surjective when  $\beta$  is injective.

The fact that the induced maps are quasiisomorphisms can be proved along the lines of Lemma 4.3, using a simple *Mittag-Leffler Criterion* for acyclicity:

**1.3.** Let  $\pi^{(i)}: \mathbf{A}^{(i)} \rightarrow \mathbf{A}^{(i-1)}$  be surjective morphisms of complexes, for  $i \geq 1$ .

If  $\text{H}_n(\mathbf{A}^{(i)}) = 0$  for all  $i \geq 0$  and all  $n \in \mathbb{Z}$ , then  $\text{H}_n(\varprojlim \mathbf{A}^{(i)}) = 0$  for all  $n \in \mathbb{Z}$ .

Indeed, the definition of inverse limit yields a sequence of morphisms

$$0 \rightarrow \varprojlim \mathbf{A}^{(i)} \rightarrow \prod_{i \geq 0} \mathbf{A}^{(i)} \xrightarrow{\lambda} \prod_{i \geq 0} \mathbf{A}^{(i)}$$

$$\text{where } \lambda(a^{(i)}) = (a^{(i)} - \pi^{(i+1)}(a^{(i+1)})).$$

The surjectivity of each  $\pi^{(i)}$  implies that  $\lambda$  is surjective. Since  $\text{H}_n(\prod_{i \geq 0} \mathbf{A}^{(i)}) = \prod_{i \geq 0} \text{H}_n(\mathbf{A}^{(i)}) = 0$ , the homology exact sequence yields  $\text{H}_n(\varprojlim \mathbf{A}^{(i)}) = 0$ .

A *resolution* (of length  $\leq q$ ) of a module  $M$  is a quasiisomorphism  $\gamma: \mathbf{G} \rightarrow M$  with  $G_n = 0$  for all  $n < 0$  (and  $G_n = 0$  for  $n > q$ ); it has an *associated exact sequence*

$$\mathbf{G}^+ = \cdots \rightarrow G_n \xrightarrow{\partial_n} G_{n-1} \rightarrow \cdots \rightarrow G_1 \xrightarrow{\partial_1} G_0 \xrightarrow{\gamma} M \rightarrow 0.$$

The next classical result [12, V §2] is sometimes called the *Horseshoe Lemma*.

**1.4.** Let  $0 \rightarrow M \xrightarrow{\mu} M' \xrightarrow{\mu'} M'' \rightarrow 0$  be an exact sequence of  $R$ -modules.

Given projective resolutions  $\pi: \mathbf{P} \rightarrow M$  and  $\pi'': \mathbf{P}'' \rightarrow M''$ , there exists a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{\mu} & M' & \xrightarrow{\mu'} & M'' & \longrightarrow & 0 \\ & & \uparrow \pi & & \uparrow \pi' & & \uparrow \pi'' & & \\ 0 & \longrightarrow & \mathbf{P} & \xrightarrow{\bar{\mu}} & \mathbf{P}' & \xrightarrow{\bar{\mu}'} & \mathbf{P}'' & \longrightarrow & 0 \end{array}$$

Given a commutative diagram of the form above with bounded below complexes of projective modules  $\mathbf{P}'$ ,  $\mathbf{P}'$ , and  $\mathbf{P}''$ , a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{\tilde{\mu}} & M' & \xrightarrow{\tilde{\mu}'} & M'' & \longrightarrow & 0 \\ & & \uparrow \gamma & & \uparrow \gamma' & & \uparrow \gamma'' & & \\ 0 & \longrightarrow & \mathbf{G} & \xrightarrow{\mu} & \mathbf{G}' & \xrightarrow{\mu'} & \mathbf{G}'' & \longrightarrow & 0 \end{array}$$

with bottom row split exact in each degree and a quasiisomorphism  $\gamma$ , a morphism  $\phi: \mathbf{P} \rightarrow \mathbf{G}$  with  $\gamma\phi = \pi$ , and a morphism  $\phi'': \mathbf{P}'' \rightarrow \mathbf{G}''$  with  $\gamma''\phi'' = \pi''$ , there exists a morphism  $\phi'$  with  $\gamma'\phi' = \pi'$  making the following diagram commute

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{G} & \xrightarrow{\tilde{\mu}} & \mathbf{G}' & \xrightarrow{\tilde{\mu}'} & \mathbf{G}'' & \longrightarrow & 0 \\ & & \uparrow \phi & & \uparrow \phi' & & \uparrow \phi'' & & \\ 0 & \longrightarrow & \mathbf{P} & \xrightarrow{\bar{\mu}} & \mathbf{P}' & \xrightarrow{\bar{\mu}'} & \mathbf{P}'' & \longrightarrow & 0 \end{array}$$

If  $\gamma: \mathbf{C} \rightarrow \mathbf{B}$  and  $\beta: \mathbf{B} \rightarrow \mathbf{C}$  are morphisms with  $\beta\gamma \sim \text{id}_{\mathbf{C}}$ , then  $\gamma$  is a *right homotopy inverse* of  $\beta$ , and  $\beta$  is a *left homotopy inverse* of  $\gamma$ . A *homotopy equivalence* is a morphism  $\gamma$  having a left homotopy inverse  $\beta$ , which itself has a left homotopy inverse  $\gamma'$ ; in this case  $\gamma\beta = \text{id}_{\mathbf{B}}$ ,  $\gamma\beta \sim \gamma'\beta\gamma\beta \sim \gamma'\text{id}_{\mathbf{C}}\beta = \gamma'\beta \sim \text{id}_{\mathbf{B}}$  so that  $\gamma$  and  $\beta$  are *homotopy inverse* morphisms. The same conclusion holds if  $\gamma$  has a right homotopy inverse  $\beta'$  which itself has a right homotopy inverse  $\gamma''$ .

Well known sources of homotopy equivalences come from 1.2 and 1.1:

**1.5.** A quasiisomorphism  $\beta: \mathbf{B} \rightarrow \mathbf{C}$  is a homotopy equivalence if

- (1)  $B_n$  and  $C_n$  are projective for all  $n$ , and vanish for all  $n \ll 0$ ; or
- (2)  $B_n$  and  $C_n$  are injective for all  $n$ , and vanish for all  $n \gg 0$ .

A complex  $\mathbf{A}$  *contractible* if  $\text{id}_{\mathbf{A}} \sim 0_{\mathbf{A}}$ . A subcomplex  $\mathbf{A}$  of a complex  $\mathbf{B}$  is *irrelevant* if  $\mathbf{A}$  is contractible and  $A_n$  is a direct summand of  $B_n$  for each  $n \in \mathbb{Z}$ .

**1.6. Lemma.** *If  $\mathbf{A}$  is an irrelevant subcomplex of a complex  $\mathbf{B}$  and  $\mathbf{C} = \mathbf{B}/\mathbf{A}$ , then the canonical morphism  $\beta: \mathbf{B} \rightarrow \mathbf{C}$  is a homotopy equivalence.*

*Proof.* The exact sequence  $0 \rightarrow \mathbf{A} \rightarrow \mathbf{B} \xrightarrow{\beta} \mathbf{C} \rightarrow 0$  induces exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_R(\mathbf{C}, \mathbf{A}) & \longrightarrow & \text{Hom}_R(\mathbf{C}, \mathbf{B}) & \xrightarrow{\text{Hom}_R(\mathbf{C}, \beta)} & \text{Hom}_R(\mathbf{C}, \mathbf{C}) \longrightarrow 0 \\ 0 & \longrightarrow & \text{Hom}_R(\mathbf{C}, \mathbf{B}) & \xrightarrow{\text{Hom}_R(\beta, \mathbf{B})} & \text{Hom}_R(\mathbf{B}, \mathbf{B}) & \longrightarrow & \text{Hom}_R(\mathbf{A}, \mathbf{B}) \longrightarrow 0 \end{array}$$

As  $\text{Hom}_R(\mathbf{C}, \mathbf{A})$  and  $\text{Hom}_R(\mathbf{A}, \mathbf{B})$  are contractible along with  $\mathbf{A}$ , their homology is trivial, so the maps induced by  $\beta$  are quasiisomorphisms. From 1.1 we get morphisms  $\gamma, \gamma': \mathbf{C} \rightarrow \mathbf{B}$  with  $\beta\gamma = \text{id}_{\mathbf{C}}$  and  $\gamma'\beta = \text{id}_{\mathbf{B}}$ .  $\square$

We introduce a concept of minimality, applicable to any complex: A complex  $\mathbf{B}$  is *minimal* if each homotopy equivalence  $\beta: \mathbf{B} \rightarrow \mathbf{B}$  is an isomorphism.

**1.7. Proposition.** *Let  $\mathbf{B}$  be a complex of  $R$ -modules.*

- (1) *The following conditions are equivalent.*
  - (i) *The complex  $\mathbf{B}$  is minimal.*
  - (ii) *Each morphism  $\beta: \mathbf{B} \rightarrow \mathbf{B}$  homotopic to  $\text{id}_{\mathbf{B}}$  is an isomorphism.*
  - (iii) *If  $\beta: \mathbf{B} \rightarrow \mathbf{C}$  and  $\gamma: \mathbf{C} \rightarrow \mathbf{B}$  are homotopy inverses, then  $\beta$  is injective,  $\gamma$  is surjective,  $\text{Ker } \gamma$  is contractible, and  $\mathbf{C} = \text{Im } \beta \oplus \text{Ker } \gamma$ .*
- (2) *Every homotopy equivalence between minimal complexes is an isomorphism.*
- (3) *If  $\mathbf{B}$  is minimal and  $\mathbf{A}$  is an irrelevant subcomplex, then  $\mathbf{A} = 0$ .*

*Proof.* (1) (i)  $\implies$  (ii). If  $\beta \sim \text{id}_{\mathbf{B}}$ , then  $\beta^2 \sim \beta \sim \text{id}_{\mathbf{B}}$ , so  $\beta$  is its own homotopy inverse. In particular,  $\beta$  is a homotopy equivalence, and so an isomorphism.

(ii)  $\implies$  (iii). The morphism  $\alpha = \gamma\beta: \mathbf{B} \rightarrow \mathbf{B}$  is homotopic to  $\text{id}_{\mathbf{B}}$ , and so is an isomorphism. Thus,  $\gamma' = \alpha^{-1}\gamma: \mathbf{C} \rightarrow \mathbf{B}$  is a homotopy equivalence with  $\gamma'\beta = \text{id}_{\mathbf{B}}$ , hence  $\mathbf{C} = \text{Im } \beta \oplus \mathbf{K}$  for  $\mathbf{K} = \text{Ker } \gamma' = \text{Ker } \gamma$ . In the exact sequence

$$0 \longrightarrow \text{Hom}_R(\mathbf{K}, \mathbf{K}) \longrightarrow \text{Hom}_R(\mathbf{K}, \mathbf{C}) \xrightarrow{\text{Hom}_R(\mathbf{K}, \gamma')} \text{Hom}_R(\mathbf{K}, \mathbf{B}) \longrightarrow 0$$

the map  $\text{Hom}_R(\mathbf{K}, \gamma')$  is a homotopy equivalence, and so a quasiisomorphism. It follows that  $\text{H Hom}_R(\mathbf{K}, \mathbf{K}) = 0$ , so the cycle  $\text{id}_{\mathbf{K}}$  is a boundary, that is,  $\text{id}_{\mathbf{K}} \sim 0_{\mathbf{K}}$ .

(iii)  $\implies$  (i) is clear.

(2) follows immediately from property (1.iii) of minimal complexes.

(3) If  $\mathbf{B}$  is minimal and  $\mathbf{A}$  is an irrelevant subcomplex, then  $\beta: \mathbf{B} \rightarrow \mathbf{B}/\mathbf{A}$  is a homotopy equivalence by Lemma 1.6, so  $\mathbf{A} = \text{Ker } \beta = 0$  by property (1.iii).  $\square$

**1.8. Example.** Let  $\mathbf{B}$  be a complex with  $B_n$  injective for all  $n$ ,  $B_n = 0$  for  $n > 0$ , and  $\text{H}_n(\mathbf{B}) = 0$  for  $n \neq 0$ . Using Proposition 1.7.1 and well known properties of injective envelopes, one sees that  $\mathbf{B}$  is minimal if and only if  $B_n$  is an injective envelope of  $\text{Ker}(\partial_n)$  for all  $n \leq 0$ . Thus, for ‘injective resolutions’ our concept of minimality agrees with the classical notion.

## 2. TOTAL REFLEXIVITY

For the rest of the paper  $R$  denotes a *left and right noetherian* ring.

Let  $G$  be a finite  $R$ -module. Recall that  $G$  is said to be *reflexive* if the canonical map  $\zeta^G: G \rightarrow G^{**}$  is bijective. We say that  $G$  is *totally reflexive*<sup>1</sup> if, in addition

$$(*) \quad \text{Ext}_R^n(G, R) = 0 = \text{Ext}_{R^\circ}^n(G^*, R) \quad \text{for all } n > 0.$$

*Problem.* Are the conditions defining total reflexivity independent?

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<sup>1</sup>Auslander and Bridger [2] introduced these modules and called them *modules of  $G$ -dimension zero*, in anticipation of their role in the construction of a homological dimension. The difficulty of using that name is quite obvious (the reader should try substituting ‘modules of projective dimension zero’ for ‘projective modules’). Another name, systematically used by Enochs and collaborators, cf. e.g. [19], is *Gorenstein projective modules*, referring to their role in a relative cohomology theory. We have opted for a name reflecting their intrinsic module-theoretic properties.

**2.1. Examples.** Let  $R$  be a commutative ring and  $G$  a finite  $R$ -module.

(1) If  $R$  is local Gorenstein, then the following are equivalent: (i)  $G$  is totally reflexive; (ii)  $G$  is *maximal Cohen-Macaulay*; (iii)  $\text{Ext}_R^n(G, R) = 0$  for all  $n > 0$ : cf. [30, (4.15)] for (iii)  $\implies$  (ii) and [10, (3.3.10.d)] for (ii)  $\implies$  (i).

(2) If  $R = \bigoplus_{i=0}^{\infty} R_i$  is graded with  $R_0$  a field,  $G$  is graded, and  $(*)$  holds, then the following are equivalent: (i)  $G$  is reflexive; (ii)  $\zeta^G$  is injective; (iii)  $\zeta^G$  is surjective.

Indeed, by the Hilbert-Serre Theorem, the Laurent series  $\sum_{i \in \mathbb{Z}} \text{rank}_{R_0}(G_i)t^i$  is the Laurent expansion around  $t = 0$  of a rational function  $H_G(t)$ . If  $N$  is a graded  $R$ -module such that  $\text{Ext}_R^n(G, N) = 0$  for  $n \gg 0$ , then [7, Theorem 1] yields

$$\sum_n (-1)^n H_{\text{Ext}_R^n(G, N)}(t) = \frac{H_G(t^{-1}) \cdot H_N(t)}{H_R(t^{-1})}.$$

Applying the formula to the pairs of modules  $(G, R)$  and  $(G^*, R)$ , we get

$$H_{G^*}(t) = \frac{H_G(t^{-1}) \cdot H_R(t)}{H_R(t^{-1})} \quad \text{and} \quad H_{G^{**}}(t) = \frac{H_{G^*}(t^{-1}) \cdot H_R(t)}{H_R(t^{-1})}.$$

hence  $H_{G^{**}}(t) = H_G(t)$ . Thus,  $\text{rank}_k G_n = \text{rank}_k G_n^{**}$  for each  $n \in \mathbb{Z}$ . Since  $\zeta^G$  is a degree 0 homomorphisms of graded vector spaces, our assertion follows.

We recall from [2] some basic properties of totally reflexive modules. Proofs are included for three reasons: not all statements appear in [2] in the form or generality needed here; some of the original arguments do not carry over to the non-commutative case; we are able to offer some substantial shortcuts.

The following full subcategories of  $\mathcal{M}(R)$  are of special interest in this paper:

- $\mathcal{F} = \mathcal{F}(R)$  whose objects are the finite  $R$ -modules.
- $\mathcal{G} = \mathcal{G}(R)$  whose objects are the totally reflexive  $R$ -modules.
- $\mathcal{P} = \mathcal{P}(R)$  whose objects are the finite projective  $R$ -modules.

The three lemmas that follow reproduce, and partly generalize, portions of [2, (4.12), (4.9), (3.11), (4.11)], sometimes with considerably shorter proofs.

**2.2. Lemma.** *If  $G$  is a totally reflexive  $R$ -module, then  $\text{Ext}_R^n(G, N) = 0$  for all  $n > 0$  and all  $R$ -modules  $N$  of finite projective dimension.*

*Proof.* As  $R$  is noetherian and  $M$  is finite, the functor  $\text{Ext}_R^n(M, \_)$  commutes with arbitrary direct sums, so  $\text{Ext}_R^n(M, Q) = 0$  for all  $n > 0$  when  $Q$  is projective. When  $1 \leq \text{proj dim}_R N = p < \infty$ , choose an exact sequence  $0 \rightarrow N' \rightarrow Q \rightarrow N \rightarrow 0$  with  $Q$  projective. By induction we have  $\text{Ext}_R^n(M, Q) = 0$  and  $\text{Ext}_R^{n+1}(M, N') = 0$  for all  $n > 0$ , so the cohomology exact sequence yields  $\text{Ext}_R^n(M, N) = 0$ .  $\square$

**2.3. Lemma.** *The category  $\mathcal{G}$  is closed under extensions and kernels of epimorphisms, and  $\mathcal{G}(R)$  contains  $\mathcal{P}(R)$ .*

*Proof.* The last assertion is elementary. Let  $\mathbf{E} = 0 \rightarrow E \rightarrow F \rightarrow G \rightarrow 0$  be an exact sequence with  $G \in \mathcal{G}$ ; we assume that one of  $E$  or  $F$  is in  $\mathcal{G}$ , and show that so is the other. The sequence  $\mathbf{E}^* = 0 \rightarrow G^* \rightarrow F^* \rightarrow E^* \rightarrow 0$  is exact because  $\text{Ext}_R^1(G, R) = 0$ . Thus, we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & E & \longrightarrow & F & \longrightarrow & G \longrightarrow 0 \\ & & \downarrow \zeta^E & & \downarrow \zeta^F & & \cong \downarrow \zeta^G \\ 0 & \longrightarrow & E^{**} & \longrightarrow & F^{**} & \longrightarrow & G^{**} \end{array}$$

By assumption  $\zeta^G$  and one of  $\zeta^E, \zeta^F$  is bijective. The Snake Lemma shows all three are bijective, so  $\mathbf{E}^{**}$  is exact. Since  $\text{Ext}_R^n(G, R) = 0$  for  $n > 0$ , the exact sequence  $\mathbf{E}$  yields  $\text{Ext}_R^n(F, R) \cong \text{Ext}_R^n(E, R)$  for each  $n > 0$ ; one of these groups vanishes by assumption, hence both do. As  $\text{Ext}_{R^\circ}^n(G^*, R) = 0$  for  $n > 0$ , the exact sequence  $\mathbf{E}^*$  shows that  $\text{Ext}_{R^\circ}^n(E^*, R) \rightarrow \text{Ext}_{R^\circ}^n(F^*, R)$  is bijective for  $n > 1$  and surjective for  $n = 1$ ; it is injective also for  $n = 1$  because  $\mathbf{E}^{**}$  is exact. By assumption, at least one of  $\text{Ext}_{R^\circ}^n(E^*, R)$  or  $\text{Ext}_{R^\circ}^n(F^*, R)$  vanishes, hence both do.  $\square$

A complex  $\mathbf{T}$  is *totally acyclic* if  $T_n \in \mathcal{P}$  and  $H_n(\mathbf{T}) = 0 = H_n(\mathbf{T}^*)$  for all  $n \in \mathbb{Z}$ .

**2.4. Lemma.** *Let  $\mathbf{T}$  be a complex with  $T_n \in \mathcal{P}$  and  $H_n(\mathbf{T}) = 0$  for all  $n \in \mathbb{Z}$ .*

*The following conditions are equivalent.*

- (i)  $H_n(\mathbf{T}^*) = 0$  for all  $n \in \mathbb{Z}$ .
- (ii)  $\Omega^n \mathbf{T}$  is totally reflexive for each  $n \in \mathbb{Z}$ .
- (iii)  $H_n \text{Hom}_R(\mathbf{T}, N) = 0$  for all  $n \in \mathbb{Z}$  and each  $N \in \mathcal{M}$  with  $\text{proj dim}_R N < \infty$ .

*Proof.* (i)  $\implies$  (ii). Fix  $n \in \mathbb{Z}$  and set  $G = \Omega^n \mathbf{T}$ . As  $\Sigma^{-n} \mathbf{T}_{\geq n}$  is a  $\mathcal{P}$ -resolution of  $G$ , we have  $\text{Ext}_R^i(G, R) \cong H_{-i-n}(\mathbf{T}^*) = 0$  for  $i > 0$  and  $G^* \cong \text{Ker}(\partial_{n+1}^{\mathbf{T}})^* = \text{Ker}(\partial_{-n}^{\mathbf{T}^*})$ . As  $\mathbf{T}^*$  is exact,  $\text{Ker}(\partial_{-n}^{\mathbf{T}^*}) = \Omega^{-n+1} \mathbf{T}^*$  and  $\Sigma^{n-1} \mathbf{T}_{\geq -n+1}^*$  is a  $\mathcal{P}$ -resolution of  $G^*$ . Since  $\mathbf{T}^{**} = \mathbf{T}$  is exact, we get  $\text{Ext}_R^i(G^*, R) \cong H_{i-n+1}(\mathbf{T}^{**}) = 0$  for  $i > 0$  and  $G^{**} \cong \text{Ker}(\partial_{-n+2}^{\mathbf{T}^*})^* = \text{Ker}(\partial_{n-1}^{\mathbf{T}^{**}}) = \text{Ker}(\partial_{n-1}^{\mathbf{T}}) = G$ .

(ii)  $\implies$  (iii). For each  $n$  the complex  $\Sigma^{-n+1} \mathbf{T}_{\geq n-1}$  is a  $\mathcal{P}$ -resolution of  $\Omega^{n-1} \mathbf{T}$ , so  $H_{-n} \text{Hom}_R(\mathbf{T}, N) \cong \text{Ext}_R^1(\Omega^{n-1} \mathbf{T}, N)$ ; this module vanishes by Lemma 2.2.

(iii)  $\implies$  (i) is clear.  $\square$

If  $\mathcal{C}$  is a full subcategory of  $\mathcal{M}$ , then a resolution  $\mathbf{G} \rightarrow M$  is called a  $\mathcal{C}$ -resolution if  $G_n$  belongs to  $\mathcal{C}$  for all  $n \in \mathbb{Z}$ .

The next result is [6, (3.13)], with a new proof we learned from Idun Reiten.

**2.5. Lemma.** *If  $M$  has a  $\mathcal{G}$ -resolution  $\mathbf{G} \rightarrow M$  of length  $\leq g$ , then in each  $\mathcal{P}$ -resolution  $\mathbf{P} \rightarrow M$  the module  $\Omega^n \mathbf{P}$  is totally reflexive for all  $n \geq g$ .*

*Proof.* Choose a comparison morphism  $\mathbf{P} \rightarrow \mathbf{G}$ . Its mapping cone is exact, so for each  $n \geq g$  we have an exact sequence of  $R$ -modules

$$0 \rightarrow \Omega^n \mathbf{P} \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_g \rightarrow P_{g-1} \oplus G_g \rightarrow \cdots \rightarrow P_0 \oplus G_1 \rightarrow G_0 \rightarrow 0$$

Using Lemma 2.3, we get first  $P_{i-1} \oplus G_i \in \mathcal{G}$  for  $i = 1, \dots, g$ , then  $\Omega^n \mathbf{P} \in \mathcal{G}$ .  $\square$

### 3. GORENSTEIN DIMENSION

In this section  $R$  is a noetherian (on both sides) ring.

For a finite  $R$ -module  $M$  we introduce several notions of resolution.

A  $\mathcal{G}$ -resolution  $\mathbf{G} \rightarrow M$  is said to be *strict* if  $G_n$  is projective for all  $n \geq 1$ .

A *complete resolution* of  $M$  is a diagram  $\mathbf{T} \xrightarrow{\vartheta} \mathbf{P} \xrightarrow{\pi} M$  where  $\pi$  is a  $\mathcal{P}$ -resolution,  $\mathbf{T}$  is a totally acyclic complex,  $\vartheta$  is a morphism, and  $\vartheta_n$  is bijective for all  $n \gg 0$ .

A  $\mathcal{G}$ -*approximation*<sup>2</sup> of  $M$  is a resolution  $\chi: \mathbf{B} \rightarrow M$  of length  $\leq 1$ , where  $B_0$  is totally reflexive and  $B_1$  has finite projective dimension.

Parts of the next theorem are known: (i)  $\iff$  (iii) is in [2, (3.13.)]; (i)  $\iff$  (iv) is in [6, (4.4.4)]; (i)  $\iff$  (ii) is proved independently by Holm [23, (4.51)].

<sup>2</sup>Complying with tradition, we also call a  $\mathcal{G}$ -approximation of  $M$  the associated exact sequence  $\mathbf{B}^+ = 0 \rightarrow B_1 \rightarrow B_0 \rightarrow M \rightarrow 0$ . By Example 2.1, when  $R$  is a Gorenstein local ring this is a *maximal Cohen-Macaulay approximation* in the sense of Auslander and Buchweitz [3].

**3.1. Theorem.** *Let  $M$  be a finite module over a noetherian ring  $R$ .*

*For each integer  $g \geq 0$  the following conditions are equivalent.*

- (i)  *$M$  has a  $\mathcal{G}$ -resolution of length  $\leq g$ .*
- (ii)  *$M$  has a strict  $\mathcal{G}$ -resolution of length  $\leq g$ .*
- (iii)  *$M$  has a  $\mathcal{P}$ -resolution  $\mathbf{P}$  with  $\Omega^g \mathbf{P} \in \mathcal{G}$ .*
- (iv)  *$M$  has a complete resolution  $\sigma: \mathbf{S} \rightarrow \mathbf{P}$  with  $\sigma_n$  bijective for all  $n \geq g$ .*
- (v)  *$M$  has a complete resolution  $\vartheta: \mathbf{T} \rightarrow \mathbf{P}$  with  $\vartheta_n$  bijective for all  $n \geq g$  and surjective for all  $n \in \mathbb{Z}$ .*
- (vi)  *$M$  has a  $\mathcal{G}$ -approximation  $\chi: \mathbf{B} \rightarrow M$  with  $\text{proj dim}_R B_1 < g$ .*

Proofs of all the results in this section are collected at the end of the section.

In [2, (3.1)] Auslander and Bridger assign to each  $R$ -module  $M$  a number  $\text{G-dim}_R M$ , called the *Gorenstein dimension*, or *G-dimension*, of  $M$ . If  $M \neq 0$  is finite, then the equivalence of conditions (a) and (c) in [2, (3.13)] shows that

$$\text{G-dim}_R M = \inf \left\{ g \in \mathbb{N} \mid \begin{array}{l} \text{there exists a } \mathcal{G}\text{-resolution} \\ \mathbf{G} \rightarrow M \text{ of length } \leq g \end{array} \right\}.$$

We take this equality as the definition of G-dimension for a finite module  $M$ , and observe the following consequences:  $\text{G-dim}_R M = -\infty$  if and only if  $M = 0$ , and  $\text{G-dim}_R M = 0$  if and only if  $M \neq 0$  and  $M$  is totally reflexive.

Recall that  $R$  is *regular* (of dimension at most  $d$ ) if each finite module, left or right, has finite projective dimension (at most  $d$ ). We say that  $R$  is *Gorenstein* (of dimension at most  $d$ ) if each finite module, left or right, has finite G-dimension (at most  $d$ ). By Auslander and Bridger [2, (4.20)] (for  $d < \infty$ ) and Goto [22], in the commutative case this definition yields the classical notion: for each maximal ideal  $\mathfrak{m}$  the injective dimension  $\text{inj dim}_{R_{\mathfrak{m}}} R_{\mathfrak{m}}$  is finite (at most  $d$ ). The following result might be ‘well known’; for completeness, we include a proof later in this section.

**3.2. Theorem.** *For a ring  $R$  and an integer  $d \geq 0$  the following are equivalent.*

- (i)  *$R$  is Gorenstein of dimension at most  $d$ .*
- (ii)  *$R$  is noetherian with  $\text{inj dim}_R R \leq d$  and  $\text{inj dim}_{R^{\circ}} R \leq d$ .*

*Remark.* Some authors call *Iwanaga-Gorenstein* rings  $R$  satisfying condition (ii) for some  $d$ ; for them Zaks [33, p. 84] proves  $\text{inj dim}_R R = \text{inj dim}_{R^{\circ}} R$ .

**3.3. Examples.** Let  $R$  be a noetherian ring.

(1) If  $d = 0$ , then by Eilenberg and Nakayama [17, Theorem 18] condition (ii) can be weakened to (ii')  $R$  is self-injective on one side, or strengthened to (ii'') is *quasi-Frobenius*, that is, left and right artinian and left and right self-injective.

(2) Let  $\mathbb{k}$  be a commutative ring,  $R$  a  $\mathbb{k}$ -algebra with a  $\mathbb{k}$ -linear involution  $x \mapsto \bar{x}$ , and let  $(\ )^{\vee}$  be the functor  $\text{Hom}_{\mathbb{k}}(\ , \mathbb{k}): \mathcal{M}(R)^{\text{op}} \rightarrow \mathcal{M}(R)$ , where for  $M \in \mathcal{M}(R)$  the ring  $R$  acts on  $M^{\vee}$  by  $(x\alpha)(y) = \alpha(\bar{x}y)$ . If  $R$  is in  $\mathcal{P}(\mathbb{k})$  and the  $R$ -module  $R^{\vee}$  is isomorphic to  $R$ , then  $\text{G-dim}_R M \leq \text{proj dim}_{\mathbb{k}} M$  for each  $M \in \mathcal{F}(R)$ .

Indeed, each  $K \in \mathcal{F}(R) \cap \mathcal{P}(\mathbb{k})$  satisfies  $K^{\vee} \in \mathcal{F}(R) \cap \mathcal{P}(\mathbb{k})$  and  $K^{\vee\vee} \cong K$ . Thus, if  $0 \rightarrow J \rightarrow L \rightarrow K^{\vee} \rightarrow 0$  is a  $\mathbb{k}$ -split exact sequence in  $\mathcal{F}(R)$  with middle term in  $\mathcal{P}(R)$ , then the dual sequence  $0 \rightarrow K \rightarrow L^{\vee} \rightarrow J^{\vee} \rightarrow 0$  has the same properties.

Each  $\mathcal{P}(R)$ -resolution  $\mathbf{P} \rightarrow M$  is a  $\mathcal{P}(\mathbb{k})$ -resolution. If  $\text{proj dim}_{\mathbb{k}} M = p < \infty$ , then  $K = \Omega^p \mathbf{P}$  is in  $\mathcal{F}(R) \cap \mathcal{P}(\mathbb{k})$ , so as above we obtain a  $\mathbb{k}$ -split exact sequence  $0 \rightarrow \Omega^p \mathbf{P} \rightarrow T' \rightarrow K' \rightarrow 0$  in  $\mathcal{F}(R)$ , with  $T' \in \mathcal{P}(R)$  and  $K' \in \mathcal{F}(R) \cap \mathcal{P}(\mathbb{k})$ . Iterating, we extend  $\mathbf{T}_{\geq p} = \mathbf{P}_{\geq p}$  to a  $\mathbb{k}$ -split exact complex  $\mathbf{T}$  with  $\Omega^p \mathbf{T} = \Omega^p \mathbf{P}$ . Finally, the isomorphisms  $\mathbf{T}^* \cong \text{Hom}_R(\mathbf{T}, R^{\vee}) \cong \text{Hom}_{\mathbb{k}}(\mathbf{T}, \mathbb{k})$  yield  $\text{H}(\mathbf{T}^*) = 0$ .

We introduce notation for two more full subcategories of  $\mathcal{F}$ :

$\tilde{\mathcal{G}} = \tilde{\mathcal{G}}(R)$  whose objects are the finite  $R$ -modules with  $\text{G-dim}_R M < \infty$ .

$\tilde{\mathcal{P}} = \tilde{\mathcal{P}}(R)$  whose objects are the finite  $R$ -modules with  $\text{proj dim}_R M < \infty$ .

We summarize basic properties of these categories, extending [2, (3.14), (3.16)].

**3.4. Proposition.** *The category  $\tilde{\mathcal{G}}(R)$  is closed under extensions, kernels of epimorphisms, cokernels of monomorphisms, and there are inclusions*

$$\tilde{\mathcal{G}}(R) \supseteq \tilde{\mathcal{P}}(R) \quad \text{and} \quad \tilde{\mathcal{P}}(R) \cap \mathcal{G}(R) = \mathcal{P}(R).$$

*Problem.* Describe the noetherian rings  $R$  for which  $\tilde{\mathcal{G}}(R) = \tilde{\mathcal{P}}(R)$ .

The following examples show that no simple answer could be expected.

**3.5. Examples.** Let  $R$  be a noetherian ring.

(1) If  $R$  is Gorenstein, then  $R$  regular if and only if  $\tilde{\mathcal{G}}(R) = \tilde{\mathcal{P}}(R)$ .

(2) If  $R$  is local, commutative, Golod, and not a hypersurface, then  $\tilde{\mathcal{G}}(R) = \tilde{\mathcal{P}}(R)$ .

Indeed, by Lescot [24, (6.5)], cf. also [5, (5.3.3.5)], if  $\mathbf{P} \rightarrow M$  is a minimal free resolution and  $\text{proj dim}_R M = \infty$ , then  $\text{rank}_R P_{n+1} > \text{rank}_R P_n$  for all  $n > d = \text{depth } R$ . Assuming  $\tilde{\mathcal{G}}(R) \neq \tilde{\mathcal{P}}(R)$ , find  $M \in \tilde{\mathcal{G}}(R) \setminus \tilde{\mathcal{P}}(R)$  by condition 3.1.iii, set  $M' = \Omega^{-2d-3}(\mathbf{P}^*)$ , and choose a minimal free resolution  $\mathbf{P}' \rightarrow M'$ . As  $\text{H}_{-n}(\mathbf{P}^*) = \text{Ext}_R^n(M, R) = 0$  for  $n > 0$ , we have  $(\Sigma^{2d+3}(\mathbf{P}^*))_{\leq 2d+2} \cong \mathbf{P}'_{\leq 2d+2}$ , so  $\text{rank}_R P'_{d+2} = \text{rank}_R P^*_{d+1} < \text{rank}_R P^*_{d+2} = \text{rank}_R P'_{d+1}$ , contradicting Lescot's Theorem.

(3) If  $R$  is commutative,  $\mathfrak{a}$  is a proper ideal,  $f$  is a non-zero-divisor contained in  $\mathfrak{a}^2$ , and  $S = R/(f)$ , then  $\tilde{\mathcal{G}}(S) \neq \tilde{\mathcal{P}}(S)$  by [8, (3.2)].

Our treatment of G-dimension is based on the following constructions.

**3.6. Construction.** Let  $\pi: \mathbf{P} \rightarrow M$  be a  $\mathcal{P}$ -resolution.

Fix an integer  $g \geq 0$ , set  $G = \Omega^g \mathbf{P}$ . and choose a  $\mathcal{P}$ -resolution  $\lambda: \mathbf{L} \rightarrow G^*$  over  $R^0$ . As  $\Sigma^{g-1}((\mathbf{P}_{<g})^*)$  is a complex of finite projectives, trivial in negative degrees and has  $\Omega^{1-g}(\mathbf{P}^*)$  as degree 0 homology, choose a lifting of the canonical map

$$\Omega^{1-g}(\mathbf{P}^*) = \text{Coker}((\partial_{g-1}^{\mathbf{P}})^*) \twoheadrightarrow \text{Im}((\partial_g^{\mathbf{P}})^*) \hookrightarrow \text{Ker}((\partial_{g+1}^{\mathbf{P}})^*) = G^*$$

to a morphism  $\kappa: \Sigma^{g-1}((\mathbf{P}_{<g})^*) \rightarrow \mathbf{L}$ . Define a graded module  $\mathbf{S}$  and homomorphisms  $\partial^{\mathbf{S}}: \mathbf{S} \rightarrow \mathbf{S}$  and  $\sigma: \mathbf{S} \rightarrow \mathbf{P}$  of degree  $-1$  and  $0$ , respectively, by

$$S_n = \begin{cases} P_n & \\ P_g & \\ L_{g-n-1}^* & \end{cases} \quad \partial_n^{\mathbf{S}} = \begin{cases} \partial_n^{\mathbf{P}} & \\ \lambda_0^* \zeta^G \omega_g^{\mathbf{P}} & \\ (\partial_{g-n}^{\mathbf{L}})^* & \end{cases} \quad \sigma_n = \begin{cases} \text{id}_{P_n} & \text{for } n > g; \\ \text{id}_{P_g} & \text{for } n = g; \\ \kappa_{g-n-1}^* & \text{for } n < g, \end{cases}$$

where  $\omega_g^{\mathbf{P}}: P_g \rightarrow G$  is the canonical surjection.

**Fact.**  $(\mathbf{S}, \partial^{\mathbf{S}})$  is a complex of finite projective  $R$ -modules,

$$\text{H}_n(\mathbf{S}) = \begin{cases} 0 & \text{for } n \geq g+1; \\ \text{Ker}(\zeta^G) & \text{for } n = g; \\ \text{Coker}(\zeta^G) & \text{for } n = g-1; \\ \text{Ext}_R^{g-1-n}(G^*, R) & \text{for } n \leq g-2; \end{cases}$$

$$\text{H}_n(\mathbf{S}^*) = \begin{cases} 0 & \text{for } n \geq -g; \\ \text{Ext}_{R^0}^{-n-g}(G, R) & \text{for } n \leq -g-1, \end{cases}$$

$\sigma$  is morphism, and  $\sigma_n$  is bijective for all  $n \geq g$ .

If  $G$  is in  $\mathcal{G}$ , then  $\mathbf{S} \xrightarrow{\sigma} \mathbf{P} \xrightarrow{\pi} M$  is a complete resolution with  $\Omega^g \mathbf{S} = G$ .

*Proof.* Easy verifications show that  $(\mathbf{S}, \partial^{\mathbf{S}})$  is a complex and  $\sigma$  is a morphism, bijective in degrees  $\geq g$ . We get an exact sequence  $0 \rightarrow \Sigma^{g-1}(\mathbf{L}^*) \rightarrow \mathbf{S} \rightarrow \mathbf{P}_{\geq g} \rightarrow 0$  of complexes. The associated homology exact sequence breaks down to produce two series of isomorphisms,  $\mathbf{H}_n(\mathbf{S}) \cong \mathbf{H}_n(\mathbf{P}) = 0$  for  $n \geq g+1$  and  $\text{Ext}_{R^{\circ}}^{g-1-n}(G^*, R) = \mathbf{H}_n(\mathbf{L}^*) \cong \mathbf{H}_n(\mathbf{S})$  for  $n \leq g-2$ , and an exact sequence

$$0 \rightarrow \mathbf{H}_g(\mathbf{S}) \rightarrow G \xrightarrow{\zeta^G} G^{**} \rightarrow \mathbf{H}_{g-1}(\mathbf{S}) \rightarrow 0.$$

A similar computation with the exact sequence  $0 \rightarrow (\mathbf{P}_{\geq g})^* \rightarrow \mathbf{S}^* \rightarrow \Sigma^{1-g}\mathbf{L} \rightarrow 0$  yields the expressions for  $\mathbf{H}_n(\mathbf{S}^*)$ . When  $G$  is totally reflexive these computations imply  $\mathbf{H}(\mathbf{S}) = 0 = \mathbf{H}(\mathbf{S}^*)$ ; each  $S_n$  is in  $\mathcal{P}$ , so  $\sigma$  is a complete resolution.  $\square$

**3.7. Construction.** Let  $\sigma: \mathbf{S} \rightarrow \mathbf{P}$  be a morphism of complexes.

Fix a number  $g \leq \infty$ , set  $\mathbf{T}_n = (\mathbf{S} \oplus \mathbf{P}_{<g} \oplus \Sigma^{-1}\mathbf{P}_{<g})_n$  for all  $n \in \mathbb{Z}$ , and for  $x \in S_n, y \in P_n, y' \in P_{n+1}$  define maps  $\partial^{\mathbf{T}}$  of degree  $-1$ , and  $\vartheta, \beta, \alpha$  of degree 0:

$$\begin{aligned} \partial^{\mathbf{T}}: \mathbf{T} \rightarrow \mathbf{T} & \quad \text{by} \quad \partial_n^{\mathbf{T}}(x, y, y') = (\partial_n^{\mathbf{S}}(x), \partial_n^{\mathbf{P}}(y), y - \partial_{n+1}^{\mathbf{P}}(y')); \\ \vartheta: \mathbf{T} \rightarrow \mathbf{P} & \quad \text{by} \quad \vartheta_n(x, y, y') = \sigma(x) + y; \\ \beta: \mathbf{T} \rightarrow \mathbf{S} & \quad \text{by} \quad \beta_n(x, y, y') = x; \\ \alpha: \mathbf{S} \rightarrow \mathbf{T} & \quad \text{by} \quad \alpha_n(x) = (x, 0, 0). \end{aligned}$$

**Fact.**  $(\mathbf{T}, \partial^{\mathbf{T}})$  is a complex,  $\vartheta$  is a morphism with  $\sigma = \vartheta \alpha$  and  $\vartheta_n$  surjective for  $n < g$ , the maps  $\alpha$  and  $\beta$  are inverse homotopy equivalences with  $\beta \alpha = \text{id}_{\mathbf{S}}$ .

If  $\mathbf{S} \xrightarrow{\sigma} \mathbf{P} \xrightarrow{\pi} M$  is a complete resolution with  $\sigma_n$  bijective for  $n \geq g$ , then  $\mathbf{T} \xrightarrow{\vartheta} \mathbf{P} \xrightarrow{\pi} M$  is a surjective complete resolution with  $\vartheta_n$  bijective for  $n \geq g$ .

*Proof.* The properties of  $\partial^{\mathbf{T}}, \vartheta$ , and  $\beta \alpha$  follow directly from the formulas. To get a homomorphism  $\xi: \mathbf{T} \rightarrow \mathbf{T}$  with  $\alpha \beta = \text{id}_{\mathbf{T}} + \partial^{\mathbf{T}} \xi + \xi \partial^{\mathbf{T}}$ , set  $\xi_n(x, y, y') = (0, y', 0)$ .

When  $\sigma$  is a complete resolution each  $T_n$  is in  $\mathcal{P}$  by definition. As  $\alpha$  is a homotopy equivalence, so is  $\alpha^*$ , hence  $\mathbf{H}(\mathbf{T}) \cong \mathbf{H}(\mathbf{S}) = 0$  and  $\mathbf{H}(\mathbf{T}^*) \cong \mathbf{H}(\mathbf{S}^*) = 0$ .  $\square$

**3.8. Construction.** Let  $\mathbf{T} \xrightarrow{\vartheta} \mathbf{P} \xrightarrow{\pi} M$  be a surjective complete resolution, with  $\vartheta_n$  bijective for  $n \geq g$ . Set  $\mathbf{Q} = \text{Ker } \vartheta$ , let  $\varkappa$  denote the inclusion  $\mathbf{Q} \subseteq \mathbf{T}$ , and set

$$\begin{aligned} G_n &= \begin{cases} Q_{n-1} & \text{for } n \geq 1; \\ \Omega^0 \mathbf{T} & \text{for } n = 0; \\ 0 & \text{for } \leq -1; \end{cases} & \partial_n^{\mathbf{G}} &= \begin{cases} -\partial_{n-1}^{\mathbf{Q}} & \text{for } n \geq 2; \\ -(\Omega^0 \varkappa) \circ \omega_0^{\mathbf{Q}} & \text{for } n = 1; \end{cases} \\ T_n^{\mathbf{b}} &= \begin{cases} T_n & \text{for } n \geq 0; \\ \Omega^0 \mathbf{T} & \text{for } n = -1; \\ 0 & \text{for } n \leq -2; \end{cases} & \partial_n^{\mathbf{T}^{\mathbf{b}}} &= \begin{cases} \partial_n^{\mathbf{T}} & \text{for } n \geq 1; \\ \omega_0^{\mathbf{T}} & \text{for } n = 0, \end{cases} \\ \varkappa_n^{\mathbf{b}} &= \begin{cases} \varkappa_n & \text{for } n \geq 0; \\ \text{id}_{\Omega^0 \mathbf{T}} & \text{for } n = -1; \\ 0 & \text{for } \leq -2; \end{cases} & \vartheta_n^{\mathbf{b}} &= \begin{cases} \vartheta_n & \text{for } n \geq 0; \\ \Omega^0 \vartheta & \text{for } n = -1; \\ 0 & \text{for } n \leq -2, \end{cases} \end{aligned}$$

where  $\omega_0^{\mathbf{Q}}: Q_0 \rightarrow \Omega^0 \mathbf{Q}$  and  $\omega_0^{\mathbf{T}}: T_0 \rightarrow \Omega^0 \mathbf{T}$  are the canonical maps.

**Fact.**  $(\mathbf{G}, \partial^{\mathbf{G}})$  and  $(\mathbf{T}^{\mathbf{b}}, \partial^{\mathbf{T}^{\mathbf{b}}})$  are complexes,  $\varkappa^{\mathbf{b}}: \Sigma^{-1} \mathbf{G} \rightarrow \mathbf{T}^{\mathbf{b}}$  and  $\vartheta^{\mathbf{b}}: \mathbf{T}^{\mathbf{b}} \rightarrow \mathbf{P}$  are morphisms, the sequence  $0 \rightarrow \Sigma^{-1} \mathbf{G} \xrightarrow{\varkappa^{\mathbf{b}}} \mathbf{T}^{\mathbf{b}} \xrightarrow{\vartheta^{\mathbf{b}}} \mathbf{P} \rightarrow 0$  is exact, and  $\gamma: \mathbf{G} \rightarrow M$  with  $\gamma_0: G_0 = \Omega^0 \mathbf{T} \xrightarrow{\Omega^0 \vartheta} \Omega^0 \mathbf{P} = M$  is a strict  $G$ -resolution of length  $g$ .

*Proof.* All but the last statement result from direct verifications. As  $H(\mathbf{T}^b) = 0$ , from the exact sequence of complexes we see that  $H_n(\mathbf{G}) = 0$  for  $n \neq 0$  and  $H_0(\gamma): H_0(\mathbf{G}) \rightarrow M$  is bijective. By construction,  $G_n = 0$  for  $n < 0$  or  $n > g$ . The module  $G_0 = \Omega^0 \mathbf{T}$  is totally reflexive by Lemma 2.4. If  $n \geq 1$ , then  $G_n$  is the kernel of  $\vartheta_{n-1}$ , an epimorphism of finite projective modules, so  $G_n$  is in  $\mathcal{P}$ .  $\square$

*Proof of Theorem 3.1.* (i)  $\implies$  (iii) by Lemma 2.5.

(iii)  $\implies$  (iv) by Construction 3.6.

(iv)  $\implies$  (v) by Construction 3.7.

(v)  $\implies$  (ii) by Construction 3.8.

(ii)  $\implies$  (vi). If  $\mathbf{G} \rightarrow M$  is a strict resolution of length  $\leq g$ , then the sequence  $\mathbf{B}^+ = 0 \rightarrow \Omega^1 \mathbf{G} \rightarrow G_0 \rightarrow M \rightarrow 0$  is a  $\mathcal{G}$ -approximation with  $\text{proj dim}_R \Omega^1 \mathbf{G} < g$ .

(vi)  $\implies$  (i). If a  $\mathcal{G}$ -approximation  $\mathbf{B} \rightarrow M$  has  $\text{proj dim}_R B_1 = p < g$ , splice it with a  $\mathcal{P}$ -resolution of  $B_1$  of length  $p$  to get a  $\mathcal{G}$ -resolution of  $M$  of length  $\leq g$ .  $\square$

*Proof of Proposition 3.4.* Let  $0 \rightarrow M \rightarrow M' \rightarrow M'' \rightarrow 0$  be an exact sequence of modules. An exact sequence of complexes provided by the Horseshoe Lemma 1.4 yields for every  $n \in \mathbb{Z}$  an exact sequence of  $R$ -modules

$$0 \rightarrow \Omega^n \mathbf{P} \rightarrow \Omega^n \mathbf{P}' \rightarrow \Omega^n \mathbf{P}'' \rightarrow 0$$

To see that  $\tilde{\mathcal{G}}$  is closed under extensions and kernels of epimorphisms we apply Lemma 2.3 and check membership in  $\tilde{\mathcal{G}}$  using Lemma 2.5 and condition 3.1.iii.

If  $M$  and  $M'$  are in  $\mathcal{G}$ , choose  $\mathcal{P}$ -resolutions  $\mathbf{P} \rightarrow M$  and  $\mathbf{P}' \rightarrow M'$ , lift  $M \rightarrow M'$  to a morphism  $\mathbf{P} \rightarrow \mathbf{P}'$ , and form its mapping cone  $\mathbf{P}''$ . The exact sequence

$$0 \rightarrow \mathbf{P}' \rightarrow \mathbf{P}'' \rightarrow \Sigma \mathbf{P} \rightarrow 0$$

shows that  $\mathbf{P}''$  is a  $\mathcal{P}$ -resolution of  $M''$  and yield an exact sequence

$$0 \rightarrow \Omega^n \mathbf{P}' \rightarrow \Omega^n \mathbf{P}'' \rightarrow \Omega^{n-1} \mathbf{P} \rightarrow 0$$

for every  $n \in \mathbb{Z}$ . As above, we conclude that  $M''$  is in  $CG$ .

For  $M \in \tilde{\mathcal{P}} \cap \mathcal{G}$  there is an exact sequence  $0 \rightarrow L \rightarrow P \rightarrow M \rightarrow 0$  with  $P \in \mathcal{P}$  and  $L \in \tilde{\mathcal{P}}$ . By Lemma 2.2 the Ext in the induced exact sequence

$$0 \rightarrow \text{Hom}_R(M, L) \rightarrow \text{Hom}_R(M, P) \rightarrow \text{Hom}_R(M, M) \rightarrow \text{Ext}_R^1(M, L)$$

vanishes, so  $M$  is a direct summand of  $P$ . Thus,  $M$  is projective, and so we have  $\tilde{\mathcal{P}} \cap \mathcal{G} \subseteq \tilde{\mathcal{P}}$ . The converse inclusion is clear from the inclusion  $\mathcal{G} \supseteq \mathcal{P}$  of Lemma 2.3, which also shows that every  $\mathcal{P}$ -resolution is a  $\mathcal{G}$ -resolution, and hence  $\tilde{\mathcal{G}} \supseteq \tilde{\mathcal{P}}$ .  $\square$

*Proof of Theorem 3.2.* (i)  $\implies$  (ii). By assumption, for each left ideal  $\mathfrak{a}$  there is a  $\mathcal{P}$ -resolution  $\mathbf{P} \rightarrow R/\mathfrak{a}$  with  $\Omega^d \mathbf{P} \in \mathcal{G}$ . Thus,  $\text{Ext}_R^{d+1}(R/\mathfrak{a}, R) \cong \text{Ext}_R^1(\Omega^d \mathbf{P}, R) = 0$ , so  $\text{inj dim}_R R \leq d$  by Baer's Criterion;  $\text{inj dim}_{R^\circ} R \leq d$  follows by symmetry.

(ii)  $\implies$  (i). For  $M \in \mathcal{F}$  choose  $\mathcal{P}$ -resolutions  $\mathbf{P} \rightarrow M$  and  $\mathbf{L} \rightarrow (\Omega^d \mathbf{P})^*$  and form a complex  $\mathbf{S}$  as in Construction 3.6. It satisfies  $H_n(\mathbf{S}^*) = 0$  for  $n \geq -d$  and

$$H_n(\mathbf{S}^*) = \text{Ext}_R^{-n-d}(\Omega^d \mathbf{P}, R) \cong \text{Ext}_R^{-n}(M, R)$$

for  $n \leq -d-1$ . Since  $\text{inj dim}_R R \leq d$ , the last module vanishes, so  $\mathbf{S}^*$  is exact and

$$H_n(\mathbf{S}) = H_n \text{Hom}_R(\mathbf{S}^*, R) = \text{Ext}_{R^\circ}^{d+1}(\Omega^{-n-d-1}(\mathbf{S}^*), R).$$

for all  $n \in \mathbb{Z}$ . The last module vanishes because  $\text{inj dim}_{R^\circ} R \leq d$ , so  $\mathbf{S}$  is exact as well. By Lemma 2.4, we conclude that  $\Omega^d \mathbf{P} = \Omega^d \mathbf{S}$  is totally reflexive.  $\square$

## 4. RELATIVE COHOMOLOGY

All relative notions below are introduced with respect to the class  $\mathcal{G}$  of totally reflexive modules, references to which are dropped when no ambiguity arises.

A sequence  $\mathbf{E}$  of homomorphisms in  $\mathcal{M}$  is *proper exact* if the induced sequence  $\mathrm{Hom}_R(G, \mathbf{E})$  is exact for all  $G \in \mathcal{G}$ ; since  $R$  is in  $\mathcal{G}$ , proper exact sequences are exact. A *proper resolution*  $\mathbf{G} \rightarrow M$  is a  $\mathcal{G}$ -resolution whose associated exact sequence  $\mathbf{G}^+$  is proper. We let  $\overline{\mathcal{G}} = \overline{\mathcal{G}}(R)$  denote the full subcategory of the category  $\mathcal{F}$  of finite  $R$ -modules whose objects are the modules admitting some proper resolution.

There are handy sufficient conditions for properness of sequences and resolutions.

- 4.1. Lemma.** (1) *Each split exact sequence is proper.*  
(2) *Each exact sequence  $0 \rightarrow N \rightarrow N' \rightarrow N'' \rightarrow 0$  with  $\mathrm{projdim}_R N < \infty$  is proper.*  
(3) *Each strict  $\mathcal{G}$ -resolution  $\mathbf{G} \rightarrow M$  of finite length is proper.*  
(4) *There is an inclusion of categories  $\overline{\mathcal{G}}(R) \supseteq \widetilde{\mathcal{G}}(R)$ .*

*Remark.* The inclusion in (4) may be strict: If  $R$  is a ring with  $\mathcal{G}(R) = \mathcal{P}(R)$ , for instance one of the rings from Example 3.5.2, then every exact sequence is proper exact, hence  $\overline{\mathcal{G}}(R) = \mathcal{F}(R) \supsetneq \widetilde{\mathcal{P}}(R) = \widetilde{\mathcal{G}}(R)$ . We thank the referee for this remark.

*Proof.* (1) is clear. (2) holds because in the induced exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(G, N) \longrightarrow \mathrm{Hom}_R(G, N') \longrightarrow \mathrm{Hom}_R(G, N'') \longrightarrow \mathrm{Ext}_R^1(G, N)$$

the Ext vanishes by Lemma 2.2. (3) follows from (2) by induction on the length of  $\mathbf{G}$ . (4) results from (3) due to Theorem 3.1.  $\square$

Choosing for each  $M \in \overline{\mathcal{G}}$  a proper resolution  $\mathbf{G} \rightarrow M$ , we define for each  $n \in \mathbb{Z}$  and each  $N \in \mathcal{M}$  a *relative cohomology group*

$$\mathrm{Ext}_{\mathcal{G}}^n(M, N) = \mathrm{H}^n \mathrm{Hom}_R(\mathbf{G}, N).$$

Choosing a projective resolution  $\pi: \mathbf{P} \rightarrow M$  and a morphism of complexes  $\phi: \mathbf{P} \rightarrow \mathbf{G}$  that lifts the identity map of  $M$ , we define a *comparison homomorphism*

$$\varepsilon_{\mathcal{G}}^n(M, N): \mathrm{Ext}_{\mathcal{G}}^n(M, N) \longrightarrow \mathrm{Ext}_R^n(M, N)$$

by setting  $\varepsilon_{\mathcal{G}}^n(M, N) = \mathrm{H}^n \mathrm{Hom}_R(\phi, N): \mathrm{H}^n \mathrm{Hom}_R(\mathbf{G}, N) \rightarrow \mathrm{H}^n \mathrm{Hom}_R(\mathbf{P}, N)$ .

- 4.2. Theorem.** *The assignment  $(M, N) \mapsto \mathrm{Ext}_{\mathcal{G}}^n(M, N)$  defines a functor*

$$\mathrm{Ext}_{\mathcal{G}}^n: \overline{\mathcal{G}}(R)^{\mathrm{op}} \times \mathcal{M}(R) \longrightarrow \mathcal{M}(\mathbb{Z})$$

*and the maps  $\varepsilon_{\mathcal{G}}^n(M, N)$  yield a morphism of functors  $\varepsilon_{\mathcal{G}}^n: \mathrm{Ext}_{\mathcal{G}}^n \rightarrow \mathrm{Ext}_R^n$  such that:*

- (1)  *$\mathrm{Ext}_{\mathcal{G}}^n$  and  $\varepsilon_{\mathcal{G}}^n$  are independent of the choices of resolutions and lifting.*  
(2.a) *For every  $M \in \overline{\mathcal{G}}$  and each  $g \geq 0$  the following conditions are equivalent.*  
(i)  $\mathrm{G-dim}_R M \leq g$ .  
(ii)  $\mathrm{Ext}_{\mathcal{G}}^n(M, \_) = 0$  for all  $n > g$ .  
(iii)  $\mathrm{Ext}_{\mathcal{G}}^{g+1}(M, \_) = 0$ .  
(iv) *Each proper resolution  $\mathbf{F} \rightarrow M$  has  $\Omega^n \mathbf{F} \in \mathcal{G}$  for all  $n \geq g$ .*  
(2.b)  $\varepsilon_{\mathcal{G}}^0: \mathrm{Ext}_{\mathcal{G}}^0 \rightarrow \mathrm{Hom}_R$  *is an isomorphism and  $\mathrm{Ext}_{\mathcal{G}}^n(M, \_) = 0$  for  $n < 0$ .*  
(3) *If  $\mathrm{projdim}_R M < \infty$ , then  $\varepsilon_{\mathcal{G}}^n(M, N)$  is bijective for all  $n \in \mathbb{Z}$ .*  
(4) *If  $\mathrm{G-dim}_R M < \infty$  and  $N$  is projective, then  $\varepsilon_{\mathcal{G}}^n(M, N)$  is bijective for all  $n \in \mathbb{Z}$ .*

*Remark.* Corollary 7.2 contains a stronger form of (4) and converses to (3) and (4).

**4.3. Lemma.** *Given proper resolutions  $\gamma: \mathbf{G} \rightarrow M$  and  $\gamma': \mathbf{G}' \rightarrow M'$ , and  $\mathcal{P}$ -resolutions  $\pi: \mathbf{P} \rightarrow M$  and  $\pi': \mathbf{P}' \rightarrow M'$ , there exist unique up to homotopy morphisms of complexes  $\phi: \mathbf{P} \rightarrow \mathbf{G}$  and  $\phi': \mathbf{P}' \rightarrow \mathbf{G}'$  with  $\pi = \gamma \phi$  and  $\pi' = \gamma' \phi'$ .*

*For each homomorphism of modules  $\mu: M \rightarrow M'$  there exists a unique up to homotopy morphism  $\tilde{\mu}$ , making the right hand square of the diagram*

$$\begin{array}{ccccc} \mathbf{P} & \xrightarrow{\phi} & \mathbf{G} & \xrightarrow{\gamma} & M \\ \downarrow \bar{\mu} & & \downarrow \tilde{\mu} & & \downarrow \mu \\ \mathbf{P}' & \xrightarrow{\phi'} & \mathbf{G}' & \xrightarrow{\gamma'} & M' \end{array}$$

*commutes; for each choice of  $\tilde{\mu}$  there exists a unique up to homotopy morphism  $\bar{\mu}$ , making the left hand square commute up to homotopy.*

*If  $\mu = \text{id}_M$ , then  $\tilde{\mu}$  and  $\bar{\mu}$  are homotopy equivalences.*

*Proof.* The sequence  $\mathbf{G}'^+$  is proper exact, hence  $\text{H Hom}_R(G_i, \mathbf{G}'^+) = 0$  for all  $i \in \mathbb{Z}$ . Note that  $\mathbf{G} = \varinjlim \mathbf{G}_{\leq i}$ , and consider the exact sequences of complexes

$$0 \rightarrow \mathbf{G}_{\leq i-1} \rightarrow \mathbf{G}_{\leq i} \rightarrow \Sigma^i G_i \rightarrow 0.$$

As  $\mathbf{G}_{\leq -1} = 0$ , using the induced exact sequence of complexes of abelian groups

$$0 \rightarrow \text{Hom}_R(\Sigma^i G_i, \mathbf{G}'^+) \rightarrow \text{Hom}_R(\mathbf{G}_{\leq i}, \mathbf{G}'^+) \rightarrow \text{Hom}_R(\mathbf{G}_{\leq i-1}, \mathbf{G}'^+) \rightarrow 0$$

and induction on  $i$  we get  $\text{H Hom}_R(\mathbf{G}_{\leq i}, \mathbf{G}'^+) = 0$  for all  $i \in \mathbb{Z}$ . Thus, we have

$$\text{H Hom}_R(\mathbf{G}, \mathbf{G}'^+) = \text{H Hom}_R(\varinjlim \mathbf{G}_{\leq i}, \mathbf{G}'^+) = \text{H}(\varprojlim \text{Hom}_R(\mathbf{G}_{\leq i}, \mathbf{G}'^+)) = 0$$

with the last equality coming from the Mittag-Leffler Criterion 1.3.

The exact sequence  $0 \rightarrow \Sigma^{-1} M' \rightarrow \mathbf{G}'^+ \rightarrow \mathbf{G}' \rightarrow 0$  induces an exact sequence

$$0 \rightarrow \text{Hom}_R(\mathbf{G}, \Sigma^{-1} M') \rightarrow \text{Hom}_R(\mathbf{G}, \mathbf{G}'^+) \rightarrow \text{Hom}_R(\mathbf{G}, \mathbf{G}') \rightarrow 0$$

For each  $n$ , the connecting map in the homology exact sequence is an isomorphism

$$\text{H}_n \text{Hom}_R(\mathbf{G}, \mathbf{G}') \cong \text{H}_{n-1} \text{Hom}_R(\mathbf{G}, \Sigma^{-1} M').$$

Composing it with  $\text{H}_{n-1} \text{Hom}_R(\mathbf{G}, \Sigma^{-1} M') = \text{H}_n \text{Hom}_R(\mathbf{G}, M')$  we obtain the map  $\text{H}_n \text{Hom}_R(\mathbf{G}, \gamma')$ , so  $\text{Hom}_R(\mathbf{G}, \gamma')$  is a quasiisomorphism. The Lifting Lemma 1.1.1 yields a unique up to homotopy morphism  $\tilde{\mu}$  with  $\gamma' \tilde{\mu} = \mu \gamma$ .

If  $\mu = \text{id}_M$ , then reversing the roles of  $M$  and  $M'$  we get a morphism  $\tilde{\mu}: \mathbf{G}' \rightarrow \mathbf{G}$  inducing  $\text{id}_M$ . Thus,  $\tilde{\mu} \tilde{\mu}': \mathbf{G}' \rightarrow \mathbf{G}'$  induces  $\text{id}_M$ , and hence is homotopic to  $\text{id}_{\mathbf{G}'}$ . By symmetry,  $\tilde{\mu}' \tilde{\mu} \sim \text{id}_{\mathbf{G}}$ , so  $\tilde{\mu}$  is a homotopy equivalence.

By 1.2.1 and 1.1.1 we get morphisms  $\phi, \phi'$  with  $\pi = \gamma \phi, \pi' = \gamma' \phi'$ , then a morphism  $\tilde{\mu}'$  with  $\phi \tilde{\mu}' \sim \tilde{\mu}' \phi'$ , all unique up to homotopy. If  $\mu = \text{id}_M$  all these maps are quasiisomorphism, so  $\bar{\mu}$  is a homotopy equivalence by *loc. cit.*  $\square$

*Proof of Theorem 4.2.* The first assertion of Lemma 4.3, applied to the chosen resolutions of  $M, M'$ , yields the naturality of  $\text{Ext}_{\mathcal{G}}^n$  and  $\varepsilon_{\mathcal{G}}^n$ .

(1) is due to the last assertion of the same lemma.

(2.a) is proved after Proposition 4.4.

(2.b) results from the left exactness of  $\text{Hom}_R(\cdot, N)$ .

(3) Note that if  $M$  has a  $\mathcal{P}$ -resolution  $\mathbf{P}$  of finite length, then this is a proper  $\mathcal{G}$ -resolution by Lemma 4.1.3, so we can take  $\mathbf{G} = \mathbf{P}$  and  $\phi = \text{id}_{\mathbf{P}}$ .

(4) is proved after Proposition 4.7.  $\square$

Exact sequences of relative Ext groups exist in certain cases.

**4.4. Proposition.** *For each  $R$ -module  $M \in \overline{\mathcal{G}}$  and each proper exact sequence  $\mathbf{N} = 0 \rightarrow N \xrightarrow{\nu} N' \xrightarrow{\nu'} N'' \rightarrow 0$  of  $R$ -modules there exist natural in  $M$  and  $\mathbf{N}$  homomorphisms  $\partial_{\mathcal{G}}^n(M, \mathbf{N})$ , such that the sequence below is exact*

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \text{Ext}_{\mathcal{G}}^n(M, N) & \xrightarrow{\text{Ext}_{\mathcal{G}}^n(M, \nu)} & \text{Ext}_{\mathcal{G}}^n(M, N') & \xrightarrow{\text{Ext}_{\mathcal{G}}^n(M, \nu')} & \text{Ext}_{\mathcal{G}}^n(M, N'') \\ & & \xrightarrow{\partial_{\mathcal{G}}^n(M, \mathbf{N})} & & \longrightarrow & & \cdots \\ & & \text{Ext}_{\mathcal{G}}^{n+1}(M, N) & & & & \end{array}$$

and the connecting maps  $\partial^n(M, \mathbf{N}): \text{Ext}_R^n(M, N'') \rightarrow \text{Ext}_R^{n+1}(M, N)$  satisfy

$$\partial^n(M, \mathbf{N}) \circ \varepsilon_{\mathcal{G}}^n(M, N'') = \varepsilon_{\mathcal{G}}^{n+1}(M, N) \circ \partial_{\mathcal{G}}^n(M, \mathbf{N}) \quad \text{for all } n \in \mathbb{Z}.$$

*Proof.* Let  $\phi: \mathbf{P} \rightarrow \mathbf{G}$  be a morphism from a projective resolution of  $M$  to a proper resolution of  $M$ , with  $H_0(\phi) = \text{id}_M$ . It induces a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{Hom}_R(\mathbf{G}, N) & \longrightarrow & \text{Hom}_R(\mathbf{G}, N') & \longrightarrow & \text{Hom}_R(\mathbf{G}, N'') & \longrightarrow & 0 \\ & & \downarrow \text{Hom}_R(\phi, N) & & \downarrow \text{Hom}_R(\phi, N') & & \downarrow \text{Hom}_R(\phi, N'') & & \\ 0 & \longrightarrow & \text{Hom}_R(\mathbf{P}, N) & \longrightarrow & \text{Hom}_R(\mathbf{P}, N') & \longrightarrow & \text{Hom}_R(\mathbf{P}, N'') & \longrightarrow & 0 \end{array}$$

of complexes of abelian groups; the bottom (respectively, top) row is exact because each  $P_n$  is in  $\mathcal{P}$  (respectively each  $G_n$  is in  $\mathcal{G}$ ) and  $\mathbf{N}$  is (respectively, proper) exact. The homology exact sequence of the top row is the desired long exact sequence. The commutativity of the diagram gives the formula for the connecting maps. Naturality in  $\mathbf{N}$  is clear; naturality in  $M$  follows from Lemma 4.3.  $\square$

*Proof of Theorem 4.2.2.b.* (ii)  $\implies$  (iii) and (iv)  $\implies$  (i) are clear.

(i)  $\implies$  (ii). In view of Lemma 4.1.3 and Theorems 4.2.1 and 3.1 we can compute  $\text{Ext}_{\mathcal{G}}^n(M, \_)$  from a proper resolution  $\gamma: \mathbf{G} \rightarrow M$  of length  $\leq g$ .

(iii)  $\implies$  (iv). Let  $\mathbf{F} \rightarrow M$  be a proper resolution and set  $G = \Omega^g \mathbf{F}$ . Since  $\Sigma^{-g} \mathbf{F}_{\geq g} \rightarrow G$  is a proper resolution, we have  $\text{Ext}_{\mathcal{G}}^1(G, \_) \cong \text{Ext}_{\mathcal{G}}^{g+1}(M, \_) = 0$ . Proposition 4.4 applied to the proper exact sequence  $0 \rightarrow \Omega^{g+1} \mathbf{F} \rightarrow F_{g+1} \xrightarrow{\omega} G \rightarrow 0$  shows that  $\text{Hom}_R(G, \omega): \text{Hom}_R(G, F_{g+1}) \rightarrow \text{Hom}_R(G, G)$  is surjective. Thus,  $G$  is isomorphic to a direct summand of  $F_{g+1}$ , so  $G$  is in  $\mathcal{G}$  by Lemma 2.3.  $\square$

A version of the Horseshoe Lemma for proper resolutions reads as follows.

**4.5. Lemma.** *Let  $\mathbf{M} = 0 \rightarrow M \xrightarrow{\mu} M' \xrightarrow{\mu'} M'' \rightarrow 0$  be a proper exact sequence of finite  $R$ -modules with  $M$  and  $M''$  in  $\overline{\mathcal{G}}$ .*

*If  $\gamma: \mathbf{G} \rightarrow M$  and  $\gamma'': \mathbf{G}'' \rightarrow M''$  are proper resolutions, while  $\pi: \mathbf{P} \rightarrow M$  and  $\pi'': \mathbf{P}'' \rightarrow M''$  are  $\mathcal{P}$ -resolutions, then there exists a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{\mu} & M' & \xrightarrow{\mu'} & M'' & \longrightarrow & 0 \\ & & \uparrow \gamma & & \uparrow \gamma' & & \uparrow \gamma'' & & \\ 0 & \longrightarrow & \mathbf{G} & \xrightarrow{\tilde{\mu}} & \mathbf{G}' & \xrightarrow{\tilde{\mu}'} & \mathbf{G}'' & \longrightarrow & 0 \\ & & \uparrow \phi & & \uparrow \phi' & & \uparrow \phi'' & & \\ 0 & \longrightarrow & \mathbf{P} & \xrightarrow{\bar{\mu}} & \mathbf{P}' & \xrightarrow{\bar{\mu}'} & \mathbf{P}'' & \longrightarrow & 0 \end{array}$$

of morphisms of complexes where the middle and bottom rows are split exact as sequences of graded modules,  $\gamma'$  is a proper resolution,  $\pi = \gamma' \phi'$  is a  $\mathcal{P}$ -resolution, and there are equalities  $\gamma \phi = \pi$  and  $\gamma'' \phi'' = \pi''$ .

*Proof.* As  $\mathbf{M}$  is proper, the map  $\text{Hom}_R(G''_0, M') \rightarrow \text{Hom}_R(G''_0, M'')$  is surjective, so there exists a homomorphism  $\nu': G''_0 \rightarrow M'$  with  $\mu'\nu' = \gamma'_0$ . In the diagram

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & M & \xrightarrow{\mu} & M' & \xrightarrow{\mu'} & M'' \longrightarrow 0 \\
& & \uparrow \gamma_0 & & \uparrow \gamma'_0 & & \uparrow \gamma''_0 \\
0 & \longrightarrow & G_0 & \xrightarrow{\tilde{\mu}_0} & G_0 \oplus G''_0 & \xrightarrow{\tilde{\mu}'_0} & G''_0 \longrightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & \Omega^1 \mathbf{G} & \longrightarrow & \text{Ker } \gamma'_0 & \longrightarrow & \Omega^1 \mathbf{G}'' \longrightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
& & 0 & & 0 & & 0
\end{array}$$

where  $\gamma'_0 = [\mu\gamma_0 \ \nu']$ , while  $\tilde{\mu}_0$  and  $\tilde{\mu}'_0$  are the canonical maps, the external columns and the top row are proper exact by hypothesis, and the middle row is proper exact by Lemma 4.1.1. Thus, for each  $G \in \mathcal{G}$  the induced diagram

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & \text{Hom}_R(G, M) & \longrightarrow & \text{Hom}_R(G, M') & \longrightarrow & \text{Hom}_R(G, M'') \longrightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & \text{Hom}_R(G, G_0) & \longrightarrow & \text{Hom}_R(G, G'_0 \oplus G''_0) & \longrightarrow & \text{Hom}_R(G, G''_0) \longrightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & \text{Hom}_R(G, \Omega^1 \mathbf{G}) & \longrightarrow & \text{Hom}_R(G, \text{Ker } \gamma'_0) & \longrightarrow & \text{Hom}_R(G, \Omega^1 \mathbf{G}'') \longrightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
& & 0 & & 0 & & 0
\end{array}$$

is commutative with exact external columns, top row, and middle row. Applying the Snake Lemma to these rows, one sees that the middle column and bottom row are exact. Thus, all rows and columns in the original diagram are proper exact.

Iterating the procedure we obtain the upper tier of the desired diagram of complexes. The lower tier is produced by applying the Horseshoe Lemma 1.4.  $\square$

**4.6. Proposition.** *For each proper exact sequence  $\mathbf{M} = 0 \rightarrow M \xrightarrow{\mu} M' \xrightarrow{\mu'} M'' \rightarrow 0$  of  $R$ -modules in  $\overline{\mathcal{G}}$  and each  $R$ -module  $N$  there are natural in  $\mathbf{M}$  and  $N$  homomorphisms  $\mathfrak{d}_{\mathcal{G}}^n(\mathbf{M}, N)$ , such that the sequence below is exact*

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \text{Ext}_{\mathcal{G}}^n(M'', N) & \xrightarrow{\text{Ext}_{\mathcal{G}}^n(\mu', N)} & \text{Ext}_{\mathcal{G}}^n(M', N) & \xrightarrow{\text{Ext}_{\mathcal{G}}^n(\mu, N)} & \text{Ext}_{\mathcal{G}}^n(M, N) \\
& & \downarrow \widehat{\mathfrak{d}}_{\mathcal{G}}^n(\mathbf{M}, N) & & & & \\
& \longrightarrow & \text{Ext}_{\mathcal{G}}^{n+1}(M'', N) & \longrightarrow & \cdots & & 
\end{array}$$

and the connecting maps  $\mathfrak{d}^n(\mathbf{M}, N): \text{Ext}_R^n(M, N) \rightarrow \text{Ext}_R^{n+1}(M'', N)$  satisfy

$$\mathfrak{d}^n(\mathbf{M}, N) \circ \varepsilon_{\mathcal{G}}^n(M, N) = \varepsilon_{\mathcal{G}}^{n+1}(M'', N) \circ \mathfrak{d}_{\mathcal{G}}^n(\mathbf{M}, N) \quad \text{for all } n \in \mathbb{Z}.$$

*Proof.* The lower tier of the diagram in Lemma 4.5 yields a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_R(\mathbf{G}'', N) & \longrightarrow & \mathrm{Hom}_R(\mathbf{G}', N) & \longrightarrow & \mathrm{Hom}_R(\mathbf{G}, N) \longrightarrow 0 \\ & & \downarrow \mathrm{Hom}_R(\phi'', N) & & \downarrow \mathrm{Hom}_R(\phi', N) & & \downarrow \mathrm{Hom}_R(\phi, N) \\ 0 & \longrightarrow & \mathrm{Hom}_R(\mathbf{P}'', N) & \longrightarrow & \mathrm{Hom}_R(\mathbf{P}', N) & \longrightarrow & \mathrm{Hom}_R(\mathbf{P}, N) \longrightarrow 0 \end{array}$$

with exact rows. The homology exact sequence of the top row is the desired long exact sequence. Commutativity of the diagram gives the formula for the connecting maps. Naturality in  $N$  is clear. To get naturality in  $\mathbf{M}$ , take a morphism  $\mathbf{M} \rightarrow \dot{\mathbf{M}}$  and apply  $\mathrm{Hom}_R(\cdot, N)$  to the commuting up to homotopy diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{G} & \xrightarrow{\tilde{\mu}} & \mathbf{G}' & \xrightarrow{\tilde{\mu}'} & \mathbf{G}'' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \dot{\mathbf{G}} & \xrightarrow{\tilde{\mu}} & \dot{\mathbf{G}}' & \xrightarrow{\tilde{\mu}'} & \dot{\mathbf{G}}'' \longrightarrow 0 \end{array}$$

with rows provided by Construction 4.5 and columns by Lemma 4.3.  $\square$

Finite direct sums of proper resolutions are proper resolutions, and for each finite  $R$ -module  $G$  the functor  $\mathrm{Hom}_R(G, \cdot)$  commutes with arbitrary direct sums, hence

**4.7. Proposition.** *For each finite set  $\{M_i \in \overline{\mathcal{G}} \mid i \in I\}$  the module  $\coprod_{i \in I} M_i$  is in  $\overline{\mathcal{G}}$ , and for any family  $\{N_j \in \mathcal{M} \mid j \in J\}$  there is a natural isomorphism*

$$\mathrm{Ext}_{\mathcal{G}}^n \left( \prod_{i \in I} M_i, \prod_{j \in J} N_j \right) \cong \prod_{(i,j) \in I \times J} \mathrm{Ext}_{\mathcal{G}}^n(M_i, N_j). \quad \square$$

*Proof of Theorem 4.2.4.* By assumption,  $\mathrm{G-dim}_R M = g < \infty$  and  $N$  is projective. The map  $\varepsilon_{\mathcal{G}}^n(M, N): \mathrm{Ext}_{\mathcal{G}}^n(M, N) \rightarrow \mathrm{Ext}_R^n(M, N)$  is bijective for  $n \leq 0$  by Theorem 4.2.2.b. By induction on  $g$  we show that it is bijective for all  $n$ . If  $g = 0$  and  $n > 0$ , then  $\mathrm{Ext}_{\mathcal{G}}^n(M, R) = 0$  by Theorem 4.2.2a, and  $\mathrm{Ext}_R^n(M, R) = 0$  by definition. If  $g > 0$  Theorem 3.1 yields a  $\mathcal{G}$ -resolution  $\mathbf{G} \rightarrow M$  of length  $g$ , so  $\mathrm{G-dim}_R \Omega^1 \mathbf{G} < g$ . Thus, each  $\varepsilon_{\mathcal{G}}^n(G_0, R)$  and  $\varepsilon_{\mathcal{G}}^n(\Omega^1 \mathbf{G}, R)$  is bijective by the induction hypothesis. The Five-Lemma, used on the commutative diagram given by Proposition 4.6 and the exact sequence  $0 \rightarrow \Omega^1 \mathbf{G} \rightarrow G_0 \rightarrow M \rightarrow 0$ , shows that  $\varepsilon_{\mathcal{G}}^n(M, R)$  is bijective.  $\square$

For each finite module  $M \neq 0$  a *relative  $\mathcal{G}$ -dimension* is defined by

$$\mathrm{rel\,dim}_{\mathcal{G}} M = \inf \left\{ g \in \mathbb{N} \mid \begin{array}{l} \text{there exists a proper } \mathcal{G}\text{-resolution} \\ \mathbf{G} \rightarrow M \text{ of length } \leq g \end{array} \right\}$$

and set  $\mathrm{rel\,dim}_{\mathcal{G}} 0 = -\infty$ ; thus,  $\mathrm{rel\,dim}_{\mathcal{G}} M = \infty$  for  $M \notin \overline{\mathcal{G}}$ .

The next result is proved by Enochs and Jenda [19, (5.2)] over commutative Gorenstein local rings; an independent general proof is given by Holm [23, (4.51)].

**4.8. Proposition.** *Every finite  $R$ -module  $M$  satisfies  $\mathrm{G-dim}_R M = \mathrm{rel\,dim}_{\mathcal{G}} M$ .*

*Proof.* Any proper resolution is a  $\mathcal{G}$ -resolution, so  $\mathrm{G-dim}_R M \leq \mathrm{rel\,dim}_{\mathcal{G}} M$ .

Equality holds trivially when  $M = 0$  or  $\mathrm{G-dim}_R M = \infty$ , so we assume that  $\mathrm{G-dim}_R M = g$  is finite. Theorem 3.1 yields a strict  $\mathcal{G}$ -resolution of length  $\leq g$ . Such a resolution is proper by Lemma 4.1.3, so we get  $\mathrm{rel\,dim}_{\mathcal{G}} M \leq g$ . Putting together the preceding relations, we obtain  $\mathrm{rel\,dim}_{\mathcal{G}} M = g = \mathrm{G-dim}_R M$ .  $\square$

The next theorem combines several results of Auslander and Bridger, cf. [2, (3.14), (4.13.a), (4.15)] (where some are proved assuming commutativity). Our proofs, based on relative cohomology, are far from the original arguments.

**4.9. Theorem.** *For any finite  $R$ -module  $M$  the following hold.*

(1) *There are inequalities*

$$\sup\{n \in \mathbb{N} \mid \text{Ext}_R^n(M, R) \neq 0\} \leq \text{G-dim}_R M \leq \text{proj dim}_R M$$

*and equalities hold to the left of any finite dimension.*

(2) *If  $0 \rightarrow M \rightarrow M' \rightarrow M'' \rightarrow 0$  is an exact sequence, then*

$$\text{G-dim}_R M' \leq \max\{\text{G-dim}_R M, \text{G-dim}_R M''\}$$

*with strict inequality possible only if  $\text{G-dim}_R M = \text{G-dim}_R M'' + 1$ .*

(3) *If  $M = M' \oplus M''$ , then  $\text{G-dim}_R(M) = \max\{\text{G-dim}_R M', \text{G-dim}_R M''\}$ .*

(4) *Each  $G$ -resolution  $\mathbf{G} \rightarrow M$  has  $\Omega^n \mathbf{G} \in \mathcal{G}$  for all  $n \geq \text{G-dim}_R M$ .*

*Proof.* (1) The inequality on the right hand side follows directly from Lemma 2.3, and Theorem 4.2.3 implies that equality holds when  $\text{proj dim}_R M$  is finite.

The inequality on the left hand side is clear, so assume  $\text{G-dim}_R M = g < \infty$  and  $\text{Ext}_R^g(M, R) = 0$ . For  $g = 0$  this means that  $M^* = 0$ ; as  $M$  is reflexive, we get  $M = 0$ , contradicting our hypothesis. When  $g \geq 1$ , Proposition 4.8 yields a proper resolution  $\mathbf{G} \rightarrow M$  of length  $g$ , and shows each proper resolution has length at least  $g$ . Our assumption means that the map  $\partial_g^*: G_{g-1}^* \rightarrow G_g^*$  is surjective. The  $R^0$ -module  $G_g^*$  is then projective, so  $\partial_g^*$  is split by a homomorphism  $\sigma: G_g^* \rightarrow G_{g-1}^*$ . Thus,  $\sigma^* \partial_g = \text{id}_{G_g}$ , hence  $\mathbf{E} = 0 \rightarrow G_g \rightarrow \partial_g(G_g) \rightarrow 0$  is an irrelevant subcomplex of  $\mathbf{G}$ . We then get a proper resolution  $\mathbf{G}/\mathbf{E} \rightarrow M$  of length  $< g$ , which is impossible.

(2) The assertion holds trivially if any one of the modules is equal to 0, so we assume that all G-dimensions involved are non-negative. If both  $\text{G-dim}_R M$  and  $\text{G-dim}_R M''$  are infinite, then there is nothing to prove. If one is finite and the other is not, then Proposition 3.4 yields  $\text{G-dim}_R M' = \infty$ , so the desired equality holds. If  $\text{G-dim}_R(M)$  and  $\text{G-dim}_R M''$  are finite, then Proposition 3.4 shows that  $\text{G-dim}_R M'$  is finite as well. To see the desired (in)equality it suffices to bear in mind (1) while staring at the cohomology exact sequence

$$\begin{aligned} \text{Ext}_R^{n-1}(M, R) &\longrightarrow \text{Ext}_R^n(M'', R) \longrightarrow \text{Ext}_R^n(M', R) \longrightarrow \\ &\text{Ext}_R^n(M, R) \longrightarrow \text{Ext}_R^{n+1}(M'', R) \end{aligned}$$

(3) In view of (2) we may assume that  $M'$ ,  $M''$ , and their direct sum have finite G-dimension. Now apply Proposition 4.7 with  $N = R$ , and use (1).

(4) is an easy consequence of (2).  $\square$

**4.10. Remark.** Yoneda congruence relations involving *only* proper short exact sequences define equivalence classes of proper exact sequences of length  $n$ , starting at  $N$  and ending at  $M$ . The result is an abelian *group of proper extensions* that depends functorially on  $M$  and  $N$ , cf. MacLane [25, XII §4]. The map sending the class of each proper sequence to its congruence class with respect to Yoneda equivalence using *all* short exact sequences is a morphism of extension functors.

If  $M$  has a proper  $\mathcal{G}$ -resolution, then there exists a canonical isomorphism of the group of proper extensions with  $\text{Ext}_{\mathcal{G}}^n(M, N)$ ; it transforms the morphism of extension functors into the morphism  $\varepsilon^n: \text{Ext}_{\mathcal{G}}^n \rightarrow \text{Ext}_R^n$ , cf. [25, XII §9].

## 5. TATE COHOMOLOGY

For each module  $M$  of finite G-dimension over a noetherian ring  $R$  choose a complete resolution  $\mathbf{T} \xrightarrow{\vartheta} \mathbf{P} \xrightarrow{\pi} M$  by Theorem 3.1, then for each  $R$ -module  $N$  and for each  $n \in \mathbb{Z}$  define a *Tate cohomology group* by the equality

$$\widehat{\mathrm{Ext}}_R^n(M, N) = \mathrm{H}^n \mathrm{Hom}_R(\mathbf{T}, N).$$

These groups come equipped with *comparison homomorphisms*

$$\varepsilon_R^n(M, N): \mathrm{Ext}_R^n(M, N) \longrightarrow \widehat{\mathrm{Ext}}_R^n(M, N)$$

given by  $\mathrm{H}^n \mathrm{Hom}_R(\vartheta, N): \mathrm{H}^n \mathrm{Hom}_R(\mathbf{P}, N) \rightarrow \mathrm{H}^n \mathrm{Hom}_R(\mathbf{T}, N)$ .

**5.1. Example.** Let  $\Pi$  be a finite group,  $R = \mathbb{Z}[\Pi]$  its group ring, and turn  $\mathbb{Z}$  into an  $R$ -module by  $gm = m$  for  $g \in \Pi$  and  $m \in \mathbb{Z}$ . The map  $\sum_{g \in \Pi} a_g g \mapsto \sum_{g \in \Pi} a_g g^{-1}$  is an involution, and the map  $h \mapsto (\sum_{g \in \Pi} a_g g \mapsto a_h)$  defines an  $R$ -linear isomorphism  $R \cong R^\vee$ , so  $\mathrm{G-dim}_R \mathbb{Z} = 0$  by Example 3.3.2, hence  $\widehat{\mathrm{Ext}}_R^n(\mathbb{Z}, N)$  is defined. This is the original  $n$ th Tate cohomology group  $\widehat{\mathrm{H}}^n(\Pi, N)$ , cf. [12, XII §3].

**5.2. Theorem.** *The assignment  $(M, N) \mapsto \widehat{\mathrm{Ext}}_R^n(M, N)$  defines a functor*

$$\widehat{\mathrm{Ext}}_R^n: \widetilde{\mathcal{G}}(R)^{\mathrm{op}} \times \mathcal{M}(R) \longrightarrow \mathcal{M}(\mathbb{Z})$$

and the maps  $\varepsilon_R^n(M, N)$  yield a morphism of functors  $\varepsilon_R^n: \mathrm{Ext}_R^n \rightarrow \widehat{\mathrm{Ext}}_R^n$  such that:

- (1)  $\widehat{\mathrm{Ext}}_R^n$  and  $\varepsilon_R^n$  are independent of the choices of resolutions and lifting.
- (2) For any integer  $g$  the following conditions are equivalent.
  - (i)  $\mathrm{G-dim}_R M \leq g$ .
  - (ii)  $\varepsilon_R^n(M, N): \mathrm{Ext}_R^n(M, N) \rightarrow \widehat{\mathrm{Ext}}_R^n(M, N)$  is bijective for all  $n > g$ .
- (3) If  $\mathrm{proj dim}_R M < \infty$ , then  $\widehat{\mathrm{Ext}}_R^n(M, \ ) = 0$  for all  $n \in \mathbb{Z}$ .
- (4) If  $\mathrm{proj dim}_R N < \infty$ , then  $\widehat{\mathrm{Ext}}_R^n(\ , N) = 0$  for all  $n \in \mathbb{Z}$ .

*Remark.* Converses to Parts (3) and (4) are proved in Theorem 5.9.

**5.3. Lemma.** *Let  $\mathbf{T} \xrightarrow{\vartheta} \mathbf{P} \xrightarrow{\pi} M$  and  $\mathbf{T}' \xrightarrow{\vartheta'} \mathbf{P}' \xrightarrow{\pi'} M'$  be complete resolutions.*

*For each homomorphism of modules  $\mu: M \rightarrow M'$  there exists a unique up to homotopy morphism  $\bar{\mu}$ , making the right hand square of the diagram*

$$\begin{array}{ccccc} \mathbf{T} & \xrightarrow{\vartheta} & \mathbf{P} & \xrightarrow{\pi} & M \\ \downarrow \widehat{\mu} & & \downarrow \bar{\mu} & & \downarrow \mu \\ \mathbf{T}' & \xrightarrow{\vartheta'} & \mathbf{P}' & \xrightarrow{\pi'} & M' \end{array}$$

*commute, and for each choice of  $\bar{\mu}$  there exists a unique up to homotopy morphism  $\widehat{\mu}$ , making the left hand square commute up to homotopy.*

*If  $\mu = \mathrm{id}_M$ , then  $\bar{\mu}$  and  $\widehat{\mu}$  are homotopy equivalences.*

*Proof.* The existence of  $\bar{\mu}$  and its uniqueness up to homotopy are classical, cf. 1.2.1 and 1.5; thus, we only have to deal with  $\widehat{\mu}$ .

Assuming first that  $\vartheta'$  is surjective, we set  $\mathbf{Q}' = \mathrm{Ker} \vartheta'$  and  $\mathbf{Q}'^{(i)} = \mathbf{Q}'_{>-i}$ . Thus, for each  $i \in \mathbb{Z}$  we get an exact sequence of complexes

$$0 \longrightarrow \Sigma^{-i}(\mathbf{Q}'_{-i}) \longrightarrow \mathbf{Q}'^{(i)} \xrightarrow{\beta'^{(i)}} \mathbf{Q}'^{(i-1)} \longrightarrow 0$$

with  $\beta^{(i)}$  the canonical epimorphism. It induces an exact sequence of complexes

$$0 \longrightarrow \mathrm{Hom}_R(\mathbf{T}, \Sigma^{-i}(Q'_{-i})) \longrightarrow \mathrm{Hom}_R(\mathbf{T}, Q^{(i)}) \longrightarrow \mathrm{Hom}_R(\mathbf{T}, Q^{(i-1)}) \longrightarrow 0$$

As  $Q^{(i)} = 0$  for  $i \ll 0$ , we assume by induction on  $i$  that  $\mathrm{H}_n \mathrm{Hom}_R(\mathbf{T}, Q^{(i-1)})$  vanishes for some  $i$  and all  $n$ . Lemma 2.2 yields  $\mathrm{H}_n \mathrm{Hom}_R(\mathbf{T}, \Sigma^{-i}Q'_{-i}) = 0$ , hence  $\mathrm{H}_n \mathrm{Hom}_R(\mathbf{T}, Q^{(i)}) = 0$ . Since  $Q' = \varprojlim Q^{(i)}$ , we get

$$\mathrm{Hom}_R(\mathbf{T}, Q') = \mathrm{Hom}_R(\mathbf{T}, \varprojlim Q^{(i)}) = \varprojlim \mathrm{Hom}_R(\mathbf{T}, Q^{(i)})$$

so  $\mathrm{H}_n \mathrm{Hom}_R(\mathbf{T}, Q') = 0$  for all  $n \in \mathbb{Z}$  by the Mittag-Leffler Criterion 1.3. The exact sequence of complexes  $0 \rightarrow Q' \rightarrow \mathbf{T}' \xrightarrow{\vartheta'} \mathbf{P}' \rightarrow 0$  induces an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(\mathbf{T}, Q') \longrightarrow \mathrm{Hom}_R(\mathbf{T}, \mathbf{T}') \xrightarrow{\mathrm{Hom}_R(\mathbf{T}, \vartheta')} \mathrm{Hom}_R(\mathbf{T}, \mathbf{P}') \longrightarrow 0$$

of complexes of abelian groups, which shows  $\mathrm{Hom}_R(\mathbf{T}, \vartheta')$  is a surjective quasiisomorphism. By 1.1.1 we get a morphism  $\widehat{\mu}$  satisfying  $\vartheta' \widehat{\mu} = \overline{\mu} \vartheta$ .

In general, factor  $\vartheta'$  as  $\mathbf{T}' \xrightarrow{\alpha} \mathbf{T}'' \xrightarrow{\vartheta''} \mathbf{P}'$ , with a homotopy equivalence  $\alpha$  and surjective morphism  $\vartheta''$ , cf. Construction 3.7. Thus,  $\mathrm{Hom}_R(\mathbf{T}, \alpha)$  is a quasiisomorphism. The map  $\mathrm{Hom}_R(\mathbf{T}, \vartheta'')$  is one by the special case above, hence so is  $\mathrm{Hom}_R(\mathbf{T}, \vartheta') = \mathrm{Hom}_R(\mathbf{T}, \vartheta'') \circ \mathrm{Hom}_R(\mathbf{T}, \alpha)$ . Now 1.1.1 yields a unique up to homotopy morphism  $\widehat{\mu}$  such that  $\vartheta' \widehat{\mu} \sim \overline{\mu} \vartheta$ .

If  $\mu = \mathrm{id}_M$ , then reversing the roles of  $M$  and  $M'$  we get a morphism  $\widehat{\mu}': \mathbf{T}' \rightarrow \mathbf{T}$  inducing  $\mathrm{id}_M$ . Thus,  $\widehat{\mu} \widehat{\mu}': \mathbf{T}' \rightarrow \mathbf{T}'$  induces  $\mathrm{id}_M$ , and hence is homotopic to  $\mathrm{id}_{\mathbf{T}'}$ . By symmetry,  $\widehat{\mu}' \widehat{\mu} \sim \mathrm{id}_{\mathbf{T}}$ , so  $\widehat{\mu}$  is a homotopy equivalence.  $\square$

*Proof of Theorem 5.2.* The naturality of  $\widehat{\mathrm{Ext}}_R^n$  and  $\varepsilon_R^n$  is obtained by applying the first assertion of Lemma 5.3 to the chosen complete resolutions of  $M$  and  $M'$ .

(1) is a consequence of the last assertion of that lemma.

(3) If  $\mathrm{proj\,dim}_R M = p < \infty$ , then in Construction 3.6 choose a  $\mathcal{P}$ -resolution  $\mathbf{P} \rightarrow M$  of length  $p$ , set  $g = p + 1$ , and resolve  $\Omega^p \mathbf{G} = 0$  with  $\mathbf{K} = 0$ ; the resulting complete resolution is  $0 \rightarrow \mathbf{P} \rightarrow M$ , hence  $\widehat{\mathrm{Ext}}_R^n(M, \ ) = 0$ .

(4) results directly from Lemma 2.4.

(2) (i)  $\implies$  (ii). By (1) and Theorem 3.1 the morphism  $\varepsilon_R^n(M, \ )$  can be computed using a Tate resolution  $\vartheta: \mathbf{T} \rightarrow \mathbf{P}$  with  $\vartheta_n$  bijective for  $n \geq \mathrm{G-dim}_R M$ , so  $\varepsilon_R^n(M, \ )$  is an isomorphism for  $n > \mathrm{G-dim}_R M$ .

(ii)  $\implies$  (i). If  $\varepsilon_R^n(M, \ )$  is bijective, then in particular  $\mathrm{Ext}_R^n(M, R) = \widehat{\mathrm{Ext}}_R^n(M, R)$ . The last group is trivial by (4), so  $\mathrm{Ext}_R^n(M, R) = 0$  for  $n > g$ . Theorem 4.9.1 now implies  $g \leq \mathrm{G-dim}_R M$ .  $\square$

With a fixed first argument,  $\widehat{\mathrm{Ext}}_R^*(M, \ )$  is a cohomology functor on  $\mathcal{M}$ .

**5.4. Proposition.** *For each  $R$ -module  $M$  of finite  $G$ -dimension and each exact sequence  $\mathbf{N} = 0 \rightarrow N \xrightarrow{\nu} N' \xrightarrow{\nu'} N'' \rightarrow 0$  of  $R$ -modules there exist natural in  $M$  and  $\mathbf{N}$  homomorphisms  $\widehat{\mathfrak{O}}^n(M, \mathbf{N})$ , such that the sequence below is exact*

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \widehat{\mathrm{Ext}}_R^n(M, N) & \xrightarrow{\widehat{\mathrm{Ext}}_R^n(M, \nu)} & \widehat{\mathrm{Ext}}_R^n(M, N') & \xrightarrow{\widehat{\mathrm{Ext}}_R^n(M, \nu')} & \widehat{\mathrm{Ext}}_R^n(M, N'') \\ & & \widehat{\mathfrak{O}}^n(M, \mathbf{N}) & \longrightarrow & \widehat{\mathrm{Ext}}_R^{n+1}(M, N) & \longrightarrow & \cdots \end{array}$$

and the connecting maps  $\widehat{\mathfrak{O}}^n(M, \mathbf{N}): \widehat{\mathrm{Ext}}_R^n(M, N'') \rightarrow \widehat{\mathrm{Ext}}_R^{n+1}(M, N)$  satisfy

$$\widehat{\mathfrak{O}}^n(M, \mathbf{N}) \circ \varepsilon_R^n(M, N'') = \varepsilon_R^{n+1}(M, N) \circ \widehat{\mathfrak{O}}^n(M, \mathbf{N}) \quad \text{for all } n \in \mathbb{Z}.$$

*Proof.* A complete resolution  $\mathbf{T} \rightarrow \mathbf{P} \rightarrow M$  induces a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_R(\mathbf{P}, N) & \longrightarrow & \mathrm{Hom}_R(\mathbf{P}, N') & \longrightarrow & \mathrm{Hom}_R(\mathbf{P}, N'') & \longrightarrow & 0 \\ & & \downarrow \mathrm{Hom}_R(\vartheta, N) & & \downarrow \mathrm{Hom}_R(\vartheta, N') & & \downarrow \mathrm{Hom}_R(\vartheta, N'') & & \\ 0 & \longrightarrow & \mathrm{Hom}_R(\mathbf{T}, N) & \longrightarrow & \mathrm{Hom}_R(\mathbf{T}, N') & \longrightarrow & \mathrm{Hom}_R(\mathbf{T}, N'') & \longrightarrow & 0 \end{array}$$

of complexes, whose rows are exact because  $P_n$  and  $T_n$  are projective for all  $n$ . The homology exact sequence of the bottom row is the desired long exact sequence. The commutativity of the diagram gives the formula for the connecting maps. Naturality in  $\mathbf{N}$  is clear; naturality in  $M$  follows from Lemma 5.3.  $\square$

We prove a version of the Horseshoe Lemma for complete resolutions.

**5.5. Lemma.** *Let  $\mathbf{M} = 0 \rightarrow M \xrightarrow{\mu} M' \xrightarrow{\mu'} M'' \rightarrow 0$  be an exact sequence of finite  $R$ -modules with  $\max\{\mathrm{G-dim}_R M, \mathrm{G-dim}_R M''\} \leq g < \infty$ .*

*If  $\pi: \mathbf{P} \rightarrow M$  and  $\pi'': \mathbf{P}'' \rightarrow M''$  are  $\mathcal{P}$ -resolutions, then there exists a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{\mu} & M' & \xrightarrow{\mu'} & M'' & \longrightarrow & 0 \\ & & \uparrow \pi & & \uparrow \pi' & & \uparrow \pi'' & & \\ 0 & \longrightarrow & \mathbf{P} & \xrightarrow{\bar{\mu}} & \mathbf{P}' & \xrightarrow{\bar{\mu}'} & \mathbf{P}'' & \longrightarrow & 0 \\ & & \uparrow \vartheta & & \uparrow \vartheta' & & \uparrow \vartheta'' & & \\ 0 & \longrightarrow & \mathbf{T} & \xrightarrow{\hat{\mu}} & \mathbf{T}' & \xrightarrow{\hat{\mu}'} & \mathbf{T}'' & \longrightarrow & 0 \end{array}$$

whose columns are surjective complete resolutions that are bijective in degrees  $\geq g$ .

*Proof.* The classical Horseshoe Lemma 1.4 provides the upper tier of the diagram. In the exact sequence  $0 \rightarrow \Omega^g \mathbf{P} \rightarrow \Omega^g \mathbf{P}' \rightarrow \Omega^g \mathbf{P}'' \rightarrow 0$  all modules are totally reflexive by Lemmas 2.5 and 2.3. Lemma 2.2 then guarantees the exactness of the top row in the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\Omega^g \mathbf{P}'')^* & \longrightarrow & (\Omega^g \mathbf{P}')^* & \longrightarrow & (\Omega^g \mathbf{P})^* & \longrightarrow & 0 \\ & & \uparrow \lambda'' & & \uparrow \lambda' & & \uparrow \lambda & & \\ 0 & \longrightarrow & \mathbf{L}'' & \xrightarrow{\psi''} & \mathbf{L}' & \xrightarrow{\psi'} & \mathbf{L} & \longrightarrow & 0 \end{array}$$

The rest of the diagram is obtained by choosing  $\mathcal{P}$ -resolutions  $\lambda$  and  $\lambda''$ , then using 1.4 again; thus, the bottom row is split exact, considered as a sequence of graded  $R$ -modules. Lift  $\omega_g^{\mathbf{P}}$  to a morphism  $\kappa: \Sigma^{g-1}((\mathbf{P}_{<g})^*) \rightarrow \mathbf{L}$ , do likewise for  $\omega_g^{\mathbf{P}''}$ , then use 1.4 for a third time to get a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{L}'' & \xrightarrow{\psi''} & \mathbf{L}' & \xrightarrow{\psi'} & \mathbf{L} & \longrightarrow & 0 \\ & & \uparrow \kappa'' & & \uparrow \kappa' & & \uparrow \kappa & & \\ 0 & \longrightarrow & \Sigma^{g-1}((\mathbf{P}''_{<g})^*) & \longrightarrow & \Sigma^{g-1}((\mathbf{P}'_{<g})^*) & \longrightarrow & \Sigma^{g-1}((\mathbf{P}_{<g})^*) & \longrightarrow & 0 \end{array}$$

where the maps in the bottom row are induced by  $\bar{\mu}$  and  $\bar{\mu}'$ .

Feeding these data to Construction 3.6, form the columns of the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{\mu} & M' & \xrightarrow{\mu'} & M'' & \longrightarrow & 0 \\
& & \uparrow \pi & & \uparrow \pi' & & \uparrow \pi'' & & \\
0 & \longrightarrow & \mathbf{P} & \xrightarrow{\bar{\mu}} & \mathbf{P}' & \xrightarrow{\bar{\mu}'} & \mathbf{P}'' & \longrightarrow & 0 \\
& & \uparrow \sigma & & \uparrow \sigma' & & \uparrow \sigma'' & & \\
0 & \longrightarrow & \mathbf{S} & \xrightarrow{\xi} & \mathbf{S}' & \xrightarrow{\xi'} & \mathbf{S}'' & \longrightarrow & 0
\end{array}$$

where  $\xi_n = \pi_n$  for  $n \geq g$ ,  $\xi_n = \psi_{g-n-1}^{i*}$  for  $n < g$ , and  $\xi'$  is defined similarly. The explicit descriptions of its maps show that the diagram commutes. Processing the columns through Construction 3.7, we obtain the columns of the desired diagram, then complete its construction by setting  $\widehat{\mu}(x, y, y') = (\xi(x), \bar{\mu}(y), \bar{\mu}(y'))$ , and defining  $\widehat{\mu}'$  is in a similar manner. Once again, the commutativity of the diagram results from the explicit formulas used in its construction.  $\square$

The next result reflects the fact that Tate cohomology is the restriction to  $\widetilde{\mathcal{G}}$  of a cohomology functor on  $\mathcal{F}$ , cf. 5.11.

**5.6. Proposition.** *For each exact sequence  $\mathbf{M} = 0 \rightarrow M \xrightarrow{\mu} M' \xrightarrow{\mu'} M'' \rightarrow 0$  of  $R$ -modules of finite  $G$ -dimension and each  $R$ -module  $N$  there exist natural in  $\mathbf{M}$  and  $N$  homomorphisms  $\widehat{\partial}^n(\mathbf{M}, N)$ , such that the sequence below is exact*

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \widehat{\text{Ext}}_R^n(M'', N) & \xrightarrow{\widehat{\text{Ext}}_R^n(\mu', N)} & \widehat{\text{Ext}}_R^n(M', N) & \xrightarrow{\widehat{\text{Ext}}_R^n(\mu, N)} & \widehat{\text{Ext}}_R^n(M, N) \\
& & \xrightarrow{\widehat{\partial}^n(\mathbf{M}, N)} & \widehat{\text{Ext}}_R^{n+1}(M'', N) & \longrightarrow & \cdots & 
\end{array}$$

and the connecting maps  $\widehat{\partial}^n(\mathbf{M}, N): \widehat{\text{Ext}}_R^n(M, N) \rightarrow \widehat{\text{Ext}}_R^{n+1}(M'', N)$  satisfy

$$\widehat{\partial}^n(\mathbf{M}, N) \circ \varepsilon_R^n(M, N) = \varepsilon_R^{n+1}(M'', N) \circ \widehat{\partial}^n(\mathbf{M}, N) \quad \text{for all } n \in \mathbb{Z}.$$

*Proof.* Since all modules  $P_n$  and  $T_n$  are projective, applying  $\text{Hom}_R(\cdot, N)$  to the lower tier of the diagram in Construction 5.5, we get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Hom}_R(\mathbf{P}'', N) & \longrightarrow & \text{Hom}_R(\mathbf{P}', N) & \longrightarrow & \text{Hom}_R(\mathbf{P}, N) & \longrightarrow & 0 \\
& & \downarrow \text{Hom}_R(\vartheta'', N) & & \downarrow \text{Hom}_R(\vartheta', N) & & \downarrow \text{Hom}_R(\vartheta, N) & & \\
0 & \longrightarrow & \text{Hom}_R(\mathbf{T}'', N) & \longrightarrow & \text{Hom}_R(\mathbf{T}', N) & \longrightarrow & \text{Hom}_R(\mathbf{T}, N) & \longrightarrow & 0
\end{array}$$

with exact rows. The homology exact sequence of the bottom row is the desired long exact sequence. Commutativity of the diagram gives the formula for the connecting maps. Naturality in  $N$  is clear. To get naturality in  $\mathbf{M}$ , take a morphism  $\mathbf{M} \rightarrow \dot{\mathbf{M}}$  and apply  $\text{Hom}_R(\cdot, N)$  to the commuting up to homotopy diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{T} & \xrightarrow{\widehat{\mu}} & \mathbf{T}' & \xrightarrow{\widehat{\mu}'} & \mathbf{T}'' & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \dot{\mathbf{T}} & \xrightarrow{\widehat{\mu}} & \dot{\mathbf{T}}' & \xrightarrow{\widehat{\mu}'} & \dot{\mathbf{T}}'' & \longrightarrow & 0
\end{array}$$

with rows provided by Construction 5.5 and columns by Lemma 5.3.  $\square$

Finite direct sums of complete resolutions are complete resolutions, and for each finite  $R$ -module  $T$  the functor  $\text{Hom}_R(T, \cdot)$  commutes with all direct sums, hence

**5.7. Proposition.** *For each finite set  $\{M_i \in \tilde{\mathcal{G}} \mid i \in I\}$  the module  $\prod_{i \in I} M_i$  is in  $\tilde{\mathcal{G}}$ , and for any family  $\{N_j \in \mathcal{M} \mid j \in J\}$  there is a natural isomorphism*

$$\widehat{\text{Ext}}_R^n \left( \prod_{i \in I} M_i, \prod_{j \in J} N_j \right) \cong \prod_{(i,j) \in I \times J} \widehat{\text{Ext}}_R^n(M_i, N_j). \quad \square$$

Let  $M$  be a finite  $R$ -module. Recall that a  $\mathcal{G}$ -approximation of  $M$  is a resolution  $\chi: \mathbf{B} \rightarrow M$ , with  $B_0 \in \mathcal{G}$ ,  $B_1 \in \tilde{\mathcal{P}}$ , and  $B_n = 0$  for  $n \geq 2$ , and that by Theorem 3.1  $M$  has a  $\mathcal{G}$ -approximation if and only if it has finite  $\mathcal{G}$ -dimension.

**5.8. Lemma.** *If  $0 \rightarrow Y \xrightarrow{\nu} X \xrightarrow{\chi} M \rightarrow 0$  is a  $\mathcal{G}$ -approximation, then for all  $R$ -modules  $L \in \tilde{\mathcal{G}}$  and  $N \in \mathcal{M}$  the following hold.*

- (1) *If  $\chi': G \rightarrow M$  is a homomorphism with  $G \in \mathcal{G}$ , then there exists a homomorphism  $\psi: G \rightarrow X$  with  $\chi\psi = \chi'$ .*
- (2) *For all  $n \in \mathbb{Z}$  the following homomorphisms are bijective:*

$$\begin{aligned} \widehat{\text{Ext}}_R^n(L, \chi): \widehat{\text{Ext}}_R^n(L, M) &\longrightarrow \widehat{\text{Ext}}_R^n(L, X); \\ \widehat{\text{Ext}}_R^n(\chi, N): \widehat{\text{Ext}}_R^n(M, N) &\longrightarrow \widehat{\text{Ext}}_R^n(X, N). \end{aligned}$$

- (3) *There is a natural exact sequence where  $\nu(\alpha \otimes y)(x) = \alpha(x)y$ :*

$$0 \longrightarrow \widehat{\text{Ext}}_R^{-1}(M, N) \longrightarrow X^* \otimes_R N \xrightarrow{\nu} \text{Hom}_R(X, N) \longrightarrow \widehat{\text{Ext}}_R^0(M, N) \longrightarrow 0$$

- (4) *For each exact sequence  $\mathbf{M} = 0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$  with  $P \in \mathcal{P}$  the maps*

$$\begin{aligned} \partial^n(L, \mathbf{M}): \widehat{\text{Ext}}_R^n(L, M) &\longrightarrow \widehat{\text{Ext}}_R^{n+1}(L, K); \\ \partial^n(\mathbf{M}, N): \widehat{\text{Ext}}_R^n(K, N) &\longrightarrow \widehat{\text{Ext}}_R^{n+1}(M, N). \end{aligned}$$

*given by Propositions 5.4 and 5.6 are bijective for all  $n \in \mathbb{Z}$ .*

*Proof.* (1) is a consequence of Lemma 4.1.2.

For (2) and (4) use Propositions 5.4 and 5.6 with Theorems 5.2.3 and 5.2.4.

(3) By (2) we may assume that  $M = X$ . Construction 3.6 yields a complete resolution  $\mathbf{S} \rightarrow \mathbf{S}_{\geq 0} \rightarrow X$ . The exactness of  $\mathbf{S}^*$  implies that  $\Omega^1 \mathbf{S}^*$  is isomorphic to  $X^*$  and that  $\Sigma^{-1}((\mathbf{S}_{<0})^*) \rightarrow X^*$  is a projective resolution. In view of the canonical isomorphism  $\text{Hom}_R(\mathbf{S}_{<0}, N) \cong (\mathbf{S}_{<0})^* \otimes_R N$  we have an exact sequence

$$0 \longrightarrow \text{Hom}_R(\mathbf{S}_{\geq 0}, N) \longrightarrow \text{Hom}_R(\mathbf{S}, N) \longrightarrow (\mathbf{S}_{<0})^* \otimes_R N \longrightarrow 0$$

A direct computation with the expression for  $\partial_0^{\mathbf{S}}$  from Construction 3.6 shows that in degrees 1, 0 the associated homology sequence is the desired exact sequence.  $\square$

Unlike absolute or relative cohomology, no numerical invariant is defined by vanishing of Tate cohomology functors. The reason is that this cohomology theory is rigid, a fact established in the next result.

**5.9. Theorem.** *For a module  $M$  of finite  $\mathcal{G}$ -dimension the following are equivalent.*

- (i)  $\text{proj dim}_R M < \infty$ .
- (ii)  $\widehat{\text{Ext}}_R^n(M, \quad) = 0$  for some  $n \in \mathbb{Z}$ .
- (ii')  $\widehat{\text{Ext}}_R^n(M, \quad) = 0$  for all  $n \in \mathbb{Z}$ .
- (iii)  $\widehat{\text{Ext}}_R^n(\quad, M) = 0$  for some  $n \in \mathbb{Z}$ .
- (iii')  $\widehat{\text{Ext}}_R^n(\quad, M) = 0$  for all  $n \in \mathbb{Z}$ .
- (iv)  $\widehat{\text{Ext}}_R^0(M, M) = 0$ .

*Proof.* Theorems 5.2.3 and 5.2.4 show that (i) implies (ii') and (iii'), from where (ii) and (iii) clearly follow. We choose a  $\mathcal{G}$ -approximation  $0 \rightarrow Y \rightarrow X \rightarrow M \rightarrow 0$  and use Lemma 5.8.2 to replace  $M$  by  $X$  in conditions (ii), (iii), and (iv). Choosing a surjective complete resolution  $\mathbf{T} \rightarrow \mathbf{P} \rightarrow X$ , from Lemma 5.8.4 we get isomorphisms  $\widehat{\text{Ext}}_R^n(X, \Omega^{-n}\mathbf{T}) \cong \widehat{\text{Ext}}_R^0(X, X) \cong \widehat{\text{Ext}}_R^n(\Omega^n\mathbf{T}, X)$ ; we see that (ii) implies (iv), and so does (iii). If (iv) holds, then  $\nu: X^* \otimes_R X \rightarrow \text{Hom}_R(X, X)$  is surjective by Lemma 5.8.3, so  $X$  is projective by [12, (VII.3.1)]; thus, (i) holds.  $\square$

The next construction goes back to Buchweitz [11] and Auslander-Buchweitz [3].

**5.10. Construction.** Let  $\mathbf{S} \xrightarrow{\sigma} \mathbf{P} \xrightarrow{\pi} M$  be a complete resolution.

Set  $U = \Omega^0\mathbf{S}$  and  $M' = \text{Coker } \Omega^0(\pi\sigma)$ . Choose an epimorphism  $\varphi': V \rightarrow M'$  with  $V \in \mathcal{P}$  and a homomorphism  $\varphi: V \rightarrow M$  with  $\mu\varphi = \varphi'$ , where  $\mu: M \rightarrow M'$  is the canonical map. Set  $X = U \oplus V$  and define a map  $\chi: X \rightarrow M$  by  $\chi(u, v) = \Omega^0(\sigma)(u) + \varphi(v)$ . Finally, set  $Y = \text{Ker } \chi$  and let  $v: Y \rightarrow X$  denote the inclusion.

**Fact.**  $\mathbf{A} = \cdots \rightarrow 0 \rightarrow Y \xrightarrow{v} U \oplus V \rightarrow 0 \rightarrow \cdots$  is a  $\mathcal{G}$ -approximation.

*Proof.* It is easy to see that  $\chi$  is surjective, so the sequence  $\mathbf{A}^+$  is exact. By Lemma 2.4 the module  $U$  is totally reflexive, hence so is  $U \oplus V$  by Proposition 2.3. Choosing a morphism  $\tau: V \rightarrow \mathbf{P}$  with  $\pi_0\tau_0 = \varphi$ , we get a commutative diagram

$$\begin{array}{ccccc} \mathbf{S} & \xrightarrow{\sigma'} & \mathbf{S}_{\geq 0} \oplus V & \xrightarrow{\pi'} & U \oplus V \\ \parallel & & \downarrow \chi' & & \downarrow \chi \\ \mathbf{S} & \xrightarrow{\sigma} & \mathbf{P} & \xrightarrow{\pi} & M \end{array}$$

where  $\sigma'_{<0} = 0$ ,  $\sigma'_n(x) = (x, 0)$  for  $n \geq 0$ ,  $\pi' = (\sigma'_{\geq 0}, \text{id}_V)$ ,  $\chi'(s, v) = \sigma_{\geq 0}(s) + \tau(v)$ . The top row is a complete resolution of  $U \oplus V$ . Thus,  $\text{H}^n \text{Hom}_R(\text{id}_{\mathbf{S}}, Y)$  represents for each  $n \in \mathbb{Z}$  the map  $\widehat{\text{Ext}}_R^n(\chi, Y)$ , which is therefore bijective. The exact sequence of Proposition 5.6 yields  $\widehat{\text{Ext}}_R^0(Y, Y) = 0$ , so  $\text{proj dim}_R Y < \infty$  by Theorem 5.9.  $\square$

**5.11. Remark.** During the 1980ies, Pierre Vogel developed, but never published, a cohomology theory that associates to each pair  $(M, N)$  of modules over an arbitrary ring  $R$  a sequence of abelian groups  $\check{\text{Ext}}_R^n(M, N)$  for  $n \in \mathbb{Z}$ , and comes equipped with a natural transformation  $\text{Ext}_R^n(M, N) \rightarrow \check{\text{Ext}}_R^n(M, N)$  of cohomology functors. He also proved that if  $G$  is a finite group, then there is a natural isomorphism  $\check{\text{Ext}}_{\mathbb{Z}[G]}^n(\mathbb{Z}, N) \cong \widehat{\text{H}}^n(G, N)$  of his theory and Tate cohomology.

Vogel's argument extends, essentially *verbatim*, to establish that if  $R$  is noetherian and  $M$  is a finite module of finite G-dimension, then there is a natural isomorphism of cohomology functors  $\check{\text{Ext}}_R^n(M, N) \cong \widehat{\text{Ext}}_R^n(M, N)$ , compatible with the natural maps from  $\text{Ext}_R^n(M, N)$ . The construction of *Vogel cohomology* and the proof of the comparison theorem are presented in [21, I], and in [26, §2].

## 6. GORENSTEIN PERFECTION

In this section we study a class of modules of finite G-dimension whose relative and Tate cohomology can be bounded by means of absolute (co)homology groups.

To each  $R$ -module  $M \in \widetilde{\mathcal{G}}$  we associate an  $R^0$ -module  $M^\dagger$ , setting  $0^\dagger = 0$  and

$$M^\dagger = \text{Ext}_R^g(M, R) \quad \text{when } M \neq 0 \quad \text{and } g = \text{G-dim}_R M.$$

Note that, in particular, if  $M$  is totally reflexive, then  $M^\dagger = M^*$ .

An  $R$ -module  $M$  is said to be  $G$ -perfect<sup>3</sup> if it is finite, has finite  $G$ -dimension, and there is an integer  $g$  such that  $\text{Ext}_R^n(M, R) = 0$  for  $n \neq g$ . In view of Theorems 4.2.2 and 4.9.1, this notion can be introduced alternatively by requiring the existence of an integer  $g$  such that  $G\text{-dim}_R M = g$  and  $\text{Ext}_R^n(M, R) = 0$  for  $n < g$ .

**6.1. Examples.** Let  $M$  be a finite  $R$ -module.

(1)  $M$  is  $G$ -perfect with  $G\text{-dim}_R M = 0$  if and only if  $M \in \mathcal{G} \setminus 0$ .

(2) If  $\text{proj dim}_R M \neq \pm\infty$  and  $\text{Ext}_R^n(M, R) = 0$  for  $n < \text{proj dim}_R M$ , then  $M$  is  $G$ -perfect with  $G\text{-dim}_R M = \text{proj dim}_R M$ , cf. Theorem 4.9.

(3) If  $R$  is commutative and Gorenstein, then  $M$  is  $G$ -perfect with  $G\text{-dim}_R M = g$  if and only if it is Cohen-Macaulay of codimension  $g$ , that is,  $\text{depth}_{R_{\mathfrak{m}}} M_{\mathfrak{m}} = \dim_{R_{\mathfrak{m}}} M_{\mathfrak{m}} = \dim R_{\mathfrak{m}} - g$  for all  $\mathfrak{m} \in \text{Max}(R) \cup \text{Supp}(M)$ , cf. [10, (3.3.10.c)].

**6.2. Construction.** Let  $M$  be a  $G$ -perfect  $R$ -module with  $G\text{-dim}_R M = g$ .

Let  $\pi: \mathbf{P} \rightarrow M$  be a  $\mathcal{P}$ -resolution, set  $G = \Omega^g \mathbf{P}$ , let  $\omega: P_g \rightarrow G$  denote the canonical surjection and  $\iota: G \rightarrow P_{g-1}$  the canonical injection.

Define a complex  $(\mathbf{G}, \partial^{\mathbf{G}})$  and a map  $\phi: \mathbf{P} \rightarrow \mathbf{G}$  of degree 0 by

$$G_n = \begin{cases} 0 & \\ G & \\ P_n & \end{cases} \quad \partial_n^{\mathbf{G}} = \begin{cases} 0 & \\ \iota & \\ \partial_n^{\mathbf{P}} & \end{cases} \quad \phi_n = \begin{cases} 0 & \text{for } n > g; \\ \omega & \text{for } n = g; \\ \text{id}_{P_n} & \text{for } n < g. \end{cases}$$

**Fact.** The map  $\phi: \mathbf{P} \rightarrow \mathbf{G}$  is a surjective quasiisomorphism, the complex  $\Sigma^g \mathbf{G}^*$  is a strict  $\mathcal{G}$ -resolution of  $M^\dagger$  and the map  $\phi^*: \mathbf{G}^* \rightarrow \mathbf{P}^*$  is a quasiisomorphism.

If  $\mathbf{T} \xrightarrow{\vartheta} \mathbf{P} \xrightarrow{\pi} M$  is a surjective complete resolution with  $\vartheta_n$  bijective for  $n \geq g$ , then the inclusion  $\iota$  of  $\mathbf{Q} = \text{Ker } \vartheta$  into  $\mathbf{T}$  defines a surjective complete resolution  $\Sigma^{g-1} \mathbf{T} \xrightarrow{\vartheta'} \Sigma^{g-1}(\mathbf{Q}^*) \rightarrow M^\dagger$  with  $\vartheta' = \Sigma^{g-1}(\iota^*)$  and  $\vartheta'_n$  bijective for  $n \geq g$ .

*Proof.* The map  $\phi$  is surjective by definition and a quasiisomorphism by inspection.

Set  $\mathbf{F} = \text{Ker } \phi$  and consider the exact sequence  $0 \rightarrow \mathbf{F} \rightarrow \mathbf{P} \rightarrow \mathbf{G} \rightarrow 0$  of complexes of  $R$ -modules. The dual sequence of complexes of  $R^0$ -modules  $0 \rightarrow \mathbf{G}^* \rightarrow \mathbf{P}^* \rightarrow \mathbf{F}^* \rightarrow 0$  is exact: there could be a problem only in degree  $-g$ , but the sequence  $0 \rightarrow G^* \rightarrow P_g^* \rightarrow F_g^* \rightarrow 0$  is exact. By Lemma 2.2. For  $n > -g$  we have  $\text{H}_n(\mathbf{F}^*) = 0$  because  $F_n = 0$ . Left exactness of  $\text{Hom}_R(\cdot, R)$  yields  $\text{H}_{-g}(\mathbf{F}^*) = 0$ . For  $n < -g$  we have  $\text{H}_n(\mathbf{F}^*) = \text{H}_n(\mathbf{P}^*) = \text{Ext}_R^{-n}(M, R) = 0$ . Thus,  $\text{H}_n(\mathbf{F}^*) = 0$  for all  $n$ , so  $\phi^*$  is a quasiisomorphism. In particular,

$$\Omega^{-g} \mathbf{G}^* = \text{H}_{-g}(\mathbf{G}^*) \cong \text{Ext}_R^{-g}(M, R) = M^\dagger.$$

The module  $G$  is totally reflexive by Lemma 2.5, so by condition 3.1.v there is a totally reflexive complex  $\mathbf{T}$  such that  $G$  is isomorphic to  $\Omega^0 \mathbf{T}$ . By definition, the complex  $\mathbf{T}^*$  is totally reflexive, and  $\Omega^0(\mathbf{T}^*)$  is isomorphic to  $G^*$ . Lemma 2.4 shows that  $G^*$  is totally reflexive. Thus,  $\Sigma^g \mathbf{G}^*$  is a strict  $\mathcal{G}$ -resolution of  $M^\dagger$ .

As each  $P_n$  is projective, the sequence of complexes  $0 \rightarrow \mathbf{P}^* \rightarrow \mathbf{T}^* \rightarrow \mathbf{Q}^* \rightarrow 0$  is exact. Because  $\text{H}(\mathbf{T}^*) = 0$ , the homology exact sequence yields  $\text{H}_{-n}(\mathbf{Q}^*) \cong \text{H}_{-n-1}(\mathbf{P}^*) = \text{Ext}_R^{n+1}(M, R)$  for all  $n \in \mathbb{Z}$ . Since  $M$  is  $G$ -perfect with  $G\text{-dim}_R M = g$ , it follows that  $\Sigma^{g-1}(\mathbf{Q}^*)$  is a  $\mathcal{P}$ -resolution of  $M^\dagger$  over  $R^0$ .  $\square$

Under condition 3.2.ii, Part (3) of the next theorem is contained in [11, (6.3.4)].

<sup>3</sup>Auslander and Bridger, who first considered such modules over commutative rings in [2, (4.34)], call them perfect. In view of Example 6.1.2 the notion of  $G$ -perfection extends that of perfection, as defined by Rees [31]. We prefer to have distinct names for different concepts.

**6.3. Theorem.** *Let  $M$  be a  $G$ -perfect  $R$ -module with  $G\text{-dim}_R M = g$ .*

- (1) *The  $R^0$ -module  $M^\dagger$  is  $G$ -perfect with  $G\text{-dim}_{R^0} M^\dagger = g$ .*
- (2) *There is a natural isomorphism of  $R$ -modules  $M \cong M^{\dagger\dagger}$ .*
- (3) *For each  $R$ -module  $N$  there exist natural in  $M$  and  $N$  homomorphisms*

$$\text{Tor}_{g-n}^R(M^\dagger, N) \longrightarrow \text{Ext}_G^n(M, N)$$

*which are bijective for  $n \geq 2$  and surjective for  $n = 1$ .*

- (4) *For each  $R$ -module  $N$  there exist natural in  $M$  and  $N$  homomorphisms*

$$\widehat{\text{Ext}}_R^n(M, N) \dashrightarrow \text{Tor}_{g-n-1}^R(M^\dagger, N)$$

*which are bijective for  $n \leq -2$  and make the following sequence exact*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \widehat{\text{Ext}}_R^{-1}(M, N) & \longrightarrow & \text{Tor}_g^R(M^\dagger, N) & & \\ & & \longrightarrow & \text{Ext}_R^0(M, N) & \longrightarrow & \cdots & \longrightarrow \text{Tor}_{g-n}^R(M^\dagger, N) \\ & & \longrightarrow & \text{Ext}_R^n(M, N) & \xrightarrow{\varepsilon_R^n(M, N)} & \widehat{\text{Ext}}_R^n(M, N) & \longrightarrow \text{Tor}_{g-n-1}^R(M^\dagger, N) \\ & & \cdots & \longrightarrow & \text{Ext}_R^g(M, N) & \longrightarrow & \widehat{\text{Ext}}_R^g(M, N) \longrightarrow 0 \end{array}$$

*Proof.* We adopt the notation used in the preceding construction.

- (1) The proper resolution  $\Sigma^g \mathbf{G}^*$  of  $M^\dagger$  yields  $G\text{-dim}_{R^0} M^\dagger \leq g$ , as well as the second isomorphism of  $R$ -modules below:

$$\text{Ext}_{R^0}^n(M^\dagger, R) \cong \text{Ext}_G^n(M^\dagger, R) \cong \text{H}_{-n}((\Sigma^g(\mathbf{G}^*))^*) \cong \text{H}_{g-n}(\mathbf{G}) = \begin{cases} 0 & \text{if } n \neq g; \\ M & \text{if } n = g; \end{cases}$$

the first isomorphism comes from Theorem 4.2.4, the third from the isomorphisms of complexes  $(\Sigma^g(\mathbf{G}^*))^* \cong \Sigma^{-g}(\mathbf{G}^{**}) \cong \Sigma^{-g}\mathbf{G}$ . As a consequence,  $G\text{-dim}_{R^0} M^\dagger = g$ .

- (2) Due to (1), for  $n = g$  the formula displayed above becomes  $M^{\dagger\dagger} \cong M$ .

(3) In view of (1), we can apply Construction 6.2 to the  $R^0$ -module  $M^\dagger$ . It yields a surjective quasiisomorphism  $\phi': \mathbf{P}' \rightarrow \mathbf{G}'$  from a  $\mathcal{P}$ -resolution  $\mathbf{P}' \rightarrow M^\dagger$  over  $R^0$ , such that  $\mathbf{G}'^*$  is a proper  $\mathcal{G}$ -resolution of  $M^{\dagger\dagger} \cong M$  over  $R$ , and  $\phi'^*: \mathbf{G}'^* \rightarrow \mathbf{P}'^*$  is a quasiisomorphism. We have a natural commutative diagram with exact row

$$\begin{array}{ccccc} (\Sigma^{-g}\mathbf{P}') \otimes_R N & \xrightarrow{(\Sigma^{-g}\phi') \otimes_R N} & (\Sigma^{-g}\mathbf{G}') \otimes_R N & \longrightarrow & 0 \\ \downarrow \xi & & \downarrow \xi & & \\ \text{Hom}_R(\Sigma^g(\mathbf{P}'^*), N) & \xrightarrow{\text{Hom}_R(\Sigma^g(\phi'^*), N)} & \text{Hom}_R(\Sigma^g(\mathbf{G}'^*), N) & & \end{array}$$

where  $\xi$  denotes the morphism of complexes given by  $(\xi(x \otimes y))(\alpha) = \alpha(x)y$ . The composition of the maps in the square induces maps  $\text{Tor}_{g-n}^R(M^\dagger, N) \rightarrow \text{Ext}_G^n(M, N)$  for all  $n \in \mathbb{Z}$ . They are bijective for  $n \geq 2$  and surjective for  $n = 1$  because the maps  $\xi$  are isomorphisms, and  $(\Sigma^{-g}\phi') \otimes_R N$  is bijective in negative degrees.

- (4) In view of the canonical isomorphism  $\xi: \mathbf{Q}^* \otimes_R M \rightarrow \text{Hom}_R(\mathbf{Q}, N)$ , the exact sequence of complexes of Construction 6.2 yields an exact sequence

$$0 \longrightarrow \Sigma^{g-1}(\mathbf{Q}^*) \otimes_R M \longrightarrow \text{Hom}_R(\Sigma^{g-1}\mathbf{T}, N) \longrightarrow \text{Hom}_R(\Sigma^{g-1}\mathbf{P}, N) \longrightarrow 0$$

The associated homology exact sequence yields the desired assertions.  $\square$

## 7. COMPARISON MORPHISMS

By Theorem 3.1 and Lemma 4.1.4, three cohomology theories—absolute, relative, and Tate—are defined on the category  $\tilde{\mathcal{G}}$  of modules of finite G-dimension. Our main result demonstrates that the morphisms of functors  $\varepsilon_{\mathcal{G}}^n$  of Theorem 4.2 and  $\varepsilon_R^n$  of Theorem 5.2 yield a very tight relation between these theories.

**7.1. Theorem.** *Let  $M$  be a finite  $R$ -module with  $\text{G-dim}_R M = g < \infty$ .*

*For each  $R$ -module  $N$  there exist natural in  $M$  and  $N$  homomorphisms  $\delta_R^n(M, N)$ , such that the following sequence is exact:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Ext}_{\mathcal{G}}^1(M, N) & \xrightarrow{\varepsilon_{\mathcal{G}}^1(M, N)} & \text{Ext}_R^1(M, N) & \longrightarrow & \cdots \\ & & \longrightarrow & \text{Ext}_{\mathcal{G}}^n(M, N) & \xrightarrow{\varepsilon_{\mathcal{G}}^n(M, N)} & \text{Ext}_R^n(M, N) & \xrightarrow{\varepsilon_R^n(M, N)} \widehat{\text{Ext}}_R^n(M, N) \\ & & \xrightarrow{\delta_R^n(M, N)} & \text{Ext}_{\mathcal{G}}^{n+1}(M, N) & \longrightarrow & \cdots & \longrightarrow \widehat{\text{Ext}}_R^g(M, N) \longrightarrow 0 \end{array}$$

Comparing the theorem with Theorems 5.2.3 and 5.2.4, we get:

**7.2. Corollary.** *For  $M \in \tilde{\mathcal{G}}$  the following conditions are equivalent.*

- (i)  $\varepsilon_{\mathcal{G}}^n(M, \_): \text{Ext}_{\mathcal{G}}^n(M, \_) \rightarrow \text{Ext}_R^n(M, \_)$  is an isomorphism for all  $n \in \mathbb{Z}$ .
- (ii)  $\varepsilon_{\mathcal{G}}^n(\_, M): \text{Ext}_{\mathcal{G}}^n(\_, M) \rightarrow \text{Ext}_R^n(\_, M)$  is an isomorphism for all  $n \in \mathbb{Z}$ .
- (iii)  $\text{proj dim}_R M < \infty$ . □

*Proof of Theorem 7.1.* The exact sequences of complexes of  $R$ -modules in Construction 3.8 is split-exact in each degree, so it induces an exact sequence

$$0 \rightarrow \text{Hom}_R(\mathbf{P}, N) \xrightarrow{\text{Hom}_R(\vartheta^b, N)} \text{Hom}_R(\mathbf{T}^b, N) \xrightarrow{\text{Hom}_R(\varkappa^b, N)} \text{Hom}_R(\Sigma^{-1}\mathbf{G}, N) \rightarrow 0$$

of complexes of abelian groups. Its cohomology exact sequence has the form

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \text{H}^n \text{Hom}_R(\mathbf{T}^b, N) & \xrightarrow{\text{H}^n \text{Hom}_R(\varkappa^b, N)} & \text{H}^n \text{Hom}_R(\Sigma^{-1}\mathbf{G}, N) & & \\ & & \xrightarrow{\delta^n} & \text{H}^{n+1} \text{Hom}_R(\mathbf{P}, N) & \xrightarrow{\text{H}^{n+1} \text{Hom}_R(\vartheta^b, N)} & \text{H}^{n+1} \text{Hom}_R(\mathbf{T}^b, N) & \longrightarrow \cdots \end{array}$$

We set out to identify the modules and maps appearing in this sequence.

Since  $\mathbf{P} \rightarrow M$  is a projective resolution, we have

$$\text{H}^{n+1} \text{Hom}_R(\mathbf{P}, N) = \text{Ext}_R^{n+1}(M, N) \quad \text{for } n \in \mathbb{Z}.$$

The left exactness of  $\text{Hom}_R(\_, N)$  yields  $\text{H}^n \text{Hom}_R(\mathbf{T}^b, N) = 0$  for  $n \leq 0$ , and

$$\text{H}^{n+1} \text{Hom}_R(\mathbf{T}^b, N) = \text{H}^{n+1} \text{Hom}_R(\mathbf{T}, N) = \widehat{\text{Ext}}_R^{n+1}(M, N) \quad \text{for } n \geq 0.$$

Since  $\text{H}^{n+1} \text{Hom}_R(\vartheta^b, N) = \text{H}^{n+1} \text{Hom}_R(\vartheta, N)$  for  $n \geq 0$ , Theorem 5.2.1 gives

$$\text{H}^n \text{Hom}_R(\vartheta^b, N) = \varepsilon_R^n(M, N): \text{Ext}_R^n(M, N) \longrightarrow \widehat{\text{Ext}}_R^n(M, N) \quad \text{for } n \geq 1.$$

As  $\mathbf{G} \rightarrow M$  is a proper resolution, from Theorem 4.2.1 we see that

$$\text{H}^n \text{Hom}_R(\Sigma^{-1}\mathbf{G}, N) = \text{H}^{n+1} \text{Hom}_R(\mathbf{G}, N) = \text{Ext}_{\mathcal{G}}^{n+1}(M, N) \quad \text{for } n \in \mathbb{Z}.$$

Next we prove that  $\delta^n = \varepsilon_G^{n+1}(M, N)$  for  $n \geq 0$ . To this end we form, following [9, §2.6], the fundamental commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Sigma^{-1}\mathbf{G} & \xrightarrow{\varkappa^b} & \mathbf{T}^b & \xrightarrow{\vartheta^b} & \mathbf{P} & \longrightarrow & 0 \\ & & \parallel & & \uparrow \dot{\varphi} & & \uparrow \varphi & & \\ 0 & \longrightarrow & \Sigma^{-1}\mathbf{G} & \xrightarrow{\dot{\varkappa}} & \mathbf{D} & \xrightarrow{\dot{\vartheta}} & \mathbf{C} & \longrightarrow & 0 \end{array}$$

where  $\mathbf{C}$  is the mapping cone of  $\varkappa^b$  and  $\mathbf{D}$  its mapping cylinder, that is:

$$\begin{aligned} C_n &= G_n \oplus T_n^b & \partial_n^{\mathbf{C}}(y, x) &= (\partial_n^{\mathbf{G}}(y), \varkappa_{n-1}^b(y) + \partial_n^{\mathbf{T}^b}(x)) \\ D_n &= G_{n+1} \oplus G_n \oplus T_n^b & \partial_n^{\mathbf{D}}(y', y, x) &= (y - \partial_{n+1}^{\mathbf{G}}(y'), \partial_n^{\mathbf{G}}(y), \varkappa_{n-1}^b(y) + \partial_n^{\mathbf{T}^b}(x)) \end{aligned}$$

and the morphisms are described as follows:

$$\begin{aligned} \dot{\varkappa}(y') &= (y', 0, 0), & \dot{\vartheta}(y', y, x) &= (y, x); \\ \dot{\varphi}(y', y, x) &= \varkappa^b(y') + \vartheta^b(x), & \varphi(y, x) &= \vartheta^b(x). \end{aligned}$$

It induces a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \rightarrow \mathrm{Hom}_R(\mathbf{P}, N) & \xrightarrow{\mathrm{Hom}_R(\vartheta^b, N)} & \mathrm{Hom}_R(\mathbf{T}^b, N) & \xrightarrow{\mathrm{Hom}_R(\varkappa^b, N)} & \mathrm{Hom}_R(\Sigma^{-1}\mathbf{G}, N) & \rightarrow & 0 \\ \downarrow \mathrm{Hom}_R(\varphi, N) & & \downarrow \mathrm{Hom}_R(\dot{\varphi}, N) & & \parallel & & \\ 0 \rightarrow \mathrm{Hom}_R(\mathbf{C}, N) & \xrightarrow{\mathrm{Hom}_R(\dot{\vartheta}, N)} & \mathrm{Hom}_R(\mathbf{D}, N) & \xrightarrow{\mathrm{Hom}_R(\dot{\varkappa}, N)} & \mathrm{Hom}_R(\Sigma^{-1}\mathbf{G}, N) & \rightarrow & 0 \end{array}$$

The connecting maps of the cohomology exact sequences yield a commutative square

$$\begin{array}{ccc} \mathrm{H}^n \mathrm{Hom}_R(\Sigma^{-1}\mathbf{G}, N) & \xrightarrow{\delta^n} & \mathrm{H}^{n+1} \mathrm{Hom}_R(\mathbf{P}, N) \\ \parallel & & \downarrow \mathrm{H}^{n+1} \mathrm{Hom}_R(\varphi, N) \\ \mathrm{H}^{n+1} \mathrm{Hom}_R(\mathbf{G}, N) & \xrightarrow{\mathrm{H}^{n+1} \mathrm{Hom}_R(\theta, N)} & \mathrm{H}^{n+1} \mathrm{Hom}_R(\mathbf{C}, N) \end{array}$$

where  $\theta: \mathbf{C} \rightarrow \mathbf{G}$  is the morphism given by  $\theta(y', x) = y'$ ; note that  $\mathrm{H}_0(\theta) = \mathrm{id}_M$ .

The morphism  $\varphi$  is clearly surjective; it is split as a homomorphism of graded  $R$ -modules because each  $P_n$  is projective. We have an exact sequence of complexes

$$\begin{aligned} 0 \longrightarrow \mathbf{E} \xrightarrow{\xi} \mathbf{C} \xrightarrow{\varphi} \mathbf{P} \longrightarrow 0 \quad \text{where} \\ E_n = G_{n+1} \oplus G_n, \quad \partial^{\mathbf{E}}(y', y) = (y, 0), \quad \xi(y', y) = (y', \varkappa(y)). \end{aligned}$$

The map  $(y', y) \mapsto (0, y')$  is a homotopy between  $\mathrm{id}_{\mathbf{E}}$  and  $0_{\mathbf{E}}$ , so  $\mathbf{E}$  is an irrelevant subcomplex of  $\mathbf{C}$ , hence  $\varphi$  is a homotopy equivalence by Lemma 1.6. We choose a homotopy inverse  $\psi: \mathbf{P} \rightarrow \mathbf{C}$  of  $\varphi$  and note that  $\mathrm{H}_0(\psi) = \mathrm{H}_0(\varphi)^{-1} = \mathrm{id}_M^{-1} = \mathrm{id}_M$ .

As  $\mathrm{Hom}_R(\varphi, N)$  and  $\mathrm{Hom}_R(\psi, N)$  are inverse homotopy equivalences, we have  $\mathrm{H}^{n+1} \mathrm{Hom}_R(\varphi, N)^{-1} = \mathrm{H}^{n+1} \mathrm{Hom}_R(\psi, N)$ . Thus, the commutative square yields

$$\begin{aligned} \delta^n &= \mathrm{H}^{n+1} \mathrm{Hom}_R(\varphi, N)^{-1} \circ \mathrm{H}^{n+1} \mathrm{Hom}_R(\theta, N) \\ &= \mathrm{H}^{n+1} \mathrm{Hom}_R(\psi, N) \circ \mathrm{H}^{n+1} \mathrm{Hom}_R(\theta, N) \\ &= \mathrm{H}^{n+1} \mathrm{Hom}_R(\theta \psi, N) \end{aligned}$$

Furthermore,  $\mathrm{H}_0(\theta \psi) = \mathrm{H}_0(\theta) \mathrm{H}_0(\psi) = \mathrm{id}_M$ , so the morphism  $\theta \psi: \mathbf{P} \rightarrow \mathbf{G}$  lifts the identity map of  $M$ , hence  $\mathrm{H}^{n+1} \mathrm{Hom}_R(\theta \psi, N) = \varepsilon_G^{n+1}(M, N)$  by Theorem 4.2.1.

To finish the construction of the exact sequence in the theorem, we set

$$\delta^n(M, N) = \mathbb{H}^{n+1} \operatorname{Hom}_R(\mathcal{X}^b, N): \widehat{\operatorname{Ext}}_R^n(M, N) \rightarrow \operatorname{Ext}_G^{n+1}(M, N) \quad \text{for } n \geq 1.$$

It is clear from the construction that the exact sequence is functorial in  $N$ .

For any homomorphism  $\mu: M \rightarrow M'$ , form a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Sigma^{-1} \mathbf{G} & \xrightarrow{\mathcal{X}^b} & \mathbf{T}^b & \xrightarrow{\vartheta^b} & \mathbf{P} & \longrightarrow & 0 \\ & & \widehat{\mu}|_{\Sigma^{-1} \mathbf{G}} \downarrow & & \widehat{\mu} \downarrow & & \overline{\mu} \downarrow & & \\ 0 & \longrightarrow & \Sigma^{-1} \mathbf{G}' & \xrightarrow{\mathcal{X}^b} & \mathbf{T}'^b & \xrightarrow{\vartheta^b} & \mathbf{P}' & \longrightarrow & 0 \end{array}$$

by first choosing  $\widehat{\mu}$  from Lemma 5.3 so that the right hand square commutes, then taking  $\alpha$  to be restriction of  $\widehat{\mu}$ . Since  $\mathbb{H}(\mathbf{T}^b)$  and  $\mathbb{H}(\mathbf{T}'^b)$ , are both trivial, the induced commutative diagram in homology shows that  $\mathbb{H}_{-1}(\alpha) = \mu$ . Thus, for all  $n \in \mathbb{Z}$  we get  $\mathbb{H}^n \operatorname{Hom}_R(\overline{\mu}, N) = \operatorname{Ext}_R^n(\mu, N)$ ,  $\mathbb{H}^n \operatorname{Hom}_R(\widehat{\mu}, N) = \widehat{\operatorname{Ext}}_R^n(\mu, N)$ , and  $\mathbb{H}^n \operatorname{Hom}_R(\alpha, N) = \operatorname{Ext}_G^{n+1}(\mu, N)$ . Applying  $\operatorname{Hom}_R(\_, N)$  to the diagram and taking homology, we obtain the desired naturality in  $M$ .  $\square$

Some extra structure comes for free with the various cohomology functors.

**7.3. Remark.** Let  $R$  be a noetherian ring which is an algebra over a commutative ring  $\mathbb{k}$ . For a finite  $R$ -module  $M$  of finite  $G$ -dimension and an  $R$ -module  $N$ , let  $F^n(M, N)$  stand for  $\operatorname{Ext}_R^n(M, N)$ , its relative or Tate variants, or  $\operatorname{Tor}_n^R(M, N)$ .

(1) For each  $n$  the group  $F^n(M, N)$  has a structure of  $\mathbb{k}$ -module, which is natural for maps induced by homomorphisms of either module argument, for comparison maps between various functors, and for connecting maps of long exact sequences.

Indeed, in each case there is an  $R$ -linear resolution  $\mathbf{P}$  and a functor  $F$  on  $\mathcal{M}(R)$  such that  $F^n(M, N) = \mathbb{H}^n F(\mathbf{P}, N)$  for all  $n \in \mathbb{Z}$ . The action of  $\mathbb{k}$  on either  $\mathbf{P}$  or  $N$  induces the same action on the complex  $F(\mathbf{P}, N)$ , and passes to homology. All maps listed above are either induced by  $\mathbb{k}$ -linear morphisms of complexes or are connecting maps in exact sequences of  $\mathbb{k}$ -linear morphisms of such complexes.

(2) The  $\mathbb{k}$ -module  $F^n(M, N)$  is annihilated by the ideal  $\operatorname{ann}_{\mathbb{k}}(M) + \operatorname{ann}_{\mathbb{k}}(N)$ .

Indeed, let  $\lambda_x$  denote any  $R$ -linear map given by left multiplication with  $x \in \mathbb{k}$ . If  $xN = 0$ , then  $\lambda_x^N = 0$ , hence  $F(\mathbf{P}, \lambda_x^N) = 0$ . If  $xM = 0$ , then  $\lambda_x^M$  is a morphism with  $\mathbb{H}_0(\lambda_x^M) = \lambda_x^M = 0$ . By the lifting property for resolutions of the corresponding type,  $\lambda_x^M$  is homotopic to  $0^M$ , so  $\mathbb{H}(F(\lambda_x^M, N)) = \mathbb{H}(F(0^M, N)) = 0$ .

(3) If  $R$  is finite over  $\mathbb{k}$  and  $N$  is finite over  $R$ , then  $F^n(M, N)$  is finite over  $\mathbb{k}$ .

Indeed, the image  $\overline{\mathbb{k}}$  of  $\mathbb{k}$  in  $R$  is noetherian by the theorem of Eakin-Nagata-Eisenbud [18]. Each  $P_n$  is finite over  $R$ , hence also over  $\overline{\mathbb{k}}$ . Being the homology of a complex of finite  $\overline{\mathbb{k}}$ -modules,  $F^n(M, N)$  is finite over  $\overline{\mathbb{k}}$ , hence over  $\mathbb{k}$ .

We close off the discussion of relative and Tate functors with a note on homology.

**7.4. Remark.** If  $M$  is a *right*  $R$ -module of finite  $G$ -dimension,  $\mathbf{G} \rightarrow M$  is a proper resolution and  $\mathbf{T} \rightarrow \mathbf{P} \rightarrow M$  is a complete resolution, one sets

$$\operatorname{Tor}_n^G(M, N) = \mathbb{H}_n(\mathbf{G} \otimes_R N) \quad \text{and} \quad \widehat{\operatorname{Tor}}_n^R(M, N) = \mathbb{H}_n(\mathbf{T} \otimes_R N).$$

for each  $R$ -module  $N$  and each  $n \in \mathbb{Z}$ . The resulting homology groups come equipped with homomorphisms  $\widehat{\operatorname{Tor}}_n^G(M, N) \rightarrow \operatorname{Tor}_n^R(M, N) \rightarrow \operatorname{Tor}_n^G(M, N)$  that are natural in both arguments. The proofs in Sections 2 through 6 dualize perfectly, so all the results in these sections have valid analogs in homology.

## 8. MINIMAL RESOLUTIONS

In this section  $R$  is a *commutative noetherian local ring* with unique maximal ideal  $\mathfrak{m}$  and residue field  $k = R/\mathfrak{m}$ , and  $M$  is a finite  $R$ -module.

Recall that  $M$  has a *free cover*, that is, a surjective homomorphism  $\varphi: F \rightarrow M$  with  $F$  free and  $\varphi \otimes_R k$  bijective. Classically, a free resolution  $\mathbf{P} \rightarrow M$  is said to be minimal if  $P_n \rightarrow \Omega^n \mathbf{P}$  is a free cover for each  $n \in \mathbb{Z}$ , that is, if  $\partial(\mathbf{P}) \subseteq \mathfrak{m}\mathbf{P}$ . This is a special case of the general concept of minimality defined in Section 1.

**8.1. Proposition.** *Over a local ring  $R$  a complex  $\mathbf{P}$  of finite free modules is minimal if and only if  $\partial(\mathbf{P}) \subseteq \mathfrak{m}\mathbf{P}$ .*

*Proof.* Assume first that  $\partial(\mathbf{P}) \subseteq \mathfrak{m}\mathbf{P}$ , and let  $\beta: \mathbf{P} \rightarrow \mathbf{P}$  be a morphism such that  $\beta = \text{id}_{\mathbf{P}} + \theta\partial + \partial\theta$ . For each  $n$  we then have  $\beta_n \otimes_R k = \text{id}_{P_n \otimes_R k}$ . As  $P_n$  is free of finite rank,  $\beta_n$  is an isomorphism, so  $\mathbf{P}$  is minimal by Proposition 1.7.1.

Assume next that  $\partial(\mathbf{P}) \not\subseteq \mathfrak{m}\mathbf{P}$ , choose  $y \in P_n$  with  $\partial(y) = x \notin \mathfrak{m}P_{n-1}$ . Setting  $E_n = Ry$ ,  $E_{n-1} = Rx$ , and  $E_i = 0$  for  $i \neq n, n-1$  we get a contractible subcomplex  $\mathbf{E} \subseteq \mathbf{P}$ . This subcomplex is irrelevant because  $y$  (respectively,  $x$ ) is part of a basis of  $P_n$  (respectively,  $P_{n-1}$ ). Thus,  $\mathbf{P}$  is not minimal by Proposition 1.7.3.  $\square$

Let  $\text{f-rank}_R M$  denote the maximal rank of a free direct summand of  $M$ .

**8.2. Lemma.** *Let  $R$  be a local ring,  $\rho: R \rightarrow k$  the canonical surjection,  $M$  a finite  $R$ -module, and  $\mathbf{P} \rightarrow M$  a minimal free resolution.*

- (1)  $\text{f-rank}_R(\Omega^n \mathbf{P}) \geq \text{rank}_k \text{Ext}_R^n(M, \rho)$  for all  $n \in \mathbb{Z}$ .
- (2)  $\text{f-rank}_R(\Omega^n \mathbf{P}) = 0$  for  $n > \max\{0, \text{depth } R - \text{depth}_R M\}$ .

*Proof.* (1) Choose homomorphisms  $\varphi_1, \dots, \varphi_r: P_n \rightarrow R$  with  $\varphi_j \partial_{n+1}^{\mathbf{P}} = 0$ , such that the images of  $\text{cls}(\varphi_1), \dots, \text{cls}(\varphi_r) \in \text{H}^n \text{Hom}_R(\mathbf{P}, R) = \text{Ext}_R^n(M, R)$  form a basis of  $\text{Im } \text{H}^n \text{Hom}_R(\mathbf{P}, \rho)$ . The map  $x \mapsto (\rho\varphi_1(x), \dots, \rho\varphi_r(x))$  is then a surjective homomorphism  $P_n \rightarrow k^r$ . It follows that  $\varphi: P_n \rightarrow R^r$ , given by  $x \mapsto (\varphi_1(x), \dots, \varphi_r(x))$ , is a surjective homomorphism. Since  $\varphi \partial_{n+1}^{\mathbf{P}} = 0$ , the map  $\varphi$  factors through a homomorphism  $\Omega^n \mathbf{P} \rightarrow R^r$ , which is necessarily surjective; thus,  $\text{f-rank}_R(\Omega^n \mathbf{P}) \geq r$ .

(2) is known, cf. e.g. [5, (1.2.5)] for a proof.  $\square$

For the following result of Auslander and Bridger [2, (4.30), (4.13.b)] the published proof of (2) contains a mistake, corrected by Mašek [29]. Our arguments are based on Auslander's early exposition [1, §3.2], which is not easily available.

**8.3. Theorem.** *Let  $R$  be a local ring,  $M$  a finite module, and  $\text{G-dim}_R M = g < \infty$ .*

- (1) *If  $x \in \mathfrak{m}$  is  $R$ -regular and  $M$ -regular, then  $\text{G-dim}_{R/Rx}(M/xM) = g$ .*
- (2)  $\text{G-dim}_R M = \text{depth } R - \text{depth}_R M$ .

*Proof.* Set  $(\overline{\quad}) = (\quad \otimes_R (R/Rx))$ , and let  $\mathbf{G} \rightarrow M$  be a  $\text{G}$ -resolution of length  $g$ .

(1) Let first  $g = 0$  and choose a totally acyclic complex  $\mathbf{S}$  with  $\Omega^0 \mathbf{S} = M$ , cf. Construction 3.6. We obtain  $\Omega^0(\overline{\mathbf{S}}) \cong \overline{M}$  and  $\text{H}(\overline{\mathbf{S}}) = 0$  from the exact sequence  $0 \rightarrow \mathbf{S} \xrightarrow{x} \mathbf{S} \rightarrow \overline{\mathbf{S}} \rightarrow 0$ , and  $\text{H}\text{Hom}_R(\mathbf{S}, \overline{R}) = 0$  from the exact sequence  $0 \rightarrow \text{Hom}_R(\mathbf{S}, R) \xrightarrow{x} \text{Hom}_R(\mathbf{S}, R) \rightarrow \text{Hom}_R(\mathbf{S}, \overline{R}) \rightarrow 0$ . Thus,  $\text{H}\text{Hom}_{\overline{R}}(\overline{\mathbf{S}}, \overline{R}) \cong \text{H}\text{Hom}_R(\mathbf{S}, \overline{R}) = 0$ , so  $\text{G-dim}_{\overline{R}} \overline{M} = 0$  by Lemma 2.4.

For any  $g$ , the sequence  $0 \rightarrow \mathbf{G} \xrightarrow{x} \mathbf{G} \rightarrow \overline{\mathbf{G}} \rightarrow 0$  is exact because each  $G_i$  is contained in a free module. Thus,  $\overline{\mathbf{G}}$  is a resolution of  $\overline{M}$ . We have just shown that each  $\overline{G}_i$  is in  $\mathcal{G}(\overline{R})$ , so  $\text{G-dim}_{\overline{R}} \overline{M} \leq g$ . By Theorem 4.9.1, in the exact sequence

$$\text{Ext}_R^g(M, R) \xrightarrow{x} \text{Ext}_R^g(M, R) \rightarrow \text{Ext}_R^g(M, \overline{R}) \rightarrow \text{Ext}_R^{g+1}(M, R)$$

the last module vanishes and the first one does not. We obtain  $\text{G-dim}_{\overline{R}} \overline{M} = g$  from the following relations, the first of which is a standard isomorphism:

$$\text{Ext}_{\overline{R}}^g(\overline{M}, \overline{R}) \cong \text{Ext}_R^g(M, \overline{R}) \cong \text{Ext}_R^g(M, R)/x \text{Ext}_R^g(M, R) \neq 0.$$

(2) Set  $t = \text{depth}_R M$  and  $d = \text{depth} R$ . Let first  $d = 0$  and pick a monomorphism  $\iota: k \rightarrow R$ . For each  $L \in \mathcal{F}(R)$  let  $L(\iota)$  denote the image of  $\text{Hom}_R(L, \iota)$  in  $L^*$ ; we remark that  $\mathfrak{m}L(\iota) = 0$ , and that  $L(\iota) = 0$  if and only if  $L = 0$ . If  $g = 0$ , then we have  $M^*(\iota) \subseteq M^{**} \cong M$ , hence  $t = 0$ . Assuming  $g > 0$ , we have an exact sequence  $0 \rightarrow G_g \xrightarrow{\partial} G_{g-1} \rightarrow \Omega^{g-1} \mathbf{G} \rightarrow 0$ . Since  $\text{Ext}_R^1(G_{g-1}, R) = 0$ , it induces an exact sequence  $G_{g-1}^* \xrightarrow{\partial^*} G_g^* \rightarrow E \rightarrow 0$  with  $E = \text{Ext}_R^1(\Omega^{g-1} \mathbf{G}, R)$ . Dualization yields  $E^* \cong \text{Ker}(\partial) = 0$ . It follows that  $E(\iota) = 0$ , hence  $E = 0$ , and thus  $\Omega^{g-1} \mathbf{G}$  is in  $\mathcal{G}$  by Theorem 4.9.1. This implies  $\text{G-dim}_R M < g$ , contradicting our assumption.

Let now  $d \geq 1$ . For  $t = 0$  the exact sequence  $0 \rightarrow \Omega^1 \mathbf{G} \rightarrow G_0 \rightarrow M \rightarrow 0$  yields  $\text{G-dim}_R \Omega^1 \mathbf{G} = g - 1$  and  $\text{depth}_R(\Omega^1 \mathbf{G}) = 1$ , so we may assume  $t \geq 1$ . Pick an  $R \oplus M$ -regular element  $x \in \mathfrak{m}$ . We get  $\text{G-dim}_{\overline{R}}(\overline{M}) = g$  from (1), and  $\text{G-dim}_{\overline{R}}(\overline{M}) = \text{depth} \overline{R} - \text{depth}_{\overline{R}}(\overline{M})$  by induction on  $d$ ; the difference equals  $d - t$ .  $\square$

We say that a complete resolution  $\mathbf{T} \rightarrow \mathbf{P} \rightarrow M$  (respectively, a proper resolution  $\mathbf{G} \rightarrow M$ , a  $\mathcal{G}$ -approximation  $\chi: \mathbf{B} \rightarrow M$ ) is minimal if the complexes  $\mathbf{T}$  and  $\mathbf{P}$  are minimal (respectively, the complex  $\mathbf{G}$  is minimal, the complex  $\mathbf{B}$  is minimal).

The results that follow establish the existence of minimal resolutions of each kind, and their uniqueness up to isomorphism.

**8.4. Theorem.** *Let  $R$  be a local ring and  $M$  a finite module of finite  $G$ -dimension.*

- (1)  *$M$  has a minimal complete resolution; more precisely:*
- (1') *A minimal complete resolution  $\mathbf{S} \xrightarrow{\sigma} \mathbf{P} \xrightarrow{\pi} M$  is produced by Construction 3.6, using a minimal free resolution  $\pi: \mathbf{P} \rightarrow M$ , a module  $G = \Omega^g \mathbf{P}$  with  $\text{G-dim}_R G = 0$  and  $\text{f-rank}_R G = 0$  (by Theorems 8.3.2 and 3.1, and Lemma 8.2.2, any  $g > \text{G-dim}_R M$  would do) and a minimal free resolution  $\lambda: \mathbf{L} \rightarrow G^*$ .*
- (2) *A complete resolution  $\mathbf{T} \rightarrow \mathbf{P} \rightarrow M$  is minimal if and only if*

$$\partial(\mathbf{T}) \subseteq \mathfrak{m}\mathbf{T} \quad \text{and} \quad \partial(\mathbf{P}) \subseteq \mathfrak{m}\mathbf{P}.$$

- (3) *Let  $\mathbf{T} \rightarrow \mathbf{P} \rightarrow M$  and  $\mathbf{T}' \rightarrow \mathbf{P}' \rightarrow M$  be complete resolutions, and let  $\widehat{\mu}: \mathbf{T} \rightarrow \mathbf{T}'$  and  $\overline{\mu}: \mathbf{P} \rightarrow \mathbf{P}'$  be morphisms over  $\text{id}_M$ , given by Lemma 5.3.*

*If  $\mathbf{T} \rightarrow \mathbf{P} \rightarrow M$  is minimal, then  $\widehat{\mu}$  and  $\overline{\mu}$  are isomorphisms onto direct summands; if both resolutions are minimal, then  $\widehat{\mu}$  and  $\overline{\mu}$  are isomorphisms.*

- (4) *In every minimal complete resolution  $\mathbf{T} \xrightarrow{\vartheta} \mathbf{P} \xrightarrow{\pi} M$  the homomorphism  $\vartheta_n$  is bijective for all  $n > \text{G-dim}_R M$ .*

*Proof.* (2) follows from Proposition 8.1, and (3) from Proposition 1.7.

(1') As  $\partial(\mathbf{P}) \subseteq \mathfrak{m}\mathbf{P}$  and  $\partial(\mathbf{L}) \subseteq \mathfrak{m}\mathbf{L}$ , Construction 3.6 yields a complete resolution  $\mathbf{S} \rightarrow \mathbf{P} \rightarrow M$  with  $\partial_n^{\mathbf{S}}(S_n) \subseteq \mathfrak{m}S_{n-1}$  for  $n \neq g$ . It also shows that  $\partial_g^{\mathbf{S}}$  factors through  $G = \Omega^g \mathbf{S}$ , so  $G \not\subseteq \mathfrak{m}S_{g-1}$  implies  $\text{f-rank}_R G > 0$ , contradicting our hypothesis. Thus,  $\mathbf{S} \rightarrow \mathbf{P} \rightarrow M$  is minimal by (2).

(4) By Theorem 5.2.2 the map  $\varepsilon_R^n(M, k): \text{Ext}_R^n(M, k) \rightarrow \widehat{\text{Ext}}_R^n(M, k)$  is bijective for all  $n > \text{G-dim}_R M$ . Furthermore, by Theorem 5.2.1 this homomorphism can be computed as  $H^n \text{Hom}_R(\vartheta, k): \text{Hom}_R(\mathbf{P}, k) \rightarrow \text{Hom}_R(\mathbf{T}, k)$ . From Part (2) above we obtain  $H^n \text{Hom}_R(\vartheta, k) = \text{Hom}_R(\vartheta_n, k)$ . Referring to Nakayama's Lemma, we conclude that  $\vartheta_n$  is bijective for all  $n > \text{G-dim}_R M$ .  $\square$

**8.5. Theorem.** *Let  $R$  be a local ring and  $M$  a finite module of finite  $G$ -dimension.*

- (1)  $M$  has a minimal proper resolution; more precisely:
- (1') Let  $\mathbf{A}^+ = 0 \rightarrow Y \xrightarrow{v} U \oplus V \xrightarrow{\chi} M \rightarrow 0$  be the  $G$ -approximation produced by Construction 5.10 from the minimal complete resolution in Theorem 8.4.1' and a free cover  $\varphi': V \rightarrow \text{Coker } \Omega^0(\pi\sigma)$ ; the resolution  $\mathbf{F} \rightarrow M$  obtained by splicing  $\mathbf{A}$  with a minimal free resolution  $\varkappa: \mathbf{K} \rightarrow Y$  is minimal.
- (2) A proper resolution  $\mathbf{G} \rightarrow M$  is minimal if and only if the following hold:
- (a)  $G_n$  is free of finite rank for  $n \geq 1$ .
  - (b)  $\partial_n(G_n) \subseteq \mathfrak{m}G_{n-1}$  for  $n \geq 2$ .
  - (c)  $\partial_1(G_1)$  contains no non-zero free direct summand of  $G_0$ .
- (3) Let  $\mathbf{G} \rightarrow M$  and  $\mathbf{G}' \rightarrow M$  be proper resolutions, and let  $\tilde{\mu}: \mathbf{G} \rightarrow \mathbf{G}'$  be a morphism over  $\text{id}_M$ , given by Lemma 4.3.
- If the first resolution is minimal, then  $\tilde{\mu}$  is an isomorphism onto a direct summand; if both resolutions are minimal, then  $\tilde{\mu}$  is an isomorphism.
- (4) If  $\mathbf{G} \rightarrow M$  is a minimal proper resolution, then

$$G\text{-dim}_R M = \sup\{n \in \mathbb{Z} \mid G_n \neq 0\}.$$

*Proof.* (3) results from Proposition 1.7, and (2) is proved in two installments.

(2') Here we show that if  $\mathbf{G} \rightarrow M$  is a proper resolution satisfying the conditions in (2), then the complex  $\mathbf{G}$  is minimal. By Proposition 1.7.1 it suffices to show that if  $\beta: \mathbf{G} \rightarrow \mathbf{G}$  is a morphism and there are maps  $\theta_n: G_n \rightarrow G_{n+1}$  with

$$\beta_n - \text{id}_{G_n} = \theta_{n-1}\partial_n + \partial_{n+1}\theta_n \quad \text{for } n \geq 0,$$

then each  $\beta_n$  is bijective. By (a) the module  $G_n$  is free of finite rank for  $n \geq 1$ , so for these  $n$  it suffices to show that the right hand side is contained in  $\mathfrak{m}G_n$ . Condition (b) guarantees this for  $n \geq 2$ , and reduces it to  $\text{Im}(\theta_0\partial_1) \subseteq \mathfrak{m}G_1$  for  $n = 1$ . Assuming the inclusion fails, we can find  $x \in \text{Im } \partial_1$  with  $\theta_0(x) \notin \mathfrak{m}G_1$ ; as  $G_1$  is free,  $Rx$  is a free direct summand of  $G_0$ , contradicting (c). Thus,  $\beta_n$  is an isomorphism for  $n \geq 1$ . It follows that in the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega^1 \mathbf{G} & \longrightarrow & G_0 & \longrightarrow & H_0(\mathbf{G}) \longrightarrow 0 \\ & & \downarrow \Omega^1(\beta) & & \downarrow \beta_0 & & \downarrow H_0(\beta) \\ 0 & \longrightarrow & \Omega^1 \mathbf{G} & \longrightarrow & G_0 & \longrightarrow & H_0(\mathbf{G}) \longrightarrow 0 \end{array}$$

the homomorphism  $\Omega^1(\beta)$  is bijective. On the other hand,  $H_0(\beta) = H_0(\text{id}_{\mathbf{G}})$  is the identity map. Thus,  $\beta_n$  is an isomorphism also for  $n = 0$ .

(1') The resolution  $\mathbf{F} \rightarrow M$  satisfies conditions (2.a) and (2.b) by construction. Assume that (2.c) fails, and choose  $y \in Y$  so that  $\partial_1(y) = (u, v) \in U \oplus V = F_0$  generates a non-zero free direct summand. We then get a surjective homomorphism  $\lambda: U \oplus V \rightarrow R$  with  $\lambda(u) + \lambda(v) = 1$ , so  $\lambda(u)$  or  $\lambda(v)$  is invertible. In the first case  $Ru \neq 0$  is a free direct summand of  $U$ , contradicting Lemma 8.2.2. In the second case  $Rv \neq 0$  is a direct summand of  $V$ ; since  $\varphi'(v) = \mu\chi(u, v) = \mu\chi\partial_1(y) = 0$ , we have  $\text{Ker}(\varphi' \otimes_R k) \ni v \otimes 1 \neq 0$ , contradicting the choice of  $\varphi': V \rightarrow M'$  as a free cover. Thus,  $\mathbf{F} \rightarrow M$  satisfies condition (2.c) as well, hence is minimal by (2').

(2'') Let  $\mathbf{G} \rightarrow M$  be a minimal proper resolution. By (3) it is isomorphic to the resolution in (1'), for which we have just checked the conditions in (2).

(4) Let  $\mathbf{G} \rightarrow M$  be a minimal proper resolution. By definition of  $G$ -dimension,  $G\text{-dim}_R M \leq \sup\{n \in \mathbb{Z} \mid G_n \neq 0\}$ . The opposite inequality comes from (3).  $\square$

By Part (3) of the next result, if  $R$  is Gorenstein then minimal  $\mathcal{G}$ -approximations are minimal Cohen-Macaulay approximations in the sense of Auslander. Thus, Part (1) contains his existence theorem for such approximations, and its extension to Cohen-Macaulay rings with dualizing module by Enochs, Jenda, and Xu [20, (5.1)].

**8.6. Theorem.** *Let  $R$  be a local ring and  $M$  a finite module of finite  $G$ -dimension.*

- (1)  $M$  has a minimal  $\mathcal{G}$ -approximation; more precisely:
- (1') The  $\mathcal{G}$ -approximation  $\mathbf{A} \rightarrow M$  described in Theorem 8.5.1' is minimal.
- (2) For a  $\mathcal{G}$ -approximation  $\chi: \mathbf{B} \rightarrow M$  the following conditions are equivalent.
  - (i)  $\mathbf{B}$  is minimal.
  - (ii)  $v(B_1)$  contains no non-zero direct summand of  $B_0$ .
  - (iii)  $v(B_1)$  contains no non-zero free direct summand of  $B_0$ .
  - (iv) Each homomorphism  $\beta_0: B_0 \rightarrow B_0$  with  $\chi\beta_0 = \chi$  is bijective.
  - (v) The complex  $\mathbf{G}$  obtained by splicing  $\mathbf{B}$  with a minimal free resolution  $\varkappa: \mathbf{K} \rightarrow B_1$  is minimal.
- (3) Let  $\chi: \mathbf{B} \rightarrow M$  and  $\chi': \mathbf{B}' \rightarrow M$  be  $\mathcal{G}$ -approximations, and let  $\beta: \mathbf{B} \rightarrow \mathbf{B}'$  be a morphism with  $\beta_0: B_0 \rightarrow B'_0$  given by Lemma 5.8.1 and  $\beta_1 = \beta_0|_{B_1}$ .  
If the first approximation is minimal, then  $\beta$  is an isomorphism onto a direct summand; if both approximations are minimal, then  $\beta$  is an isomorphism.
- (4) If  $0 \rightarrow Y \rightarrow X \rightarrow M \rightarrow 0$  is a minimal  $\mathcal{G}$ -approximation and  $M \neq 0$ , then

$$G\text{-dim}_R M = \sup\{0, \text{proj dim}_R Y + 1\}.$$

*Proof.* We start by proving that the conditions in (2) are equivalent.

(i)  $\implies$  (ii). If  $C$  is a direct summand of  $B_0$  contained in  $v(B_1)$ , then setting  $E_0 = C$ ,  $E_1 = v^{-1}(C)$ , and  $E_n = 0$  for  $n \neq 0, 1$  we get an irrelevant subcomplex of  $\mathbf{B}$ . Since  $\mathbf{B}$  is minimal, we have  $\mathbf{E} = 0$  by Proposition 1.7.2.

(ii)  $\implies$  (iii) is clear.

(iii)  $\implies$  (v). We have  $\mathbf{G}_{\geq 1} \cong \Sigma \mathbf{K}$  by definition, so  $G_n$  is free for  $n \geq 1$  and  $\partial_n(G_n) \subseteq \mathfrak{m}G_{n-1}$  for  $n \geq 2$ . Since  $\text{Im } \partial_1 = \text{Im } v$ , this module contains no non-zero free direct summand of  $G_0$  by hypothesis. Thus,  $\mathbf{G}$  is minimal by Theorem 8.5.2.

(v)  $\implies$  (iv). Let  $\beta_0: B_0 \rightarrow B_0$  be a homomorphism with  $\chi\beta_0 = \chi$ , and choose a morphism  $\tilde{\beta}: \mathbf{K} \rightarrow \mathbf{K}$  lifting the induced map  $B_1 \rightarrow B_1$ . Setting  $\beta_n = \tilde{\beta}_{n-1}$  for  $n \geq 1$ , we obtain a morphism  $\beta: \mathbf{G} \rightarrow \mathbf{G}$  with  $H_0(\beta) = \text{id}_M$ . The resolution  $\mathbf{G} \rightarrow M$  being proper by Lemma 4.1.3, the morphism  $\beta$  is a homotopy equivalence by Lemma 4.3. By hypothesis,  $\mathbf{G}$  is a minimal complex, so  $\beta$  is an isomorphism; in particular,  $\beta_0$  is bijective.

(iv)  $\implies$  (i). Let  $\beta: \mathbf{B} \rightarrow \mathbf{B}$  be a morphism with  $\beta \sim \text{id}_{\mathbf{B}}$ . As  $H_0(\beta) = H_0(\text{id}_{\mathbf{B}}) = \text{id}_M$ , we have  $\chi\beta_0 = \chi$ , so  $\beta_0$  is an isomorphism; as  $v$  is injective,  $\beta_1$  is an isomorphism as well. Thus,  $\mathbf{B}$  is minimal by Proposition 1.7.1.

(1') and (4) follow from the corresponding parts of Theorem 8.5, using (2.v).

(3) Follows from Proposition 1.7.  $\square$

For a module  $M$  of finite  $G$ -dimension choose a minimal free resolution  $\mathbf{P} \rightarrow M$ , minimal  $\mathcal{G}$ -approximations  $0 \rightarrow Y_n \rightarrow X_n \rightarrow \Omega^n \mathbf{P} \rightarrow 0$ . The uniqueness of minimal free resolutions and Theorem 8.6.3 imply that all choices yield isomorphic modules  $X_n$ . We introduce the  $n$ th  $\mathcal{G}$ -delta invariant of  $M$  by the formula

$$\delta_{\mathcal{G}}^n(M) = \text{f-rank}_R(X_n) \quad \text{for } n \in \mathbb{Z}.$$

When  $R$  is Gorenstein this is Auslander's definition of  $\delta^n(M)$ , cf. [4, p. 314].

**8.7. Theorem.** *If  $(R, \mathfrak{m}, k)$  is a local ring,  $M$  is a finite  $R$ -module,  $\text{G-dim}_R M$  is finite, and  $\varepsilon_R^n(M, k): \text{Ext}_R^n(M, k) \rightarrow \widehat{\text{Ext}}_R^n(M, k)$  is the comparison map, then*

$$\delta_{\mathcal{G}}^n(M) = \text{rank}_k \text{Ker } \varepsilon_R^n(M, k) \quad \text{for all } n \in \mathbb{Z}.$$

Furthermore,  $\delta_{\mathcal{G}}^n(M) = 0$  for all  $n < 0$  and all  $n > \text{G-dim}_R M$ .

*Proof.* In view of Theorem 5.2.2 the second assertion follows from the first.

Let  $\mathbf{P} \rightarrow M$  be a *minimal* free resolution. By Proposition 5.6 and Theorem 5.9, the exact sequence  $0 \rightarrow \Omega^1 \mathbf{P} \rightarrow P_0 \rightarrow M \rightarrow 0$  induces a commutative diagram

$$\begin{array}{ccc} \text{Ext}_R^n(\Omega^1 \mathbf{P}, k) & \xrightarrow{\cong} & \text{Ext}_R^{n+1}(M, k) \\ \downarrow \varepsilon_R^n(\Omega^1 \mathbf{P}, k) & & \downarrow \varepsilon_R^{n+1}(M, k) \\ \widehat{\text{Ext}}_R^n(\Omega^1 \mathbf{P}) & \xrightarrow{\cong} & \widehat{\text{Ext}}_R^{n+1}(M, k) \end{array}$$

Thus, it suffices to prove the equality  $\delta_{\mathcal{G}}(M) = \text{rank}_k (\text{Ker } \varepsilon_R^0(M, k))$ .

Let  $\mathbf{B} = 0 \rightarrow Y \xrightarrow{\nu} X \xrightarrow{\chi} M \rightarrow 0$  be a minimal  $\mathcal{G}$ -approximation. Decompose  $X$  as  $U \oplus V$  where  $V$  is free with  $\text{rank}_R V = \delta_{\mathcal{G}}(M)$ , and let  $\iota: U \rightarrow X$  and  $\pi: X \rightarrow V$  be the canonical maps. Since  $\text{f-rank}_R U = 0$ , Theorem 8.4.1 yields a complete resolution  $\mathbf{S} \xrightarrow{\vartheta} \mathbf{P} \rightarrow U$  with  $\partial(\mathbf{S}) \subseteq \mathfrak{m}\mathbf{S}$  and  $\vartheta_n = \text{id}_{S_n}$  for  $n \geq 0$ . Thus,  $\varepsilon_R^n(U, k): \text{Ext}_R^n(U, k) \rightarrow \widehat{\text{Ext}}_R^n(U, k)$  is bijective for  $n \geq 0$ , so in the diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & \text{Ker } \varepsilon_R^0(M, k) & \dashrightarrow & \text{Hom}_R(V, k) & & \\ & & \downarrow & & \downarrow \text{Hom}_R(\pi, k) & & \\ 0 & \longrightarrow & \text{Hom}_R(M, k) & \xrightarrow{\text{Hom}_R(\chi, k)} & \text{Hom}_R(X, k) & \xrightarrow{\text{Hom}_R(\nu, k)} & \text{Hom}_R(Y, k) \\ & & \downarrow \varepsilon_R^0(M, k) & & \downarrow \varepsilon_R^0(U, k) \circ \text{Hom}_R(\iota, k) & & \\ & & \widehat{\text{Ext}}_R^0(M, k) & \xrightarrow{\widehat{\text{Ext}}_R^0(\iota \chi, k)} & \widehat{\text{Ext}}_R^0(U, k) & & \end{array}$$

the right hand column is exact. The left hand column is exact by definition and the bottom square commutes by Proposition 5.6. This defines the hyphenated arrow, which is necessarily injective. It is surjective if  $\text{Im } \text{Hom}_R(\pi, k) \subseteq \text{Im } \text{Hom}_R(\chi, k)$ . In view of the exactness of the middle row, this condition can be rewritten as  $\text{Hom}_R(\nu, k) \circ \text{Hom}_R(\pi, k) = 0$ . The composition is equal to  $\text{Hom}_R(\pi \nu, k)$ , and the last map is trivial because  $\pi \nu(Y)$  is contained in  $\mathfrak{m}V$  by the minimality of  $\mathbf{B}$ .  $\square$

As illustration and for later use, we record some computations of Auslander.

**8.8. Examples.** Let  $R$  be a non-regular Gorenstein local ring.

(1) The residue field has trivial delta invariants:  $\delta_{\mathcal{G}}^n(k) = 0$  for all  $n \in \mathbb{Z}$ ; this is announced in [4, (5.7)], and follows also from the more general [27, Theorem 6].

(2) If  $\ell(R)$  denotes the *Loewy length* of  $R$ , that is, the smallest integer  $r$  such that  $\mathfrak{m}^r$  is contained in an ideal generated by some system of parameters of  $R$ , then  $\delta_{\mathcal{G}}^0(R/\mathfrak{m}^s) = 1$  for all  $s \geq \ell(R)$ : a proof may be found in [15, p. 273].

## 9. BETTI NUMBERS

In this section  $R$  is a commutative noetherian local ring with residue field  $k$ . The rank of the  $n$ th free module in a minimal free resolution of a finite  $R$ -module  $M$  is called the  $n$ th *Betti number* of  $M$ ; it is denoted  $\beta_n^R(M)$ , and is equal to  $\text{rank}_k \text{Ext}_R^n(M, k)$ . If  $\text{G-dim}_R M < \infty$ , then by 7.3 the numbers

$$\beta_n^G(M) = \text{rank}_k \text{Ext}_G^n(M, k) \quad \text{and} \quad \widehat{\beta}_n^R(M) = \text{rank}_k \widehat{\text{Ext}}_R^n(M, k).$$

are non-negative integers, that we call the  $n$ th *relative Betti number* and  $n$ th *stable Betti number* of  $M$ , respectively. We relate them to their absolute counterparts.

**9.1. Theorem.** *Let  $M$  be a finite  $R$ -module with  $\text{G-dim}_R M = g < \infty$ . Let  $\mathbf{T} \rightarrow \mathbf{P} \rightarrow M$  be a complete resolution and  $0 \rightarrow Y \rightarrow X \rightarrow M \rightarrow 0$  a  $\mathcal{G}$ -approximation.*

(1)  $\widehat{\beta}_n^R(M) = \widehat{\beta}_n^R(X) = \widehat{\beta}_{n-i}^R(\Omega^i \mathbf{T}) = \widehat{\beta}_{-n-1}^R(X^*)$  for all  $n \in \mathbb{Z}$  and all  $i \in \mathbb{Z}$ .

*If the complete resolution is minimal, then  $\widehat{\beta}_n^R(M) = \text{rank}_R T_n$  for all  $n \in \mathbb{Z}$ .*

*If  $M$  is  $G$ -perfect, then  $\widehat{\beta}_n^R(M) = \widehat{\beta}_{g-1-n}^R(M^\dagger)$  for all  $n \in \mathbb{Z}$ .*

(2) *If  $\text{proj dim}_R M < \infty$ , then  $\beta_n^G(M) = \beta_n^R(M)$  and  $\widehat{\beta}_n^R(M) = 0$  for all  $n \in \mathbb{Z}$ .*

(3) *If  $\text{proj dim}_R M = \infty$ , then the following tables express the relative Betti numbers and the stable Betti numbers of  $M$ .*

$n$	...	0	1	...	g + 1	...
$\beta_n^G(M)$	0	$\beta_0^R(M)$	$\beta_0^R(M) - \beta_0^R(X) + \beta_0^R(Y) = \delta_G^1(M) \leq \beta_1^R(M)$	$\beta_{n-1}^R(Y)$	0	

  

$n$	...	-1	0	...	g	...
$\widehat{\beta}_n^R(M)$	$\beta_{-n-1}^R(X^*)$	$\beta_0^R(X^*) - \delta_G^0(X)$	$\beta_0^R(X) - \delta_G^0(X)$	$\beta_n^R(X)$	$\beta_g^R(X) - \delta_G^g(X)$	$\beta_n^R(M)$

Furthermore,  $\beta_n^G(M) > 0$  if  $n = 0$  or  $2 \leq n \leq g$ , and  $\widehat{\beta}_n^R(M) > 0$  for all  $n \in \mathbb{Z}$ .

*Proof.* (1) Lemmas 5.8.2 and 5.8.4 yield equalities  $\widehat{\beta}_n^R(X) = \widehat{\beta}_n^R(M) = \widehat{\beta}_{n-i}^R(\Omega^i \mathbf{T})$ . If  $\mathbf{T}$  is minimal, then  $\widehat{\beta}_n^R(M) = \text{rank}_R T_n$  by Theorem 8.4.2. If  $M$  is  $G$ -perfect, then  $\widehat{\beta}_n^R(M) = \widehat{\beta}_{g-1-n}^R(M^\dagger)$  for all  $n$  by the last computation and Construction 6.2. As  $X$  is  $G$ -perfect with  $X^\dagger = X^*$ , we get  $\widehat{\beta}_n^R(X) = \widehat{\beta}_{-n-1}^R(X^*)$  for all  $n \in \mathbb{Z}$ .

(2) results from Theorems 4.2.3 and 5.2.3.

(3) Theorem 4.2.2 yields the expressions for  $\beta_n^G(M)$  when  $n \leq 0$  or  $n > g$ .  $\mathcal{G}$ -approximations are proper exact sequences and  $\text{Ext}_G^n(X, k) = 0$  for  $n \neq 0$  by Theorem 4.2.2, so Proposition 4.6 yields isomorphisms and an exact sequence:

$$\text{Ext}_G^{n-1}(Y, k) \cong \text{Ext}_G^n(M, k) \quad \text{for all } n \geq 2;$$

$$0 \longrightarrow \text{Ext}_G^0(M, k) \longrightarrow \text{Ext}_G^0(X, k) \longrightarrow \text{Ext}_G^0(Y, k) \longrightarrow \text{Ext}_G^1(M, k) \longrightarrow 0.$$

By Theorems 4.2.2.b and 4.2.3 all relative Ext modules above, except for  $\text{Ext}_G^n(M, k)$  with  $n \geq 1$ , are isomorphic to the corresponding absolute Ext modules. This gives the expressions for  $\beta_n^G(M)$  for  $n \geq 2$ , and the first expression for  $\beta_1^G(M)$ . The second expression and the inequality for  $\beta_1^G(M)$  come from Theorem 7.1.

Observe that the expression for  $\widehat{\beta}_n^R(M)$  is given by Theorem 5.2.2 for  $n > g$  and by Theorem 7.1 for  $n = g$ . Due to (1), this observation applied to  $X$  and to  $X^*$

gives the tabulated values of  $\widehat{\beta}_n^R(M)$  for  $0 \leq n \leq g$  and for  $n \leq -1$ , along with an equality  $\widehat{\beta}_0^R(M) = \beta_0^R(X^*) - \delta_G^0(X^*)$ ; it remains to note that  $\delta_G^0(X^*) = \delta_G^0(X)$ .

Assume  $\widehat{\beta}^n(M) = 0$  for some  $n \in \mathbb{Z}$ . In view of (1), this yields  $T_n = 0$  in a minimal complete resolution  $\mathbf{T} \rightarrow \mathbf{P} \rightarrow M$ , hence  $\widehat{\text{Ext}}_R^n(M, N) = 0$  for all  $R$ -modules  $N$ . By Theorem 5.9 this implies  $\text{proj dim}_R M < \infty$ , a contradiction.  $\square$

The theorem suggests how to find a Betti sequence with a hole at  $n = 1$ .

**9.2. Example.** If  $R$  is a non-regular Gorenstein local ring with  $\dim R = d$ , then

$$\beta_n^G(k) = \begin{cases} 0 & \text{for } n < 0, n = 1, \text{ and } n > d; \\ 1 & \text{for } n = 0; \\ \beta_{d-n}^R(k) & \text{for } 2 \leq n \leq d; \end{cases}$$

$$\widehat{\beta}_n^R(k) = \beta_n^R(k) + \beta_{d-1-n}^R(k) \quad \text{for all } n \in \mathbb{Z}.$$

Indeed, the value of  $\beta_n^G(k)$  is given by Theorem 4.2.2.b for  $n \leq 0$ , and by Theorem 6.3.3 for  $n \geq 2$ . Theorems 6.3.4 and 5.2.2 compute  $\widehat{\beta}_n^R(k)$  for  $n < -1$  and for  $n > d$ . Theorem 8.7 and Remark 8.8.1 imply that the exact sequences of Theorems 7.1 and 6.3.4 split, yielding  $\beta_1^G(k) = 0$ , and  $\widehat{\beta}_n^R(k) = \beta_n^R(k) + \beta_{d-1-n}^R(k)$  for  $-1 \leq n \leq d$ .

In order to compare the asymptotes of a complete resolution we raise the

*Problem.* If  $\text{G-dim}_R M < \text{proj dim}_R M$ , does  $\lim_{n \rightarrow \infty} \frac{\widehat{\beta}_n^R(M)}{\widehat{\beta}_{i-n}^R(M)}$  exist for some  $i \in \mathbb{Z}$ ?

**9.3. Remarks.** (1) By Theorem 9.1 it suffices to treat the case when  $M$  is totally reflexive with  $\text{f-rank}_R M = 0$ . For such modules Theorems 6.3.4 and 5.2.2 yield equalities  $\widehat{\beta}_n^R(M)/\widehat{\beta}_{i-n}^R(M) = \beta_n^R(M)/\beta_{n+g-i-1}^R(M^*)$  for all  $n \geq 0$ .

(2) For  $R$  Gorenstein and  $M = k$  the limit is known to exist if  $i = d - 1$ , cf. Example 9.2, but not in general, cf. [5, §4.3]. Also, limits for various  $i$  may differ: if  $R = k[x, y, z]/(x^2 - y^2, y^2 - z^2, xy, yz, xz)$ , then  $\beta_n^R(k) = (b^{n+1} - b^{-n-1})/\sqrt{5}$  for all  $n \geq 0$  and  $b = (3 + \sqrt{5})/2$ , cf. [32, Satz 9], so  $\lim_{n \rightarrow \infty} \widehat{\beta}_n^R(k)/\widehat{\beta}_{i-n}^R(k) = b^{i+1}$ .

For more examples, consider the *complete intersection dimension*, defined by

$$\text{CI-dim}_R M = \inf \{ \text{proj dim}_Q(M \otimes_R R') - \text{proj dim}_Q R' \}$$

with infimum taken over all diagrams of local homomorphisms  $R \rightarrow R' \leftarrow Q$  where the first map is flat and the second is surjective with kernel generated by a regular sequence. There are inequalities  $\text{G-dim}_R M \leq \text{CI-dim}_R M \leq \text{proj dim}_R M$ , with equalities to the left of any finite dimension, cf. [8, (1.4)]. The ring  $R$  is a complete intersection if and only if  $\text{CI-dim}_R M < \infty$  for each  $M \in \mathcal{F}(R)$ , cf. [8, (1.3)].

**9.4. Example.** If  $\text{CI-dim}_R M < \text{proj dim}_R M$ , then  $\lim_{n \rightarrow \infty} \widehat{\beta}_n^R(M)/\widehat{\beta}_{i-n}^R(M)$  exists for each  $i \in \mathbb{Z}$ , does not depend on  $i$ , and is a rational number; when the sequence  $(\beta_n^R(M))_{n \geq 0}$  is bounded the sequence  $(\widehat{\beta}_n^R(M))_{n \in \mathbb{Z}}$  is constant.

Indeed, by [8, (8.1)] there exist polynomials  $b_{\pm}^M(t) \in \mathbb{Q}[t]$ , with equal leading terms, such that  $\beta_n^R(M) = b_+^M(n)$  for all even  $n \gg 0$  and  $\beta_n^R(M) = b_-^M(n)$  for all odd  $n \gg 0$ . We may assume  $M \in \mathcal{G}$ , cf. Remark 9.3.1, and then by [6, (2.6.2), (3.3)] the polynomials for  $M$  and  $M^*$  have the same degree, say  $c$ . If  $c = 0$ , then  $\beta_n^R(M) = b^M$  and  $\beta_n^R(M^*) = b^{M^*}$  for  $n \geq 0$  by [8, (7.3.2)]. If  $\mathbf{T}$  is a complete resolution of  $M$ , then each  $\Omega^i \mathbf{T}$  is in  $\mathcal{G}$  by Lemma 2.4, so Theorem 9.1 yields

$$\beta_0^R(M^*) = \widehat{\beta}_0^R(M^*) = \widehat{\beta}_0^R(\Omega^{-1} \mathbf{T}) = \beta_0^R(\Omega^{-1} \mathbf{T}) = \beta_1^R(\Omega^{-1} \mathbf{T}) = \beta_0^R(M).$$

## 10. BASS NUMBERS

Let  $R$  be a local ring with residue field  $k$ . The  $n$ th Bass number of a finite  $R$ -module  $N$  is the number of copies of the injective envelope of  $k$  in the  $n$ th module of a minimal injective resolution of  $N$ ; it is denoted  $\mu_n^R(N)$  and is equal to  $\text{rank}_k \text{Ext}_R^n(k, N)$ . To use a similar expression to define stable and relative analogs of Bass numbers, we need  $\text{G-dim}_R k$  to be finite. This imposes restrictions on  $R$ .

**10.1. Remark.** For a local ring  $R$  with residue field  $k$  the following are equivalent.

- (i)  $R$  is Gorenstein.
- (ii)  $\text{G-dim}_R N \leq \dim R$  for each finite  $R$ -module  $N$ .
- (iii)  $\text{G-dim}_R N < \infty$  for each finite  $R$ -module  $N$ .
- (iv)  $k$  is a G-perfect  $R$ -module with  $\text{G-dim}_R k = \dim R$  and  $k^\dagger \cong k$ .
- (v)  $\text{G-dim}_R k < \infty$ .

Indeed, the equivalence of (i) and (ii) comes from Theorem 3.2. By Theorem 4.2.2 a condition (iv) can be restated as  $\mu_n^R(R) = 0$  for  $n \neq \dim R$  and  $\mu_{\dim R}^{\dim R}(R) = 1$ , and condition (v) as  $\mu_n^R(R) = 0$  for  $n \gg 0$ . Thus, (i), (iv), and (v) are equivalent by the classical characterization of Gorenstein local rings by Bass, cf. [10, §3.3]

For the rest of this section  $R$  is a *commutative noetherian local Gorenstein ring* and set  $d = \dim R$ . For each finite  $R$ -module  $N$  we introduce an  $n$ th relative Bass number and an  $n$ th stable Bass number by

$$\mu_G^n(N) = \text{rank}_k \text{Ext}_G^n(k, N) \quad \text{and} \quad \widehat{\mu}_R^n(N) = \text{rank}_k \widehat{\text{Ext}}_R^n(k, N).$$

**10.2. Remark.** The Bass numbers of the  $R$ -module  $k$  are equal to its Betti numbers, so Example 9.2 expresses  $\mu_G^n(k)$  and  $\widehat{\mu}_R^n(k)$  in terms of  $\mu_R^n(k)$ .

**10.3. Theorem.** Let  $N$  be a finite  $R$ -module with  $\text{G-dim}_R N = g < \infty$ . Let  $\mathbf{T} \rightarrow \mathbf{P} \rightarrow N$  be a complete resolution and  $0 \rightarrow Y \rightarrow X \rightarrow N \rightarrow 0$  a  $\mathcal{G}$ -approximation.

- (1)  $\widehat{\mu}_R^n(N) = \widehat{\mu}_R^n(X) = \widehat{\mu}_R^{n+i}(\Omega^i \mathbf{T}) = \widehat{\beta}_{d-1-n-i}^R(\Omega^i \mathbf{T})$  for all  $n \in \mathbb{Z}$  and all  $i \in \mathbb{Z}$ .
- (2) If  $\text{proj dim}_R N < \infty$ , then  $\mu_G^n(N) = \mu_R^n(N)$  and  $\widehat{\mu}_R^n(N) = 0$  for all  $n \in \mathbb{Z}$ .
- (3) If  $\text{proj dim}_R N = \infty$  then the following tables, where  $t = \text{depth}_R N$ , express the relative Bass numbers and the stable Bass numbers of  $N$ .

$n$	...	0	1	...	$d$	...
$\mu_G^n(N)$	0	$\mu_R^0(N)$	$\leq \mu_R^1(N)$	$\beta_{d-n}^R(N)$	0	

$n$	...	$t-2$	...	$d$	$d+1$	...
$\widehat{\mu}_R^n(N)$	$\beta_{d-n-1}^R(N)$	$\mu_R^d(\Omega^{d-n} \mathbf{T})$	$\mu_R^d(\Omega^0 \mathbf{T}) \leq \mu_R^d(N)$	$\mu_R^n(N)$		

Furthermore,  $\widehat{\mu}_n^R(N) > 0$  for all  $n$ , and  $\mu_n^G(N) > 0$  for  $2 \leq n \leq d$ .

*Proof.* (1) The first two equalities come from Lemmas 5.8.2 and 5.8.4. We obtain the third equality directly from Theorem 6.3.4 when  $n+i \leq -2$ , then shifting degrees by means of Lemma 5.8.4 we extend it to all values of  $n$  and  $i$ .

(2) results from Theorems 4.2.4 and 5.2.4.

(3) The first table comes from Theorem 4.2.2.b for  $n \leq 0$  and from Theorem 7.1 for  $n = 1$ . Since  $k$  is  $G$ -perfect with  $G\text{-dim}_R k = d$ , cf. Remark 10.1, the tabulated values of  $\mu_G^n(N)$  for  $n \geq 2$  are given by Theorem 6.3.3.

Recalling that  $\text{Ext}_R^n(k, N) = 0$  for  $n < t$ , we fill in the second table using Theorem 6.3.4 for  $n \leq t - 2$ , Part (1) for  $t - 2 < n < d$ , Part (1) with Theorem 7.1 for  $n = d$ , and Theorem 5.2.2 for  $n \geq d + 1$ .

If  $\widehat{\mu}_R^n(N) = 0$  for some  $n$ , then (1) yields  $\widehat{\beta}_{d-1-n}^R(\Omega^0 \mathbf{T}) = 0$ . Using alternatively parts (3) and (1) of Theorem 9.1, for  $i > G\text{-dim}_R N$  we get equalities  $0 = \widehat{\beta}_i^R(\Omega^0 \mathbf{T}) = \widehat{\beta}_i^R(N) = \beta_i^R(N)$  contradicting the hypothesis  $\text{proj dim}_R N = \infty$ .  $\square$

The number  $\text{depth}_G N = \inf\{n \in \mathbb{Z} \mid \mu_n^G(N) \neq 0\}$  was introduced in [28, §2.3], in search of an equality of Auslander-Buchsbaum type for relative invariants. We obtain an amusing catalog of all cases where such an equality holds.

**10.4. Corollary.** *An equality  $\text{rel dim}_G N = \text{depth}_G R - \text{depth}_G N$  holds if and only if  $N$  satisfies one of the following disjoint conditions:  $\text{depth}_R N = 0$ , or  $\text{depth}_G N = 1$ , or  $\text{depth}_R N = 2$ , or  $\text{proj dim}_R N \leq \text{depth } R - 3$ .*

*Proof.* Applying Proposition 4.8, Theorem 8.3, and Theorem 4.2.4, one obtains

$$\text{rel dim}_G N = G\text{-dim}_R N = \text{depth } R - \text{depth}_R N = \text{depth}_G R - \text{depth}_R N.$$

Thus,  $N$  satisfies a ‘relative Auslander-Buchsbaum Equality’ if and only if  $\text{depth}_G N = \text{depth}_R N$ . From Theorem 10.3 one reads off the implications:

$$\text{depth}_G N = 0 \iff \text{depth}_R N = 0.$$

$$\text{depth}_G N = 1 \implies \text{depth}_R N = 1.$$

$$\text{depth}_G N = 2 \iff \text{depth}_R N = 2.$$

If  $n \geq 3$ , then  $\text{depth}_G N = n \iff \text{depth}_R N = n$  and  $\text{proj dim}_R N < \infty$ .

The list above is just another format of the desired assertion.  $\square$

The balance of this section is devoted to showing that all admissible configurations of vanishing Betti numbers and Bass numbers do occur. We construct relevant examples and analyze them through the results of the preceding sections. It may come as a surprise that relative Bass sequences and relative Betti sequences can have a gap at  $n = 1$ , or that the relative Bass sequence of a nontrivial module can be identically equal to zero: no such pattern occurs in absolute cohomology.

**10.5. Examples.** Let  $R$  be a non-regular Gorenstein ring of dimension  $d$ .

The following table lists examples of all possible patterns of vanishing in pairs  $(\mu_G^0(N), \mu_G^1(N))$  for an *indecomposable*  $R$ -module  $N$  with  $\text{proj dim}_R N = \infty$ .

	$\mu_G^0 = 0 = \mu_G^1$	$\mu_G^0 > 0 = \mu_G^1$	$\mu_G^0 = 0 < \mu_G^1$	$\mu_G^0 > 0 < \mu_G^1$
$d = 0$	No such $N$	Each $N$	No such $N$	
$d = 1$	Each torsion-free $N$	$k$		
$d \geq 2$	$\mathfrak{m}$		$N_{d-1}$	$N_d$

The assertions on  $k$ ,  $\mathfrak{m}$ , and  $N_i$  come from Remark 10.2, and Propositions 10.6, 10.8, respectively; the rest follows from Proposition 10.7 and Example 8.8.2.

**10.6. Proposition.** *If  $R$  is a non-regular Gorenstein local ring with  $\dim R = d \geq 1$ , then the following hold:*

$$\mu_{\mathcal{G}}^n(\mathfrak{m}) = \begin{cases} 0 & \text{for } n \leq 1 \text{ and } n \geq d+1; \\ \beta_{n-d}^R(\mathfrak{m}) & \text{for } 2 \leq n \leq d; \end{cases}$$

$$\widehat{\mu}_R^n(\mathfrak{m}) = \begin{cases} \mu_R^n(\mathfrak{m}) + \mu_R^{d-n+1}(\mathfrak{m}) & \text{for } n \neq 1, d; \\ \mu_R^d(\mathfrak{m}) & \text{for } n = 1, d. \end{cases}$$

*Proof.* Since  $R$  is Gorenstein, we have  $\text{Ext}_R^n(k, R) = 0$  for  $n \neq d$  and  $\text{Ext}_R^d(k, R) \cong k$ . Let  $\rho: R \rightarrow k$  be the canonical surjection, let  $\mathbf{P} \rightarrow k$  be a minimal free resolution, and let  $X \rightarrow \Omega^n \mathbf{P}$  be a  $\mathcal{G}$ -approximation. We then have

$$\text{rank}_k \text{Ext}_R^d(k, \rho) \leq \text{f-rank}_R \Omega^n \mathbf{P} \leq \text{f-rank}_R X = \delta_{\mathcal{G}}^d(k) = 0.$$

where the first inequality comes from Lemma 8.2.1 and the equality from Remark 8.8.1. Thus, the cohomology exact sequence induced by the short exact sequence  $0 \rightarrow \mathfrak{m} \rightarrow R \xrightarrow{\rho} k \rightarrow 0$  yields equalities

$$\widehat{\mu}_R^n(\mathfrak{m}) = \widehat{\mu}_R^{n-1}(k) \quad \text{for all } n \in \mathbb{Z}; \quad \mu_R^n(\mathfrak{m}) = \begin{cases} \mu_R^{n-1}(k) & \text{for } n \neq d; \\ \mu_R^{d-1}(k) + 1 & \text{for } n = d. \end{cases}$$

The expressions for  $\widehat{\mu}_R^n(\mathfrak{m})$  now follow from those for  $\widehat{\mu}_R^n(k)$ , cf. Remark 10.2.

For  $n \neq 1$  the values of  $\mu_{\mathcal{G}}^n(\mathfrak{m})$  are given by Theorem 10.3. To prove that  $\mu_{\mathcal{G}}^1(\mathfrak{m})$  vanishes we use an exact sequence given by Theorem 7.1:

$$0 \longrightarrow \text{Ext}_{\mathcal{G}}^1(k, \mathfrak{m}) \longrightarrow \text{Ext}_R^1(k, \mathfrak{m}) \xrightarrow{\varepsilon_R^1(k, \mathfrak{m})} \widehat{\text{Ext}}_R^1(k, \mathfrak{m}) \longrightarrow \text{Ext}_{\mathcal{G}}^2(k, \mathfrak{m}).$$

If  $d = 1$ , then  $\text{Ext}_{\mathcal{G}}^2(k, \mathfrak{m}) = 0$  by Theorem 4.2.2, so  $\varepsilon_R^1$  is surjective, hence  $\mu_{\mathcal{G}}^1(\mathfrak{m}) = \mu_R^1(\mathfrak{m}) - \widehat{\mu}_R^1(\mathfrak{m})$ ; the difference vanishes by the computation above.

If  $d \geq 2$ , then Proposition 5.4 yields a commutative diagram

$$\begin{array}{ccccccc} \text{Hom}_R(k, R) & \longrightarrow & \text{Hom}_R(k, k) & \longrightarrow & \text{Ext}_R^1(k, \mathfrak{m}) & \longrightarrow & \text{Ext}_R^1(k, R) \\ & & \varepsilon_R^0(k, k) \downarrow & & \downarrow \varepsilon_R^1(k, \mathfrak{m}) & & \\ \widehat{\text{Ext}}_R^0(k, R) & \longrightarrow & \widehat{\text{Ext}}_R^0(k, k) & \longrightarrow & \widehat{\text{Ext}}_R^1(k, \mathfrak{m}) & \longrightarrow & \widehat{\text{Ext}}_R^1(k, R) \end{array}$$

The upper ends vanish by the depth hypothesis, the lower ones by Theorem 5.2.4. As  $\varepsilon_R^0(k, \mathfrak{m})$  is injective by Theorem 8.7 and Example 8.8.1, so is  $\varepsilon_R^1(k, \mathfrak{m})$ .  $\square$

Next we take a look at dimension one.

**10.7. Proposition.** *If  $R$  is a non-regular Gorenstein local ring with  $\dim R = 1$  and  $N$  is a finite  $R$ -module, then  $\mu_{\mathcal{G}}^1(N) = \delta_{\mathcal{G}}^0(N)$ .*

*Remark.* The module  $N$  is torsion-free if and only if it is maximal Cohen-Macaulay.

*Proof.* For convenience of notation, we set  $r = \delta_{\mathcal{G}}^0(N)$ .

If  $\text{depth } N = 1$ , then  $N$  is maximal Cohen-Macaulay, so  $N \cong R^r \oplus G$  with  $G \in \mathcal{G}$  and  $\text{f-rank}_R G = 0$ . From Proposition 4.7 and Theorem 4.2.4 we obtain

$$\begin{aligned} \text{Ext}_{\mathcal{G}}^1(k, N) &\cong \text{Ext}_{\mathcal{G}}^1(k, R^r) \oplus \text{Ext}_{\mathcal{G}}^1(k, G) \\ &\cong \text{Ext}_R^1(k, R^r) \oplus \text{Ext}_{\mathcal{G}}^1(k, G) \\ &\cong k^r \oplus \text{Ext}_{\mathcal{G}}^1(k, G). \end{aligned}$$

Since  $0 \rightarrow \mathfrak{m} \xrightarrow{\iota} R \rightarrow 0$  is a  $\mathcal{G}$ -resolution of  $k$ , from Construction 6.2 we obtain a proper resolution  $0 \rightarrow R^* \xrightarrow{\iota^*} \mathfrak{m}^* \rightarrow 0$  of  $k^\dagger = k$ , and hence an exact sequence

$$\mathrm{Hom}_R(\mathfrak{m}^*, G) \xrightarrow{\mathrm{Hom}_R(\iota^*, G)} \mathrm{Hom}_R(R^*, G) \rightarrow \mathrm{Ext}_G^1(k, G) \rightarrow 0.$$

Take any  $\alpha \in \mathrm{Hom}_R(R^*, G)$ . If  $\alpha^*: G^* \rightarrow R^{**} = R$  is surjective, then  $R$  splits off as a direct summand of  $G^*$ , so  $G \cong G^{**}$  has a direct summand isomorphic to  $R$ , a contradiction. Thus,  $\alpha^*(G^*) \subseteq \mathfrak{m}$ , hence  $\alpha^* = \iota\beta$  for some  $\beta: G^* \rightarrow \mathfrak{m}$ . Dualizing, we get  $\alpha = \alpha^{**} = \mathrm{Hom}_R(\iota^*, G)(\beta^*)$ , hence  $\mathrm{Ext}_G^1(k, G) = 0$ , and so  $\mu_G^1(N) = r$ .

If  $\mathrm{depth} N = 0$ , then Theorem 8.5 yields a proper exact sequence

$$0 \rightarrow R^s \xrightarrow{\partial} R^r \oplus G \rightarrow N \rightarrow 0$$

where  $G$  is maximal Cohen-Macaulay with  $\mathrm{f}\text{-rank}_R G = 0$  and  $\partial(R^s)$  contains no nontrivial free direct summand of  $R^r$ . Thus, the composition  $\sigma: R^s \rightarrow R^r \oplus G \rightarrow R^r$  satisfies  $\sigma(R^s) \subseteq \mathfrak{m}R^r$ . We have  $\mathrm{Ext}_G^1(k, G) = 0$  by the case treated above, and  $\mathrm{Ext}_G^2(k, R^s) = 0$  by Proposition 4.2.2.a, so by Proposition 4.6 the sequence

$$\mathrm{Ext}_G^1(k, R^s) \xrightarrow{\mathrm{Ext}_G^1(k, \sigma)} \mathrm{Ext}_G^1(k, R^r) \rightarrow \mathrm{Ext}_G^1(k, N) \rightarrow 0$$

is exact. Now  $\mathrm{Ext}_G^1(k, \sigma) = 0$  by the minimality of  $\sigma$ , so  $\mu_G^1(N) = r$ , as desired.  $\square$

The missing patterns in dimensions  $\geq 2$  are provided by

**10.8. Proposition.** *Let  $R$  be a non-regular Gorenstein local ring with  $\dim R = d \geq 2$ , let  $x_1, \dots, x_d$  be a maximal regular sequence, and set  $R_i = R/(x_1, \dots, x_i)$ .*

*For  $i = d - 1$  and  $i = d$  there exist exact sequences*

$$0 \rightarrow R_i \rightarrow N_i \rightarrow \mathfrak{m} \rightarrow 0$$

where  $N_i$  is indecomposable,  $\mathrm{proj} \dim_R N_i = \infty$ , and

$$\mu_G^n(N_i) = \binom{i}{d-n} \quad \text{for } n = 0, 1.$$

*Proof.* The exact sequences  $0 \rightarrow R_i \xrightarrow{x_{i+1}} R_i \rightarrow R_{i+1} \rightarrow 0$  induce exact sequences

$$\dots \xrightarrow{x_{i+1}} \mathrm{Ext}_R^n(k, R_i) \rightarrow \mathrm{Ext}_R^n(k, R_{i+1}) \rightarrow \mathrm{Ext}_R^{n+1}(k, R_i) \xrightarrow{x_{i+1}} \dots$$

where multiplication by  $x_{i+1}$  is the zero map. As  $R_0 = R$  and  $R$  is Gorenstein, we have  $\mu_R^n(R_0) = \binom{0}{d-n}$ . Induction on  $i$  yields  $\mu_R^n(R_i) = \binom{i}{d-n}$  for all  $0 \leq i \leq d$  and  $n \in \mathbb{Z}$ . If  $i \geq d - 1$  then  $\mu_R^2(R_i) \neq 0$ , so  $\mathrm{Ext}_R^1(\mathfrak{m}, R_i) \cong \mathrm{Ext}_R^2(k, R_i) \neq 0$ , hence for  $i \geq d - 1$  there exists a non-split exact sequence  $0 \rightarrow R_i \rightarrow N_i \rightarrow \mathfrak{m} \rightarrow 0$ . It is proper by Lemma 4.1.2, so Proposition 4.4 and Corollary 7.2 give an exact sequence

$$\mathrm{Ext}_G^0(k, \mathfrak{m}) \rightarrow \mathrm{Ext}_R^1(k, R_i) \rightarrow \mathrm{Ext}_G^1(k, N_i) \rightarrow \mathrm{Ext}_G^1(k, \mathfrak{m})$$

Both ends vanish by Proposition 10.6, hence  $\mu_G^1(N_i) = \mu_R^1(R_i) = \binom{i}{d-n}$  for  $n = 0, 1$ .

Denote  $T_i$  the torsion submodule of  $N_i$ . The defining exact sequence shows that  $\mathrm{proj} \dim N_i$  is infinite, that  $N_i$  has rank 1, and that  $T_i$  is isomorphic to  $R_i$ ; in particular,  $T_i$  is indecomposable. If  $N_i = V_i \oplus W_i$  with  $V_i \neq 0$  and  $W_i \neq 0$ , we may assume that  $\mathrm{rank} V_i = 0$ , and hence  $V_i \subseteq T_i$ . As  $V_i$  is a direct summand of  $N_i$ , it is also a direct summand of  $T_i$ , hence  $V_i = T_i$ . This implies that the defining sequence splits, contrary to our choice. We conclude that  $N_i$  is indecomposable.  $\square$

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DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907  
*E-mail address:* [avramov@math.purdue.edu](mailto:avramov@math.purdue.edu)

DEPARTMENT OF MATHEMATICS, NORTHEASTERN UNIVERSITY, BOSTON, MASSACHUSETTS 02115  
*E-mail address:* [alexmart@neu.edu](mailto:alexmart@neu.edu)