STABLE PROJECTIVE HOMOTOPY THEORY OF MODULES, TAILS, AND KOSZUL DUALITY

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To Kent Fuller, a friend and a teacher

Abstract. A contravariant functor is constructed from the stable projective homotopy theory of finitely generated graded modules over a finite-dimensional algebra to the derived category of its Yoneda algebra modulo finite complexes of modules of finite length. If the algebra is Koszul with a noetherian Yoneda algebra, then the constructed functor is a duality between triangulated categories. If the algebra is self-injective, then stable homotopy theory specializes trivially to stable module theory. In particular, for an exterior algebra the constructed duality specializes to (a contravariant analog of) the Bernstein-Gelfand-Gelfand correspondence.

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1. Introduction

The Bernstein-Gelfand-Gelfand correspondence is a triangulated equivalence between the category of finitely generated graded modules over the exterior algebra on $n+1$ letters modulo projectives and the bounded derived category of coherent sheaves on the n-dimensional projective space. The goal of this paper is to show that this result admits a far-reaching generalization.

In colloquial terms, we construct a contravariant triangulated functor from the stable projective homotopy[1] theory of graded modules over a finite-dimensional algebra Λ to the bounded derived category of graded modules over the Yoneda algebra Γ of Λ modulo complexes of modules of finite length. If Λ is Koszul and Γ is noetherian, then the constructed functor is a duality. In particular, by specializing to the case when Λ is an exterior algebra, we have a contravariant analog of the BGG correspondence.[2]

A generalization of the original BGG correspondence using a covariant functor can be found in [10], but we believe that the approach presented here, with emphasis on universal constructions and, especially, new methods of linear approximations of complexes introduced here, is much more transparent. In addition, a special effort has been made to make this paper self-contained. (Standard facts from homological algebra and triangulated categories can be found in [2], [3], and [13]. For a concise survey on Koszul algebras, see [4]; for a detailed exposition, see [5] and [6].)

The paper is organized as follows. We begin by reviewing, in Sec. 2, a construction from [10] (see also the more recent preprint [12]) that takes a graded Λ-module and turns it into a linear complex of projective graded Γ-modules. This is in fact a functor and when Λ is quadratic, it becomes a duality. In Sec. 3 we specialize to Koszul algebras and show that a module is Koszul if and only if the above linearization procedure yields a minimal projective resolution of the Koszul-dual module. In Sec. 4 we recall basic facts about linear complexes and show how to approximate complexes of graded modules over a Koszul algebra of finite global dimension by linear complexes of projectives. It is well-known that Koszul duality interchanges the functors of taking the syzygy module of a module and the radical of a module. Under appropriate assumptions, starting with any finite module and passing to a power of the radical of the module one obtains a Koszul module. Unfortunately, the Koszul-dual statement is not true: passing to the higher syzygy modules one may never obtain Koszul modules. However those high syzygy modules turn out to be what is known as weakly Koszul and in Sec. 5 we recall the relevant basic results. In Sec. 6 we show how the construction of Sec. 2 gives rise to a functor from the category of finitely generated graded Λ-modules modulo projectives to the bounded derived category of Γ-tails. In Sec. 7 we invert, following R.-O. Buchweitz, the syzygy endofunctor Ω which allows us to extend, in Sec. 8, the just constructed functor to the Ω-stabilization of the category of finitely generated graded Λ-modules modulo projectives. In the same section we state the

[1]The projective homotopy is in the sense of Eckmann-Hilton. The stable homotopy is meant to be the stabilization of the homotopy by the loop space, again in the sense of Eckmann-Hilton, functor.

[2]Using the fact that the exterior algebra is self-injective, one immediately recovers an equivalence between the original categories by composing our duality and the self-duality with respect to the base field on the stable category of finite modules over the exterior algebra.

main theorem that the constructed functor is a triangulated duality. The proof of this result is given in Sec. 9.

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2. From graded modules to linear complexes of projectives

In this paper, unless stated otherwise, a module over an algebra will always be assumed to be a left module. Most of our modules will be graded. A category of graded modules will always have homomorphisms of degree zero as morphisms. We will continue to use the notation and terminology introduced in [8]. In particular, graded modules will be visualized as segmented vertical strips with the increasing order of degrees going down. Shifts of grading in graded modules will be referred to as *vertical* shifts. This is done to distinguish them from shifts of degrees indicating the positions of graded modules within a complex[3] of graded modules. In the latter case we may refer to *horizontal* shifts. Notice that a vertical shift can also be applied to a complex by simultaneously applying it to all the modules in the complex. We will use brackets to denote vertical shifts and parentheses for horizontal shifts. Thus if M is a graded module, then $M[n]_i := M_{i+n}$, where the right-hand side denotes the degree $i+n$ part of M. On the other hand, if $C_\bullet$ is a complex of graded modules, then $C(n)_i := C_{i+n}$, where the right-hand side denotes the graded module in $C_\bullet$ indexed by $i+n$.

Let $\Lambda := \coprod_{i=0}^{n} \Lambda_i$ be a *finite-dimensional* graded algebra over a field $\Bbbk$, generated in degree one. We shall assume that Λ_0, viewed as a subring of Λ, is a product of copies of $\Bbbk$; consequently, the structure homomorphism diagonally embeds $\Bbbk$ in Λ_0. Notice that while Λ_0 is commutative, it is not necessarily contained in the center of Λ. Let $J := \coprod_{k=1}^{\infty} \Lambda_k$ denote the graded radical of Λ (by assumption, this sum is actually finite). We use the canonical vector space isomorphism $\Lambda/J \cong \Lambda_0$ to view Λ_0 as a left Λ-module. Let $\Gamma := \coprod_{j=0}^{\infty} \operatorname{Ext}_{\Lambda}^{j}(\Lambda_0, \Lambda_0)$ be the Yoneda algebra of Λ.[4]

Definition 1. *A complex of graded modules is said to be linear if, for each i, the ith module is generated in degree i, provided it is nonzero.*

It was shown in [10] that there is a functor

$$\Phi : \operatorname{gr} \Lambda \to \mathcal{LCP}^b(\operatorname{gr} \Gamma)$$

between the category of finitely generated graded Λ-modules and the category of bounded linear complexes of finitely generated *projective* graded Γ-modules. We now recall the construction of Φ.

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[3] All of our complexes are homological.

[4] To unambiguously refer to the dexterity of Γ-modules, we adhere to the convention that the composition fg of two elements in Γ is given by first performing a chain map in the class of g, followed by a chain map in the class of f.

Let $M := \coprod_{k=n_0}^{n_1} M_k$ be a finitely generated graded Λ-module. The multiplication map $\Lambda_1 \otimes_{\Bbbk} M_k \to M_{k+1}$ gives rise to a homomorphism of left Λ_0-modules

$$\mu : \Lambda_1 \otimes_{\Lambda_0} M_k \to M_{k+1}$$

Since M_k is projective over Λ_0, we have a homomorphism of Λ_0-modules

$$D(\mu) : D(M_{k+1}) \to D(M_k) \otimes_{\Lambda_0} D(\Lambda_1),$$

where $D(-) := \mathrm{Hom}_{\Lambda_0}(-, \Lambda_0)$. Applying the functor $\mathrm{Hom}_{\Lambda}(-, \Lambda_0)$ to the canonical exact sequence

$$0 \to J \to \Lambda \to \Lambda_0 \to 0,$$

we have an exact sequence

$$0 \to \mathrm{Hom}_{\Lambda}(\Lambda_0, \Lambda_0) \to \mathrm{Hom}_{\Lambda}(\Lambda, \Lambda_0) \to \mathrm{Hom}_{\Lambda}(J, \Lambda_0) \to \mathrm{Ext}^1_{\Lambda}(\Lambda_0, \Lambda_0) \to 0$$

in which the middle map is zero since the preceding map is obviously an isomorphism. Thus we have an isomorphism $\mathrm{Hom}_{\Lambda}(J, \Lambda_0) \cong \mathrm{Ext}^1_{\Lambda}(\Lambda_0, \Lambda_0) = \Gamma_1$. Since Λ is generated in degree one, we have an isomorphism of $\Bbbk$-vector spaces $J/J^2 \cong \Lambda_1$, which allows us to view Λ_1 as a Λ-module. Thus

$$\mathrm{Hom}_{\Lambda}(J, \Lambda_0) \cong \mathrm{Hom}_{\Lambda}(J/J^2, \Lambda_0) \cong \mathrm{Hom}_{\Lambda}(\Lambda_1, \Lambda_0),$$

and, as a result, we have an isomorphism $\mathrm{Hom}_{\Lambda}(\Lambda_1, \Lambda_0) \to \Gamma_1$ of Λ_0-modules.[5] Since

$$D(\Lambda_1) = \mathrm{Hom}_{\Lambda_0}(\Lambda_1, \Lambda_0) = \mathrm{Hom}_{\Lambda}(\Lambda_1, \Lambda_0)$$

we now have a Λ_0-isomorphism $\gamma : D(\Lambda_1) \to \Gamma_1$. Thus we also have a Λ_0-linear map

$$d_{k0} : D(M_{k+1}) \to D(M_k) \otimes_{\Lambda_0} \Gamma_1.$$

Tensoring it (over $\Lambda_0 = \Gamma_0$) with Γ_l, we can compose the result with $1 \otimes \upsilon$, where υ is multiplication in Γ, to obtain a new map d_{kl}, as shown by the diagram

$$\begin{array}{ccc} D(M_{k+1}) \otimes_{\Gamma_0} \Gamma_l & \xrightarrow{d_{k0} \otimes 1} & D(M_k) \otimes_{\Gamma_0} \Gamma_1 \otimes_{\Gamma_0} \Gamma_l \\ & \searrow^{d_{kl}} & \downarrow{\scriptstyle 1 \otimes \upsilon} \\ & & D(M_k) \otimes_{\Gamma_0} \Gamma_{l+1} \end{array}$$

In a slightly more compact notation, we have defined a map

$$d_k : D(M_{k+1}) \otimes_{\Gamma_0} \Gamma \to D(M_k) \otimes_{\Gamma_0} \Gamma[1].$$

Shifting it by $-k-1$ and keeping the same notation, we have a map

$$d_k : D(M_{k+1}) \otimes_{\Gamma_0} \Gamma[-k-1] \to D(M_k) \otimes_{\Gamma_0} \Gamma[-k]$$

Proposition 2. *The sequence*

$$\Phi(M) := \{D(M_{k+1}) \otimes_{\Gamma_0} \Gamma[-k-1], d_k\},$$

where the Γ-module on the right is placed in degree $k+1$, is a bounded linear complex of finitely generated projective graded Γ-modules.

Remark 3. *It is immediate from the definition of Φ that $\Phi(M[n]) \simeq \Phi(M)[n](n)$.*

[5] The inverse of this map is just restriction to Λ_1.

It is easy to check that a morphism $f : M \to N$ of graded Λ-modules gives rise to a morphism $\Phi(f) : \Phi(N) \to \Phi(M)$ of the corresponding complexes and that Φ becomes a contravariant functor. One can also show that Φ is full and faithful. Suppose now that Λ is quadratic i.e., when it is written as a quotient of a free algebra, the generating letters are all of degree one and the ideal of relations is generated by elements of degree two. We then have

Proposition 4 ([10], Cor. 3.4)**.** *Λ is quadratic if and only if $\Phi : \operatorname{gr}\Lambda \to \mathcal{LCP}^b(\operatorname{gr}\Gamma)$ is a duality.*

3. The linearization functor and Koszul duality

Recall that Λ is said to be a Koszul algebra if the minimal projective resolution of Λ_0 is a linear complex. Likewise, a Λ-module is said to be Koszul if its minimal projective resolution is a linear complex. The goal of this section is to give further properties of the functor Φ when Λ is a Koszul algebra. More precisely, we have

Theorem 5. *Suppose that Λ is Koszul and M is a finitely generated Λ-module. Then M is Koszul if and only if $\Phi(M)$ is exact except at the minimal degree; in that case $\Phi(M)$ is a minimal projective resolution of (the appropriate horizontal shift of) the Koszul-dual Γ-module $F(M) := \coprod_{j=0}^{\infty} \operatorname{Ext}_{\Lambda}^{j}(M, \Lambda_0)$.*

The proof of this theorem will be given later in this section, after we have established several preparatory results. Let Λ be any ring and M a finitely generated projective Λ-module. Choose an epimorphism $q : \coprod \Lambda \to M$ with domain a finite sum of copies of Λ and a splitting $s : M \to \coprod \Lambda$. These data give rise to a dual basis $\{m_j, f_j\}$ of M, defined as follows: $m_j := q(e_j)$, where e_j is the jth standard basis vector in $\coprod \Lambda$, and $f_j := p_j s$, where p_j is the projection of $\coprod \Lambda$ onto its jth component. Now let N be an arbitrary Λ-module.

Lemma 6. *Under the above assumptions, the map*

$$\alpha : \operatorname{Hom}_{\Lambda}(M, N) \longrightarrow \operatorname{Hom}_{\Lambda}(M, \Lambda) \otimes_{\Lambda} N$$

given by $\alpha(g) := \sum f_j \otimes g(m_j)$, is an isomorphism natural in M and N. Moreover α is canonical, i.e., it does not depend on the choice of the dual basis.

Proof. The naturality of α is clear. To construct the inverse of α, define a map

$$\beta : \operatorname{Hom}_{\Lambda}(M, \Lambda) \otimes_{\Lambda} N \longrightarrow \operatorname{Hom}_{\Lambda}(M, N)$$

by $\beta(f \otimes n)(m) := f(m)n$. Given any $m \in M$, we have $m = \sum f_j(m) m_j$. Applying an arbitrary $f \in \operatorname{Hom}_{\Lambda}(M, \Lambda)$ we also have $f = \sum f_j f(m_j)$. Using the former equality, we have that $\beta\alpha = 1$ and using the latter, we have $\alpha\beta = 1$. Finally, since α is the inverse of the canonical map β, it is also canonical. $\square$

Lemma 7. *Let Λ be any ring, J an ideal, $\Lambda_0 := \Lambda/J$, and M a finitely generated projective Λ_0-module. Then, for any integer $k \geq 0$, the Yoneda product induces a natural in M isomorphism*

$$\upsilon_M : \operatorname{Hom}_{\Lambda}(M, \Lambda_0) \otimes_{\Lambda_0} \operatorname{Ext}_{\Lambda}^{k}(\Lambda_0, \Lambda_0) \longrightarrow \operatorname{Ext}_{\Lambda}^{k}(M, \Lambda_0)$$

of right Λ_0-modules.

Proof. The naturality of υ_M is clear. To show that υ_M is an isomorphism assume first that $M = \Lambda_0$. In that case we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_\Lambda(\Lambda_0, \Lambda_0) \otimes_{\Lambda_0} \operatorname{Ext}^k_\Lambda(\Lambda_0, \Lambda_0) & \xrightarrow{\upsilon_{\Lambda_0}} & \operatorname{Ext}^k_\Lambda(\Lambda_0, \Lambda_0) \\ \downarrow e\otimes 1 & & \| \\ \Lambda_0^{op} \otimes_{\Lambda_0} \operatorname{Ext}^k_\Lambda(\Lambda_0, \Lambda_0) & \longrightarrow & \operatorname{Ext}^k_\Lambda(\Lambda_0, \Lambda_0) \end{array}$$

of right Λ_0-modules. In this diagram $e : \operatorname{Hom}_\Lambda(\Lambda_0, \Lambda_0) \to \Lambda_0^{op}$ is the canonical ring isomorphism evaluating maps at the identity and the bottom arrow is the canonical isomorphism of left Λ_0^{op}-modules (i.e., of right Λ_0-modules). Thus υ_{Λ_0} is also an isomorphism. Since υ_M is natural in M and since both the source and the target are additive in M, it now follows that the lemma is true when M is a finitely generated free Λ_0-module. In general, let

$$G \longrightarrow F \longrightarrow M \longrightarrow 0$$

be a split presentation of M by finitely generated free Λ_0-modules F and G. Applying the additive functors $(-, \Lambda_0) := \operatorname{Hom}_\Lambda(-, \Lambda_0)$ and $- \otimes_{\Lambda_0} Y^k$, where $Y^k := \operatorname{Ext}^k_\Lambda(\Lambda_0, \Lambda_0)$, and using the naturality of υ, we obtain a commutative diagram whose top row is split-exact:

$$\begin{array}{ccccccc} 0 \longrightarrow & (M, \Lambda_0) \otimes_{\Lambda_0} Y^k & \longrightarrow & (F, \Lambda_0) \otimes_{\Lambda_0} Y^k & \longrightarrow & (G, \Lambda_0) \otimes_{\Lambda_0} Y^k \\ & \downarrow \upsilon_M & & \downarrow \upsilon_F & & \downarrow \upsilon_G \\ 0 \longrightarrow & \operatorname{Ext}^k_\Lambda(M, \Lambda_0) & \longrightarrow & \operatorname{Ext}^k_\Lambda(F, \Lambda_0) & \longrightarrow & \operatorname{Ext}^k_\Lambda(G, \Lambda_0) \end{array}$$

Since $\operatorname{Ext}^k_\Lambda(-, \Lambda_0)$ is an additive functor, the bottom row is also split-exact. It follows that υ_M is an isomorphism. □

Taking the direct sum over all k of the Ext-groups in υ_M we have an isomorphism, denoted by the same letter,

$$\upsilon_M : \operatorname{Hom}_\Lambda(M, \Lambda_0) \otimes_{\Lambda_0} \coprod \operatorname{Ext}^k_\Lambda(\Lambda_0, \Lambda_0) \longrightarrow \coprod \operatorname{Ext}^k_\Lambda(M, \Lambda_0)$$

The Yoneda product gives rise to a left action of Γ on both direct sums. Since the chain maps representing elements of Γ are homomorphisms of left Λ-modules, the left actions of Γ and Λ on $\Gamma = \coprod \operatorname{Ext}^k_\Lambda(\Lambda_0, \Lambda_0)$ commute and therefore not only the codomain but also the domain of υ_M are left Γ-modules. Since the Yoneda product is associative, υ_M is a homomorphism of left Γ-modules. We thus have

Lemma 8. *The map*

$$\upsilon_M : \operatorname{Hom}_\Lambda(M, \Lambda_0) \otimes_{\Lambda_0} \Gamma \longrightarrow \coprod \operatorname{Ext}^k_\Lambda(M, \Lambda_0)$$

is an isomorphism of left Γ*-modules.*

Now assume that Λ is a non-negatively graded algebra over $\Bbbk$ generated in degree one with graded radical J and that Λ_0 is a finite product of copies of $\Bbbk$. Let M be a locally finite graded Λ-module (i.e., $\dim_\Bbbk M_k$ is finite for each k). For a given k, choose a dual basis $\{m_j, f_j\}$ of M_k and consider the composition ε of canonical maps

$$\begin{array}{ccccc} D(M_{k+1}) & \xrightarrow{D(\mu)} & D(\Lambda_1 \otimes_{\Lambda_0} M_k) & \xrightarrow{\sigma} & \operatorname{Hom}_{\Lambda_0}(M_k, D(\Lambda_1)) \\ \Big\downarrow{\scriptstyle \varepsilon} & & & & \Big\downarrow{\scriptstyle \alpha} \\ \operatorname{Ext}^1_{\Lambda}(M_k, \Lambda_0) & \xleftarrow{\upsilon_{M_k}} & D(M_k) \otimes_{\Lambda_0} \Gamma_1 & \xleftarrow{1\otimes\gamma} & D(M_k) \otimes D(\Lambda_1) \end{array}$$

where $D(-) = \operatorname{Hom}_{\Lambda_0}(-, \Lambda_0) \cong \operatorname{Hom}_{\Lambda}(-, \Lambda_0)$, the isomorphism σ arises from the adjointness of the Hom and the tensor product functors, and the isomorphism γ was established on p. 4. The next result provides an explicit description of ε.

Lemma 9. *Under the above assumptions, $\varepsilon(g) = \sum_j g\mu_{m_j}\pi(\Omega f_j)$ for any $g \in D(M_{k+1})$. Here $\mu_{m_j} := \mu(- \otimes_{\Lambda_0} m_j) : \Lambda_1 \to M_{k+1}$, Ω stands for the syzygy operation over Λ, and $\pi : J \to \Lambda_1$ is the canonical projection.*

Proof. The proof is a straightforward verification based on the explicit descriptions of the involved maps given in Lemmas 6 and 7. □

Remark 10. *Notice that $\varepsilon = \upsilon_{M_k} \circ d_{k0}$, where d_{k0} was constructed in Sec. 2. Thus we have a commutative diagram*

$$\begin{array}{ccc} D(M_{k+1}) \otimes_{\Lambda_0} \operatorname{Ext}^l_{\Lambda}(\Lambda_0, \Lambda_0) & \xrightarrow{d_{k0}\otimes 1} & D(M_k) \otimes_{\Lambda_0} \operatorname{Ext}^1_{\Lambda}(\Lambda_0, \Lambda_0) \otimes_{\Lambda_0} \operatorname{Ext}^l_{\Lambda}(\Lambda_0, \Lambda_0) \\ & \searrow^{\varepsilon\otimes 1} & \cong \Big\downarrow{\scriptstyle \upsilon_{M_k}\otimes 1} \\ & & \operatorname{Ext}^1(M_k, \Lambda_0) \otimes_{\Lambda_0} \operatorname{Ext}^l(\Lambda_0, \Lambda_0) \end{array}$$

Our next goal is to interpret the just constructed map ε in terms of extensions. To this end, it is convenient to first build certain canonical extensions associated with graded modules. Given a graded Λ-module M and an integer k, we have a short exact sequence

$$\theta_k \qquad 0 \longrightarrow M_{\geq k+1} \xrightarrow{\iota_k} M_{\geq k} \xrightarrow{\rho_k} M_k \longrightarrow 0$$

where ι_k is the canonical inclusion and ρ_k is the canonical projection. The Yoneda product $\zeta_k := \rho_{k+1}\theta_k$ is represented by the bottom row in the push-out diagram

$$\begin{array}{ccccccccc} \theta_k & 0 & \longrightarrow & M_{\geq k+1} & \xrightarrow{\iota_k} & M_{\geq k} & \xrightarrow{\rho_k} & M_k & \longrightarrow 0 \\ & & & \Big\downarrow{\scriptstyle \rho_{k+1}} & & \Big\downarrow & & \Big\| & \\ \zeta_k & 0 & \longrightarrow & M_{k+1} & \longrightarrow & E_k & \longrightarrow & M_k & \longrightarrow 0 \end{array}$$

We can now give an alternative description of the map ε constructed in Lemma 9.

Proposition 11. *In the above notation, $\varepsilon(g) = g\zeta_k = g\rho_{k+1}\theta_k$ for any integer k and any $g \in D(M_{k+1})$. In other words, $\varepsilon(g)$ is represented by the bottom row in the push-out diagram*

$$\begin{array}{ccccccccc} \zeta_k & 0 & \longrightarrow & M_{k+1} & \longrightarrow & E_k & \longrightarrow & M_k & \longrightarrow 0 \\ & & & \Big\downarrow{\scriptstyle g} & & \Big\downarrow & & \Big\| & \\ g\zeta_k & 0 & \longrightarrow & \Lambda_0 & \longrightarrow & E & \longrightarrow & M_k & \longrightarrow 0 \end{array}$$

Proof. Earlier in this section we constructed a dual basis $\{m_j, f_j\}$ of the projective Λ_0-module M_k using a splitting diagram

$$\Lambda_0 \xleftarrow{\;p_j\;} \coprod \Lambda_0 \underset{s}{\overset{q}{\rightleftarrows}} M_k$$

and setting $f_j = p_j s$. The commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J \otimes_{\Lambda_0} M_k & \longrightarrow & \Lambda \otimes_{\Lambda_0} M_k & \longrightarrow & M_k & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle 1\otimes s} & & \big\downarrow{\scriptstyle 1\otimes s} & & \big\downarrow{\scriptstyle s} & & \\
0 & \longrightarrow & J \otimes_{\Lambda_0} \coprod \Lambda_0 & \longrightarrow & \Lambda \otimes_{\Lambda_0} \coprod \Lambda_0 & \longrightarrow & \coprod \Lambda_0 & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle 1\otimes p_j} & & \big\downarrow{\scriptstyle 1\otimes p_j} & & \big\downarrow{\scriptstyle p_j} & & \\
0 & \longrightarrow & J \otimes_{\Lambda_0} \Lambda_0 & \longrightarrow & \Lambda \otimes_{\Lambda_0} \Lambda_0 & \longrightarrow & \Lambda_0 & \longrightarrow & 0 \\
 & & {\scriptstyle \cong}\big\downarrow{\scriptstyle \nu} & & {\scriptstyle \cong}\big\downarrow{\scriptstyle \nu} & & \big\| & & \\
0 & \longrightarrow & J & \longrightarrow & \Lambda & \longrightarrow & \Lambda_0 & \longrightarrow & 0
\end{array}$$

where ν denotes the multiplication map, shows that $\Omega f_j = \nu(1_J \otimes f_j)$. Let $\tau : \Lambda_1 \to \Lambda_1 \otimes_{\Lambda_0} \Lambda_0$ be the isomorphism $\lambda \mapsto \lambda \otimes 1$ and $i_j : \Lambda_0 \to \coprod \Lambda_0$ the canonical inclusion of Λ_0 as the jth component. Then the composition

$$\Lambda_1 \underset{\cong}{\xrightarrow{\;\tau\;}} \Lambda_1 \otimes_{\Lambda_0} \Lambda_0 \xrightarrow{1\otimes i_j} \Lambda_1 \otimes_{\Lambda_0} \coprod \Lambda_0 \xrightarrow{1\otimes q} \Lambda_1 \otimes_{\Lambda_0} M_k \xrightarrow{\;\mu\;} M_{k+1}$$

is just the map μ_{m_j}. It follows that $\mu_{m_j}\pi(\Omega f_j) = \mu(1 \otimes q i_j)\tau\pi\nu(1_J \otimes f_j)$. Picking arbitrary elements $\lambda \in J$ and $m \in M_k$ and applying this map we have

$$\mu_{m_j}\pi(\Omega f_j)(\lambda \otimes m) = \overline{\lambda} f_j(m) m_j$$

where $\overline{\lambda}$ is the image of λ in Λ_1 under π. Therefore $\sum_j \mu_{m_j}\pi(\Omega f_j)(\lambda \otimes m) = \sum_j \overline{\lambda} f_j(m) m_j = \overline{\lambda} m = \mu(\pi \otimes 1)(\lambda \otimes m)$. We have thus proved that

$$\textstyle\sum_j \mu_{m_j}\pi(\Omega f_j) = \mu(\pi \otimes 1)$$

Picking an arbitrary $g \in D(M_{k+1})$ and taking account of the commutative diagram

$$\begin{array}{ccc}
J \otimes_{\Lambda_0} M_k & \xrightarrow{\;\mu'\;} & M_{\geq k+1} \\
\big\downarrow{\scriptstyle \pi\otimes 1} & & \big\downarrow{\scriptstyle \rho_{k+1}} \\
\Lambda_1 \otimes_{\Lambda_0} M_k & \xrightarrow{\;\mu\;} & M_{k+1}
\end{array}$$

we have $\varepsilon(g) = \sum_j g\mu_{m_j}\pi(\Omega f_j) = g\mu(\pi \otimes 1) = g\rho_{k+1}\mu'$. Since M_k is a projective Λ_0-module, the sequence

$$0 \longrightarrow J \otimes_{\Lambda_0} M_k \longrightarrow \Lambda \otimes_{\Lambda_0} M_k \longrightarrow M_k \longrightarrow 0$$

is exact with a projective Λ-module in the middle. Therefore $J \otimes_{\Lambda_0} M_k$ is a first syzygy module of M_k. The commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J \otimes_{\Lambda_0} M_k & \longrightarrow & \Lambda \otimes_{\Lambda_0} M_k & \longrightarrow & M_k & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mu'} & & \downarrow{\scriptstyle \mu'} & & \| & & \\
0 & \longrightarrow & M_{\geq k+1} & \xrightarrow{\iota_k} & M_{\geq k} & \xrightarrow{\rho_k} & M_k & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \rho_{k+1}} & & \downarrow & & \| & & \\
0 & \longrightarrow & M_{k+1} & \longrightarrow & E_k & \longrightarrow & M_k & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g} & & \downarrow & & \| & & \\
0 & \longrightarrow & \Lambda_0 & \longrightarrow & E & \longrightarrow & M_k & \longrightarrow & 0
\end{array}$$

now shows that $\varepsilon(g)$ is represented by the bottom row. □

Dualizing the extensions θ_k and θ_{k+1} into Λ_0 we have the composition

$$e_k : \operatorname{Ext}^l_\Lambda(M_{k+1}, \Lambda_0) \xrightarrow{\operatorname{Ext}^l(\rho_{k+1}, \Lambda_0)} \operatorname{Ext}^l_\Lambda(M_{\geq k+1}, \Lambda_0) \xrightarrow{\delta} \operatorname{Ext}^{l+1}_\Lambda(M_k, \Lambda_0)$$

where the first map is the composition with ρ_{k+1} on the right and δ is the composition with θ_k on the right.

Corollary 12. *The diagram*

$$\begin{array}{ccc}
D(M_{k+1}) \otimes_{\Lambda_0} \operatorname{Ext}^l_\Lambda(\Lambda_0, \Lambda_0) & \xrightarrow{\varepsilon \otimes 1} & \operatorname{Ext}^1_\Lambda(M_k, \Lambda_0) \otimes_{\Lambda_0} \operatorname{Ext}^l_\Lambda(\Lambda_0, \Lambda_0) \\
\downarrow{\scriptstyle \upsilon_{M_{k+1}}} & & \downarrow{\scriptstyle \upsilon} \\
\operatorname{Ext}^l_\Lambda(M_{k+1}, \Lambda_0) & \xrightarrow{e_k} & \operatorname{Ext}^{l+1}_\Lambda(M_k, \Lambda_0)
\end{array}$$

where υ is given by the Yoneda product, is commutative.

Proof. For $g \in D(M_{k+1})$ and $y \in \operatorname{Ext}^l_\Lambda(\Lambda_0, \Lambda_0)$ we have $e_k \upsilon_{M_{k+1}}(g \otimes y) = e_k(yg) = \delta(yg\rho_{k+1}) = yg\rho_{k+1}\theta_k$. On the other hand, in view of Prop. 11, $\upsilon(\varepsilon \otimes 1)(g \otimes y) = \upsilon(\varepsilon(g) \otimes y) = y\varepsilon(g) = yg\rho_{k+1}\theta_k$. □

We are now ready to prove Theorem 5.

Proof of Theorem 5. We begin with the "only if" part. Assuming that M is Koszul, one can construct a minimal projective resolution

$$\cdots \longrightarrow F(J^2M/J^3M)[-2] \xrightarrow{d_1} F(JM/J^2M)[-1] \xrightarrow{d_0} F(M/JM) \twoheadrightarrow F(M)$$

of the Γ-module $F(M)$ as in [6], p. 238. We quickly recall the definition of the differential. Applying $\operatorname{Hom}_\Lambda(-, \Lambda_0)$ to the short exact sequence

$$\theta_0 \qquad 0 \longrightarrow JM \longrightarrow M \longrightarrow M/JM \longrightarrow 0$$

we have, for each l, a short exact sequence

$$0 \longrightarrow \operatorname{Ext}^l_\Lambda(JM, \Lambda_0) \xrightarrow{\delta} \operatorname{Ext}^{l+1}_\Lambda(M/JM, \Lambda_0) \longrightarrow \operatorname{Ext}^{l+1}_\Lambda(M, \Lambda_0) \longrightarrow 0$$

Taking the sum over k, we have a short exact sequence

$$0 \longrightarrow F(JM)[-1] \xrightarrow{\delta} F(M/JM) \longrightarrow F(M) \longrightarrow 0$$

and therefore an isomorphism $F(JM)[-1] \simeq \Omega F(M)$. Let $\rho_1 : JM \to JM/J^2M$ be the canonical projection. Then the map d_0 is defined as the composition

$$F(JM/J^2M)[-1] \xrightarrow{F(\rho_1)[-1]} F(JM)[-1] \xrightarrow{\delta} F(M/JM)$$

In other words, d_0 is the direct sum of the compositions

$$\operatorname{Ext}^l_\Lambda(JM/J^2M, \Lambda_0) \xrightarrow{\operatorname{Ext}^l(\rho_1,\Lambda_0)} \operatorname{Ext}^l_\Lambda(JM, \Lambda_0) \xrightarrow{\delta} \operatorname{Ext}^{l+1}_\Lambda(M/JM, \Lambda_0)$$

In terms of the Yoneda product, $d_0(a) = a\rho_1\theta_0$ for any $a \in \operatorname{Ext}^l_\Lambda(JM/J^2M, \Lambda_0)$.

The other maps d_k are defined similarly and we have $d_k(a) = a\rho_{k+1}\theta_k$, where $a \in \operatorname{Ext}^l_\Lambda(J^{k+1}M/J^{k+2}M, \Lambda_0)$ is arbitrary, $\rho_{k+1} : J^{k+1}M \to J^{k+1}M/J^{k+2}M$ is the canonical projection, and θ_k is the extension

$$\theta_k \qquad 0 \longrightarrow J^{k+1}M \longrightarrow J^kM \longrightarrow J^kM/J^{k+1}M \longrightarrow 0$$

Since M is Koszul, it is generated in the top degree and we may assume that that degree is zero. Then $J^kM = M_{\geq k}$ and $J^kM/J^{k+1}M \cong M_k$. The diagram

$$\begin{array}{ccccc}
D(M_{k+1}) \otimes \Gamma_l & \xrightarrow{d_{k0}\otimes 1} & D(M_k) \otimes \Gamma_1 \otimes \Gamma_l & \xrightarrow{1\otimes \upsilon} & D(M_k) \otimes \Gamma_{l+1} \\
\| & & \downarrow{\scriptstyle \upsilon_{M_k}\otimes 1} & & \downarrow{\scriptstyle \upsilon_{M_k}} \\
D(M_{k+1}) \otimes \Gamma_l & \xrightarrow{\varepsilon\otimes 1} & \operatorname{Ext}^1_\Lambda(M_k, \Lambda_0) \otimes \Gamma_l & \xrightarrow{\upsilon} & \operatorname{Ext}^{l+1}_\Lambda(M_k, \Lambda_0) \\
\downarrow{\scriptstyle \upsilon_{M_{k+1}}} & & & & \| \\
\operatorname{Ext}^l_\Lambda(M_{k+1}, \Lambda_0) & & \xrightarrow{\qquad d_k \qquad} & & \operatorname{Ext}^{l+1}_\Lambda(M_k, \Lambda_0)
\end{array}$$

is constructed as follows. All tensor products are taken over Λ_0, all maps labeled by various υ are Yoneda products, the top row is a direct summand of the differential in $\Phi(M)$, and the bottom row is a direct summand of the differential in the projective resolution of $F(M)$ constructed above. By Lemma 7, the vertical maps are isomorphisms. By Remark 10, the top left square is commutative. By the associativity of the Yoneda product, the top right square is commutative. By Cor. 12, the lower part of the diagram commutes (since $d_k = e_k$). Therefore the two peripheral paths commute. We now conclude that the complex $\Phi(M)$ is isomorphic to the minimal projective resolution of $F(M)$ constructed above, thus proving the "only if" part of the theorem.

For the "if" part we assume that $\Phi(M)$ is exact except at the rightmost degree, which we may assume to be zero. We claim that the map $d_{k0} : D(M_{k+1}) \to D(M_k) \otimes_{\Lambda_0} \Gamma_1$ constructed in Sec. 2 is a monomorphism for each $k \geq 0$. Indeed, this is the restriction to the top vertical degree of the differential in a positive horizontal degree. If it were not a monomorphism, there would be a cycle in the top vertical degree and a positive horizontal degree. By the exactness of the complex in the positive horizontal degrees, that cycle would also be a boundary. But this is impossible in a linear complex. Thus d_{k0} is a monomorphism. Since this map is a composition of maps beginning with $D(\mu)$, we conclude that $D(\mu)$ is also a monomorphism. Therefore the multiplication map $\mu : \Lambda_1 \otimes_{\Lambda_0} M_k \to M_{k+1}$ is an epimorphism for each $k \geq 0$. If not, the cokernel of μ would be a nonzero finitely generated projective Λ_0-module (since Λ_0 is semisimple). As such, it would be reflexive and, since the Hom functor is left-exact, the kernel of $D(\mu)$ would be nonzero, a contradiction. It now follows that M is generated in the top degree. Hence, $M_{\geq k} = J^k M$ and $M_k \cong J^k M/J^{k+1}M$ for each k.

Let X be the degree zero homology of $\Phi(M)$. The triple-decker diagram constructed in the proof of the "only if" part above only required that M be generated in the top degree. Therefore the Γ-projective resolution $\Phi(M) \to X$ is isomorphic to the projective resolution

$$\cdots \longrightarrow E_{k+1}[-k-1] \xrightarrow{d_k} E_k[-k] \longrightarrow \cdots \longrightarrow E_0 \longrightarrow X \longrightarrow 0$$

where $E_{k+1} = \coprod_l \operatorname{Ext}^l_\Lambda(J^{k+1}M/J^{k+2}M, \Lambda_0)$ and the differential is given by the Yoneda product: $d_k(a) = a\rho_{k+1}\theta_k$ for a homogeneous element a.

Let $G : K_\Gamma \to K_\Lambda$ denote the Koszul duality between the categories of Koszul Γ-modules and Koszul Λ-modules. Our goal is to show that the Koszul Λ-module $G(X)$ is isomorphic to M. Let's compute $G(X)$ (disregarding the horizontal shifts):

$$\begin{aligned}
G(X) &= \coprod_k \operatorname{Ext}^k_\Gamma(X, \Gamma_0) \\
&\simeq \coprod_k \operatorname{Hom}_\Gamma(\Omega^k(X), \Gamma_0) && \text{(since } \Gamma_0 \text{ is semisimple)} \\
&\cong \coprod_k \operatorname{Hom}_\Gamma(\Omega^k(X)/\Omega^k(X)J, \Gamma_0) && \text{(since } \Omega^k X \text{ is generated in degree } k) \\
&\simeq \coprod_k \operatorname{Hom}_\Gamma(\operatorname{Hom}_\Lambda(J^kM/J^{k+1}M, \Lambda_0), \Gamma_0)
\end{aligned}$$

To establish the last isomorphism, notice that $\Omega^k(X)/\Omega^k(X)J$, being the top of $\Omega^k(X)$, is isomorphic to the top $\operatorname{Hom}_\Lambda(J^kM/J^{k+1}M, \Lambda_0)$ of E_k (since $\Phi(M)$ is a minimal resolution of X). The above maps are isomorphisms of left Λ_0-modules and also of left Γ-modules (since they all arise from maps on the contravariant arguments and the action of Γ is given by composition through the covariant argument.) As $\Gamma_0 \cong \Lambda_0^{op}$ and $J^kM/J^{k+1}M$ is a finitely generated projective Λ_0-module, it follows that the canonical evaluation map

$$\tau : J^kM/J^{k+1}M \longrightarrow \operatorname{Hom}_\Gamma(\operatorname{Hom}_\Lambda(J^kM/J^{k+1}M, \Lambda_0), \Lambda_0^{op})$$

with $\tau(x)(f) := f(x)$ for all $x \in J^kM/J^{k+1}M$ and $f \in \operatorname{Hom}_\Lambda(J^kM/J^{k+1}M, \Lambda_0)$, is an isomorphism of left Λ_0-modules.

An element $y \in \operatorname{Ext}^k_\Gamma(X, \Gamma_0)$ corresponds to a map

$$\Omega^k(X) \xrightarrow{\pi} \Omega^k(X)/\Omega^k(X)J \xrightarrow{\kappa} \Gamma_0$$

In view of the isomorphisms mentioned above, κ can be written as

$$\kappa : \operatorname{Hom}_\Lambda(J^kM/J^{k+1}M, \Lambda_0) \longrightarrow \Lambda_0^{op}$$

and hence is of the form $\kappa = \tau(x)$ for some $x \in J^kM/J^{k+1}M$. Thus the isomorphism $\operatorname{Ext}^k_\Gamma(X, \Gamma_0) \to J^kM/J^{k+1}M$ is given by $y \mapsto x$, where x is the element such that $\kappa = \tau(x)$.

We want to establish an isomorphism $G(X) \simeq \coprod J^kM/J^{k+1}M$ of graded Λ-modules under the identification $\Lambda = E(\Gamma) := \coprod \operatorname{Ext}^k_\Gamma(\Gamma_0, \Gamma_0)$. In other words we want to show that, under the correspondence $y \mapsto x$ and the identification $\Lambda = E(\Gamma)$, the Yoneda product corresponds to the multiplication by elements of Λ.

Suppose $\gamma \in \operatorname{Ext}^1_\Gamma(\Gamma_0, \Gamma_0)$ is represented by a short exact sequence

$$0 \longrightarrow \Gamma_0 \longrightarrow E \longrightarrow \Gamma_0 \longrightarrow 0$$

which therefore fits in the push-out diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & \Gamma & \longrightarrow & \Gamma_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \kappa'\pi} & & \downarrow & & \| & & \\ 0 & \longrightarrow & \Gamma_0 & \longrightarrow & E & \longrightarrow & \Gamma_0 & \longrightarrow & 0 \end{array}$$

Since Γ_0 is semisimple, the leftmost vertical map is the composition of the canonical projection $\pi : J \to J_1$ and a map $\kappa' : J_1 \to \Gamma_0$. But $J_1 = \operatorname{Ext}^1_\Lambda(\Lambda_0, \Lambda_0) \simeq \operatorname{Hom}_{\Lambda_0}(\Lambda_1, \Lambda_0)$ and the map $\kappa' : \operatorname{Hom}_{\Lambda_0}(\Lambda_1, \Lambda_0) \to \Lambda_0^{op}$ is of the form $\kappa' = \tau(\lambda_1)$ for some $\lambda_1 \in \Lambda_1$. We shall show that $\lambda_1 x$ corresponds to the Yoneda product γy.

As before, we have commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \frac{\Omega^k(X)}{\Omega^k(X)J} \otimes J & \longrightarrow & \frac{\Omega^k(X)}{\Omega^k(X)J} \otimes \Gamma & \longrightarrow & \frac{\Omega^k(X)}{\Omega^k(X)J} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tau(x)\otimes 1} & & \downarrow{\scriptstyle \tau(x)\otimes 1} & & \downarrow{\scriptstyle \tau(x)} & & \\ 0 & \longrightarrow & \Gamma_0 \otimes J & \longrightarrow & \Gamma_0 \otimes \Gamma & \longrightarrow & \Gamma_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \nu} & & \| & & \\ 0 & \longrightarrow & J & \longrightarrow & \Gamma & \longrightarrow & \Gamma_0 & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^{k+1}(X) & \longrightarrow & P & \longrightarrow & \Omega^k(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle t} & & \simeq\downarrow{\scriptstyle t'} & & \downarrow{\scriptstyle \pi} & & \\ 0 & \longrightarrow & \frac{\Omega^k(X)}{\Omega^k(X)J} \otimes J & \longrightarrow & \frac{\Omega^k(X)}{\Omega^k(X)J} \otimes \Gamma & \longrightarrow & \frac{\Omega^k(X)}{\Omega^k(X)J} & \longrightarrow & 0 \end{array}$$

Here for t' we can take the established earlier isomorphism

$$\textstyle\coprod_l \operatorname{Hom}_{\Lambda_0}(J^k M/J^{k+1}M, \Lambda_0) \otimes \operatorname{Ext}^l_\Lambda(\Lambda_0, \Lambda_0) \longrightarrow \coprod_l \operatorname{Ext}^l_\Lambda(J^{k+1}M/J^{k+2}M, \Lambda_0)$$

Splicing the two diagrams, we have $\Omega(\tau(x)\pi) = \nu(\tau(x) \otimes 1)t$. The extension γy corresponds to the homomorphism $\tau(\lambda_1)\pi\nu(\tau(x) \otimes 1)t : \Omega^{k+1}(X) \to \Gamma_0$. In horizontal degree $k+1$ we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_\Lambda(\frac{J^{k+1}M}{J^{k+2}M}, \Lambda_0) & \longrightarrow & \operatorname{Ext}^1_\Lambda(\frac{J^k M}{J^{k+1}M}, \Lambda_0) \\ {\scriptstyle D(\mu)}\searrow & & \nearrow{\scriptstyle t'}\ \simeq \\ & \operatorname{Hom}_\Lambda(\frac{J^k M}{J^{k+1}M}, \Lambda_0) \otimes D(\Lambda_1) & \end{array}$$

where $\mu : \Lambda_1 \otimes \frac{J^kM}{J^{k+1}M} \to \frac{J^{k+1}M}{J^{k+2}M}$ is multiplication. Then we have a commutative diagram

$$\begin{array}{ccccccccc}
\Omega^{k+1}(X) & \xrightarrow{t} & \frac{\Omega^k(X)}{\Omega^k(X)J} \otimes J & \xrightarrow{\tau(x)\otimes 1} & \Gamma_0 \otimes J & \xrightarrow{\nu} & J & \xrightarrow{\pi} & J_1 \\
\downarrow & & \downarrow{\scriptstyle 1\otimes\pi} & & & & & & \downarrow{\scriptstyle \tau(\lambda_1)} \\
D(\frac{J^{k+1}M}{J^{k+2}M}) & \xrightarrow{D(\mu)} & D(\frac{J^kM}{J^{k+1}M}) \otimes J_1 & & & \xrightarrow{\psi} & & & \Gamma_0
\end{array}$$

where the leftmost vertical map is the composition of the reduction modulo J followed by an isomorphism. Since $J_1 \simeq D(\Lambda_1)$, the map ψ is the tensor product of maps $\psi_1 : D(J^{k+1}M/J^{k+2}M) \to \Lambda_0^{op}$ and $\psi_2 : D(\Lambda_1) \to \Lambda_0^{op}$. Therefore $\psi_1 = \tau(x')$, $\psi_2 = \tau(\lambda')$, and $\psi = \tau(x') \otimes \tau(\lambda')$. We will show that $x' = x$ and $\lambda' = \lambda_1$.

Let $w \in J$ and $f \in \operatorname{Hom}_{\Lambda_0}(J^{k+1}M/J^{k+2}M, \Lambda_0)$. Then $\psi(1 \otimes \pi)(f \otimes w) = \tau(x')(f) \otimes \tau(\lambda')(\overline{w}) = f(x')\overline{w}(\lambda')$, with $\overline{w} = \pi(w)$. Also, $\tau(\lambda_1)\pi\nu(\tau(x) \otimes 1)(f \otimes w) = \tau(\lambda_1)\pi(f(x)w) = \tau(\lambda_1)(f(x)\overline{w}) = f(x)\overline{w}(\lambda_1)$. By the commutativity of the diagram, we have that $f(x)\overline{w}(\lambda_1) = f(x')\overline{w}(\lambda')$ for all f and all y. It follows that $x' = x$ and $\lambda' = \lambda_1$ and therefore $\psi = \tau(x) \otimes \tau(\lambda_1)$.

Now let $z \in \operatorname{Hom}_{\Lambda_0}(J^{k+1}M/J^{k+2}M, \Lambda_0)$. Then $D(\mu)(z) = z\mu$ and $\psi(z\mu) = z(\lambda_1 x) = \tau(\lambda_1 x)(z)$. We have thus proved that $\psi D(\mu) = \tau(\lambda_1 x)$. It follows that the element corresponding to γy is $\lambda_1 x$, as claimed.

Therefore the Λ-module $G(X)$ is isomorphic to the associated graded module $\coprod J^kM/J^{k+1}M$ of M. But M, being graded, is isomorphic to its associated graded module. This shows that M is Koszul. □

4. Approximations by linear complexes

We now assume that Λ is a finite-dimensional Koszul algebra. Then the Yoneda algebra Γ of Λ is of finite global dimension and we shall also assume that Γ is noetherian. Then the category $K^b(\operatorname{gr} P_\Gamma)$ of bounded complexes of finitely generated projective graded Γ-modules modulo homotopy is equivalent to the derived category $D^b(\operatorname{gr}\Gamma)$ of finitely generated graded Γ-modules.

The next lemma is a simple observation about complexes over a category of graded modules and degree zero maps between them. To state it, it is convenient to have the following definition.

Definition 13. *A complex of graded modules over a graded algebra is said to be* sub-diagonal*, if for each i, the ith module is generated in degrees at least i, provided it is nonzero.*

Thus linear complexes are sub-diagonal.

Lemma 14. *Let M and N be complexes of graded modules over a graded algebra and $f : M \to N$ a null-homotopic chain map. If M is linear and N is sub-diagonal, then $f = 0$.*

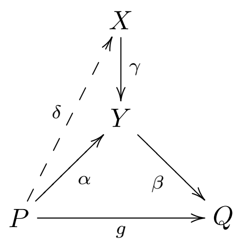
Proof. Let $\{s_j\}$ be a degree zero null-homotopy for f:

$$\begin{array}{ccccc}
 & & M_j & \xrightarrow{d_j} & M_{j-1} \\
 & {\scriptstyle s_j}\swarrow & \downarrow{\scriptstyle f_j} & \swarrow{\scriptstyle s_{j-1}} & \\
N_{j+1} & \xrightarrow{d'_{j+1}} & N_j & &
\end{array}$$

Since N_j is generated in degrees at least j, the map s_{j-1}, being of degree zero, must be zero for each j. Thus $f_j = s_{j-1}d_j + d'_{j+1}s_j = 0$. □

Proposition 15. *Let P be a linear complex of projectives bounded on the right, Q a sub-diagonal complex, and $g : P \to Q$ a chain map. Let $\gamma : X \to Y$ be a quasi-isomorphism. If g factors through Y, then it factors through X.*

Proof. We are given a commutative diagram



where γ is a quasi-isomorphism. As the functor $(P, -)$ preserves quasi-isomorphisms ([2], Prop. 5.2.4), $(P, \gamma) : (P, X) \to (P, Y)$ is a quasi-isomorphism. In these complexes the zero-cycles are chain maps and the zero-boundaries are null-homotopic chain maps. Therefore we can identify the degree zero homology as homotopy classes of chain maps. Hence there is a chain map $\delta : P \to X$ such that α is homotopic to $\gamma\delta$. Therefore g is homotopic to $\beta\gamma\delta$. By Lemma 14, the two chain maps must be equal. □

Corollary 16. *Let $g : P \to Q$ be a chain map, where P is a linear complex of projectives bounded on the right and Q a sub-diagonal complex. Then g lifts along any quasi-isomorphism $\gamma : X \to Q$.*

Proof. Set $Y := Q$ and $\beta := 1_Q$. □

Corollary 17. *Let X be an arbitrary complex, P a linear complex of projectives bounded on the right, and $\gamma : X \to P$ a quasi-isomorphism. Then γ is a retraction in the category of complexes and chain maps.*

Proof. Set $Y := P$ and $\alpha := 1_P$. □

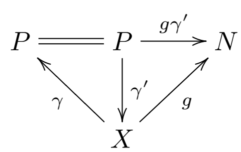
The following consequence of the just proved result will be used to clear denominators in derived categories.

Corollary 18. *Any morphism in a (suitable) derived category of modules whose domain is a bounded on the right linear complex of projectives can be represented by a chain map.*

Proof. Suppose that the morphism is represented by the diagram

$$P \xleftarrow{\gamma} X \xrightarrow{g} N$$

By Cor. 17, γ has a section γ'. Then the diagram



gives the desired representation. □

Lemma 19. *Let $f : P \to Y$ be a chain map, where P is a bounded on the right linear complex of projectives. If f represents a zero morphism in a (suitable) derived category, then f is null-homotopic. If, in addition, Y is sub-diagonal, then $f = 0$.*

Proof. As f represents zero, there is a commutative up to homotopy diagram of chain maps

$$\begin{array}{ccc} T & \xrightarrow{0} & Y \\ {\scriptstyle\gamma}\downarrow & \nearrow{\scriptstyle f} & \\ P & & \end{array}$$

where γ is a quasi-isomorphism. By Cor. 17, γ has a section γ'. Then $f = f\gamma\gamma'$ is null-homotopic. The last assertion of the lemma follows at once from Lemma 14. $\square$

Reflecting the fact that in the rest of the paper all algebras are Koszul, we now give a more narrow definition of linearity.

Definition 20. *A complex is said to be* `totally linear` *if it is linear and each of its terms has a linear projective resolution.*

Remark. It is immediate that a linear complex of projectives is totally linear.

Our next goal is to show that, over a Koszul algebra of finite global dimension, bounded totally linear complexes can be "approximated" by bounded linear complexes of projectives. To this end we need the following construction ([14], Lemme III 2.1.2).

Lemma 21. *Let $X_\bullet := \{X_i, d_i\}$ be a complex of objects in an abelian category, $\mu_i : Y_i \to X_i$ a morphism, and*

$$\begin{array}{ccc} X'_{i+1} & \xrightarrow{v_2} & Y_i \\ {\scriptstyle\mu_{i+1}}\downarrow & & \downarrow{\scriptstyle\mu_i} \\ X_{i+1} & \xrightarrow{d_{i+1}} & X_i \end{array}$$

a pull-back of μ_i. Then there exists a morphism $v_1 : X_{i+2} \to X'_{i+1}$ such that $v_2 v_1 = 0$ and $d_{i+2} = \mu_{i+1} v_1$. We now have a complex

$$Y_\bullet \qquad \ldots \to X_{i+2} \overset{v_1}{\to} X'_{i+1} \overset{v_2}{\to} Y_i \overset{d_i}{\to} X_{i-1} \to \ldots$$

and an obvious chain map $\mu : Y_\bullet \to X_\bullet$, where μ_j is the identity map for $j \neq i, i+1$. This map μ is a quasi-isomorphism if and only if $\operatorname{Im} \mu_i + \operatorname{Im} d_{i+1} = X_i$.

We now use Verdier's lemma to construct the promised approximations.

Proposition 22. *Let Γ be a noetherian graded ring of finite global dimension and $M_\bullet := \{M_j, d_j\}_{j=0}^n$ a bounded totally linear complex of finitely generated graded Γ-modules. Then there exist a bounded linear complex $P_\bullet$ of finitely generated graded Γ-projectives and a quasi-isomorphism $\mu : P_\bullet \to M_\bullet$, such that $\mu_i : P_i \to M_i$ is an epimorphism for each i.*

Proof. The desired approximation will be constructed inductively. We start with the exact sequence $0 \to B_0 \to M_0 \to H_0 \to 0$. By assumption, there is a projective P_0 generated in degree zero and an epimorphism $P_0 \to M_0$ whose kernel has a linear projective resolution. We have a commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Omega(M_0) & = & \Omega(M_0) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \Omega(H_0) & \to & P_0 & \to & H_0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & B_0 & \to & M_0 & \to & H_0 & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

In the pull-back diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Omega(M_0) & = & \Omega(M_0) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & Z_1 & \to & W_1 & \to & \Omega(H_0) & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & Z_1 & \to & M_1 & \to & B_0 & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

$\Omega(M_0)$ and M_1 have linear resolutions, hence the same is true for W_1. Notice that W_1 is finitely generated. By Verdier's Lemma, the complex

$$0 \to M_n \to \ldots \to M_2 \to W_1 \to P_0 \to 0$$

is quasi-isomorphic to the complex

$$0 \to M_n \to \ldots \to M_2 \to M_1 \to M_0 \to 0$$

and is totally linear. Moreover the quasi-isomorphism is an epimorphism in each degree. Now assume that we have constructed a totally linear complex

$$0 \to M_n \to \ldots \to M_{j+1} \to W_j \to P_{j-1} \to \ldots \to P_0 \to 0$$

(of finitely generated graded modules) together with a quasi-isomorphism μ from it to the original complex $\{M_i\}$ which is an epimorphism in each degree. By assumption, there is a finitely generated projective P_j generated in degree j and an epimorphism $P_j \to W_j$. We have a commutative diagram with exact rows and columns

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Omega(W_j) & = & \Omega(W_j) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \rightarrow & K & \rightarrow & P_j & \rightarrow & W_j/B_j & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \rightarrow & B_j & \rightarrow & W_j & \rightarrow & W_j/Bj & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

In the pullback diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Omega(W_j) & = & \Omega(W_j) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \rightarrow & Z_{j+1} & \rightarrow & W_{j+1} & \rightarrow & K & \rightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \rightarrow & Z_{j+1} & \rightarrow & M_{j+1} & \rightarrow & B_j & \rightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

M_{j+1} and $\Omega(W_j)$ have linear resolutions, hence the same is true for W_{j+1}. By Verdier's Lemma, we have a complex

$$P_\bullet^{(j)} \qquad 0 \to M_n \to \ldots \to M_{j+2} \to W_{j+1} \to P_j \to P_{j-1} \to \ldots \to P_0 \to 0$$

(of finitely generated graded modules) and a quasi-isomorphism $\mu : P_\bullet^{(j)} \to M_\bullet$ which is an epimorphism in each degree. The nth iteration of this procedure will result in a complex of the form

$$0 \to W_n \to P_{n-1} \to \ldots \to P_0 \to 0,$$

where the modules $P_0, \ldots, P_j$ are finitely generated and projective. Replacing W_n by its linear projective resolution we obtain the desired complex. $\square$

A similar result holds for sub-diagonal complexes. Before stating it we recall the well-known

Lemma 23. *Let Γ be a non-negatively graded ring and assume that Γ_0 is a finite product of copies of $\Bbbk$. Then Γ is graded semiperfect.*

Proof. Notice that Γ/J_Γ, being a product of fields, is semisimple and that idempotents lift in an obvious way along the radical. □

Thus, if Γ is noetherian, finitely generated graded Γ-modules have minimal projective resolutions.

Proposition 24. *Let Γ be a noetherian graded ring of finite global dimension with Γ_0 a finite product of fields and $M_\bullet := \{M_j, d_j\}_{j=0}^n$ a bounded sub-diagonal complex of finitely generated graded Γ-modules. Then there exist a bounded sub-diagonal complex $P_\bullet$ of finitely generated graded Γ-projectives and a quasi-isomorphism $\mu : P_\bullet \to M_\bullet$, such that $\mu_i : P_i \to M_i$ is an epimorphism for each i.*

Proof. By Lemma 23, finitely generated modules have projective covers. We now repeat the proof of Prop. 22 where we choose the maps $P_0 \to M_0$ and $P_j \to W_j$ to be projective covers and make the obvious modifications. □

Our next result will show that, under suitable conditions, a bounded complex of finite modules can be "approximated" by a subcomplex which is a vertical shift of a totally linear complex.

Proposition 25. *Let Γ be a noetherian Koszul algebra of finite global dimension and $M_\bullet$ a bounded complex of finitely generated graded Γ-modules. Then there exists a subcomplex $L_\bullet$ of $M_\bullet$ such that $L_\bullet$ is a vertical shift of a totally linear complex and $M_\bullet/L_\bullet$ is a bounded complex of modules of finite length.*

Proof. Let $M_\bullet := \{M_j \,|\, 0 \leq j \leq p\}$. Our assumptions guarantee that for any finitely generated Γ-module X there exists an integer n such that $X_{\geq n}$ is Koszul ([7], Prop. 2.4). Since there are only finitely many nontrivial modules M_j, we can choose an integer n such that $N_j := (M_j)_{\geq n}$ are all (shifts of) Koszul modules, generated in the same degree. Thus we have a subcomplex $N_\bullet$ of $M_\bullet$ such that $M_\bullet/N_\bullet$ is a complex of modules of finite length. We now define another subcomplex $L_\bullet$ of $M_\bullet$ by setting $L_j := (N_j)_{\geq n+j}$. Since each N_j is Koszul, each L_j is generated in degree $n + j$. This means that $L_\bullet$ is a shift of a totally linear complex, giving us the desired subcomplex. □

5. Weakly Koszul modules

In this section we recall some results from [11] about weakly Koszul modules which allow to approximate arbitrary finitely generated modules by Koszul modules. We keep the notation from the previous sections.

Definition 26. *A finitely generated graded Λ-module M is said to be weakly Koszul if there is a minimal projective resolution*

$$\ldots \to P_n \stackrel{d_n}{\to} P_{n-1} \to \ldots \to P_0 \stackrel{d_0}{\to} M \to 0$$

such that $J^{k+1}P_i \cap \operatorname{Ker} d_i = J^k \operatorname{Ker} d_i$ for all non-negative integers i and k.

It is not difficult to see that a Koszul module is weakly Koszul. Also, it is immediate from the definition that any syzygy module of a weakly Koszul module is again weakly Koszul. The first approximation result that we need shows that under certain assumptions a high enough syzygy module of any module is weakly Koszul.

Theorem 27 ([11], Th. 4.5). *Let Λ be a finite-dimensional Koszul algebra with a noetherian Yoneda algebra and M a finitely generated Λ-module. Then there is a non-negative integer k such that $\Omega^k M$ is weakly Koszul.*

The next result provides a connection between Koszul and weakly Koszul modules.

Theorem 28 ([11], Th. 2.4). *Let Λ be a Koszul algebra, $M = M_0 \coprod M_1 \coprod \dots$ a graded weakly Koszul Λ-module with $M_0 \neq 0$, and $\langle M_0 \rangle$ the submodule generated by M_0. Then:*

(i) *$\langle M_0 \rangle$ is Koszul.*
(ii) *$J^k M \cap \langle M_0 \rangle = J^k \langle M_0 \rangle$ for each non-negative integer k.*
(iii) *$M/\langle M_0 \rangle$ is weakly Koszul.*

Recurrently applying part (iii) of this theorem, one obtains a *canonical* finite increasing filtration of M by weakly Koszul modules whose successive quotients are Koszul. Let $w(M)$ be the length of this filtration. Thus $w(M) = 1$ if and only if M is Koszul. We also set $w(0) := 0$.

Corollary 29. *If M is a nonzero weakly Koszul Λ-module, then*

$$w(M/\langle M_0 \rangle) = w(M) - 1.$$

Definition 30. *A short exact sequence $0 \to A \to B \to C \to 0$ in $\operatorname{gr} \Lambda$ is called a relative extension if $J^k A = J^k B \cap A$ for all non-negative integers k.*

Corollary-Definition 31. *The short exact sequence*

$$0 \to \langle M_0 \rangle \xrightarrow{\iota} M \to M/\langle M_0 \rangle \to 0$$

is a relative extension. We shall call it the canonical relative extension associated with M.

Proposition 32 ([11], Prop. 3.2). *Let Λ be a Koszul algebra and*

$$0 \to A \to B \to C \to 0$$

a relative extension in $\operatorname{gr} \Lambda$ such that A is (weakly) Koszul. Then, for each non-negative integer n,

$$0 \to \Omega^n A \to \Omega^n B \to \Omega^n C \to 0$$

is a relative extension and $\Omega^n A$ is (weakly) Koszul.[6]

The next result provides a useful inequality relating the values of w on a weakly Koszul module and its syzygy module.

Proposition 33. *Let M be a weakly Koszul Λ-module. Then $w(\Omega M) \leq w(M)$.*

Proof. We induct on $w(M)$. If $w(M) = 1$, then M is Koszul and so is ΩM. Hence $w(\Omega M) = 1 = w(M)$ and we are done. Suppose now that $w(M) \geq 2$ and that the desired inequality holds for all weakly Koszul modules whose w-invariant does not exceed $w(M) - 1$. Applying Ω to the canonical relative extension associated with M we have, by Prop. 32, an exact sequence

$$0 \to \Omega \langle M_0 \rangle \to \Omega M \to \Omega(M/\langle M_0 \rangle) \to 0$$

[6] The symbol Ω here stands for a syzygy module in a minimal projective resolution. See Lemma 23.

If $\Omega\langle M_0\rangle = 0$, then $\Omega M \simeq \Omega(M/\langle M_0\rangle)$. Hence, by the induction hypothesis and Cor. 29,

$$w(\Omega M) = w(\Omega(M/\langle M_0\rangle)) \leq w(M/\langle M_0\rangle) = w(M) - 1,$$

and we are done. If $\Omega\langle M_0\rangle \neq 0$, then the short exact sequence above is isomorphic to the canonical relative extension associated with ΩM. By Cor. 29 and the induction hypothesis,

$$w(\Omega M) = w(\Omega(M/\langle M_0\rangle)) + 1 \leq w(M/\langle M_0\rangle) + 1 = w(M),$$

and we are done. □

We end this section with a simple observation about the short exact sequence

$$0 \to \langle M_0\rangle \xrightarrow{\iota} M \to M/\langle M_0\rangle \to 0$$

of Cor. 31 in a general setting.

Lemma 34. *Let $M = M_0 \coprod M_1 \coprod \ldots$ be an arbitrary (in particular, not necessarily weakly Koszul) graded Λ-module and $\langle M_0\rangle$ the submodule generated by M_0. Then $H_0(\Phi(\iota))$ is an isomorphism.*

Proof. Apply the functor Φ to the above sequence and use the Snake Lemma on the short exact sequences in horizontal degrees 0 and 1. □

6. Constructing the linearization functor on modules modulo projectives

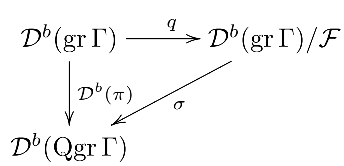
Let Γ be a noetherian Koszul algebra. Recall that $\operatorname{gr}\Gamma$ denotes the category of finitely generated graded Γ-modules. Let $\operatorname{Qgr}\Gamma$ be the quotient category of $\operatorname{gr}\Gamma$ by the Serre subcategory of modules of finite length. This is an abelian category and we have an exact functor $\pi : \operatorname{gr}\Gamma \to \operatorname{Qgr}\Gamma$. Then π induces a functor

$$\mathcal{D}^b(\pi) : \mathcal{D}^b(\operatorname{gr}\Gamma) \to \mathcal{D}^b(\operatorname{Qgr}\Gamma)$$

between the corresponding derived categories. Let $\mathcal{F} \subset \mathcal{D}^b(\operatorname{gr}\Gamma)$ be the full subcategory consisting of all complexes isomorphic to complexes of modules of finite length and

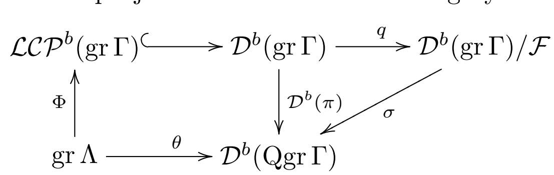
$$q : \mathcal{D}^b(\operatorname{gr}\Gamma) \to \mathcal{D}^b(\operatorname{gr}\Gamma)/\mathcal{F}$$

the corresponding Verdier quotient. It was shown in ([10], Th. 4.4) that there is a commutative diagram of functors



where σ is an equivalence of categories.

Now we want to examine the composition θ of Φ and the tautological functor from linear complexes of projectives to the derived category:



We shall refer to θ as the linearization functor.

Now let P be a finitely generated projective graded Λ-module. Then P is isomorphic to $\coprod P_i[n_i]$, where each P_i is generated in degree zero. By Remark 3, $\Phi(P)$ is isomorphic in the category of complexes over $\operatorname{gr}\Gamma$ to $\coprod \Phi(P_i)[n_i](n_i)$, where each $\Phi(P_i)$ is a projective resolution of a semisimple Γ-module S_i. Hence $\Phi(P)$ is isomorphic in $\mathcal{D}^b(\operatorname{gr}\Gamma)$ to a direct sum of complexes each of which is isomorphic to a vertical shift of a horizontal shift of a semisimple Γ-module. It follows that θ sends any map factoring through a projective Λ-module to a zero map in $\mathcal{D}^b(\operatorname{Qgr}\Gamma)$. Consequently, θ induces a functor, which we also call the linearization,

$$\underline{\theta} : \underline{\operatorname{gr}}\,\Lambda \to \mathcal{D}^b(\operatorname{Qgr}\Gamma)$$

where $\underline{\operatorname{gr}}\,\Lambda$ denotes the category of finitely generated graded Λ modules modulo projectives. In that category, the syzygy operation Ω becomes an endofunctor. We now examine the result of applying $\underline{\theta}$ to Ω. To this end, we recall a procedure that universally inverts an endofunctor.

7. Inverting the syzygy endofunctor

Let the pair $(\mathcal{A}, \phi)$ denote a category $\mathcal{A}$ with an endofunctor $\phi : \mathcal{A} \to \mathcal{A}$. If $(\mathcal{B}, \psi)$ is another such pair, then a functor $F : \mathcal{A} \to \mathcal{B}$ is said to be a morphism of the pairs if it makes the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\phi} & \mathcal{A} \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\ \mathcal{B} & \xrightarrow{\psi} & \mathcal{B} \end{array}$$

commute, i.e. if the functors $F\phi$ and ψF are isomorphic. If ψ happens to be an autoequivalence, we shall then say that the morphism F inverts ϕ.[7] This naturally leads to the following universal problem. Given a pair $(\mathcal{A}, \phi)$, find a pair $(\mathcal{A}[\phi^{-1}], \rho)$ and a morphism of pairs $G : (\mathcal{A}, \phi) \to (\mathcal{A}[\phi^{-1}], \rho)$ such that G inverts ϕ (i.e. ρ is an autoequivalence) and for any morphism of pairs $F : (\mathcal{A}, \phi) \to (\mathcal{B}, \psi)$ such that F inverts ϕ, there exists a unique, up to isomorphism, morphism of pairs $F' : (\mathcal{A}[\phi^{-1}], \rho) \to (\mathcal{B}, \psi)$ making the diagram

$$\begin{array}{ccc} (\mathcal{A}, \phi) & \xrightarrow{\quad F \quad} & (\mathcal{B}, \psi) \\ & {\scriptstyle G}\searrow \qquad \nearrow{\scriptstyle F'\ \exists!} & \\ & (\mathcal{A}[\phi^{-1}], \rho) & \end{array}$$

commute. A very short solution to this problem was given by R.-O. Buchweitz. We now recall his construction.

The objects of $\mathcal{A}[\phi^{-1}]$ consist of the objects of $\mathcal{A}$ and of the formal symbols $\phi^{-n}M$, where M runs through all objects of $\mathcal{A}$ and $n = 0, 1, \ldots$. For each object M the symbol $\phi^0 M$ is identified with M. If M and N are objects of $\mathcal{A}[\phi^{-1}]$, define morphisms from M to N by setting

$$\operatorname{Mor}_{\mathcal{A}[\phi^{-1}]}(M, N) := \varinjlim_{k} \operatorname{Mor}_{\mathcal{A}}(\phi^k M, \phi^k N),$$

[7] Thus if F inverts ϕ, then so does any other morphism between the same pairs.

where we assume that if $M = \phi^{-m}M'$ and $N = \phi^{-n}N'$, then $k \geq \max\{m, n\}$. We also tacitly assume that each morphism is labeled by its domain and codomain, thus providing for unique identification of those. It is convenient, for fixed objects M and N in $\mathcal{A}[\phi^{-1}]$, to visualize a morphism from $f : M \to N$ as represented by an $\mathcal{A}$-morphism $f_n : \phi^n M \to \phi^n N$ for a sufficiently large integer n.[8] The composition of morphisms and the identity morphisms in $\mathcal{A}[\phi^{-1}]$ are defined in an obvious way, which shows that $\mathcal{A}[\phi^{-1}]$ is indeed a category.

Now we define the endofunctor $\rho : \mathcal{A}[\phi^{-1}] \to \mathcal{A}[\phi^{-1}]$ by setting $\rho(M) := \phi(M)$ and $\rho(\phi^{-n}M) := \phi^{-n+1}(M)$ for any M in $\mathcal{A}$ and any natural number n. If f is a morphism in $\mathcal{A}[\phi^{-1}]$ represented by some f_n in $\mathcal{A}$, then $\rho(f)$ is defined as the appropriate morphism represented by $\phi(f_n)$. The verification that ρ is a functor is immediate. It is equally straightforward to check that ρ is an autoequivalence. Notice that symbolically ρ operates exactly like ϕ does. For this reason it makes sense to change notation and replace the passive ρ by the evocative ϕ. As a result, we have a pair $(\mathcal{A}[\phi^{-1}], \phi)$ with an invertible endofunctor. Moreover, we also have the desired morphism of pairs

$$G : (\mathcal{A}, \phi) \to (\mathcal{A}[\phi^{-1}], \phi),$$

defined tautologically. Thus G inverts ϕ. The verification that G is a universal inverter of ϕ is left to the reader.

Next, we apply the just described construction to the pair $(\mathcal{A}, \Omega)$, where $\mathcal{A}$ is a module category modulo projectives and Ω is the syzygy endofunctor on $\mathcal{A}$. Notice that both $\mathcal{A}$ and Ω are additive. Moreover, the resulting category $\mathcal{A}[\Omega^{-1}]$ has a triangulated structure. The shift is given by Ω^{-1}. To define the distinguished triangles, let $f : M \to N$ be a morphism in $\mathcal{A}[\Omega^{-1}]$. We want to embed this morphism into a distinguished triangle and define distinguished triangles at the same time. Since Ω is invertible, we may assume without loss of generality that both M and N are modules and that f is (a projective equivalence class of) a homomorphism of modules. Moreover, we may also assume that M is a first syzygy module $\Omega M'$ of some module $M' = \Omega^{-1}M$, i.e. we have an exact sequence of modules

$$0 \to M \to P \to \Omega^{-1}M \to 0$$

with P a projective module. Together with f this gives rise to a push-out diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & P & \longrightarrow & \Omega^{-1}M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & X & \xrightarrow{p} & \Omega^{-1}M & \longrightarrow & 0
\end{array}$$

The triangle

$$M \xrightarrow{f} N \longrightarrow X \xrightarrow{p} \Omega^{-1}M$$

is now declared distinguished. The verification that $\mathcal{A}[\Omega^{-1}]$ with shift Ω^{-1} and the just defined distinguished triangles is triangulated is standard and left to the reader.

---

[8] Because of our lame notation, caution should be exercised with such representatives as the same $\mathcal{A}$-morphism can "represent" many morphisms in $\mathcal{A}[\phi^{-1}]$; for example, f_n represents a morphism $\phi^m M \to \phi^m N$ for each integer m.

8. Constructing the linearization functor on the stabilized category and main theorem

We now return to the diagram and the linearization functor

$$\underline{\theta} : \underline{\mathrm{gr}}\,\Lambda \longrightarrow \mathcal{D}^b(\mathrm{Qgr}\,\Gamma)$$

constructed in Sec. 6.

Let $0 \to \Omega M \to P \xrightarrow{r} M \to 0$ be an exact sequence in $\mathrm{gr}\,\Lambda$ with P a projective cover. Upon applying Φ, we have a distinguished triangle

$$\Phi(M) \xrightarrow{\Phi(r)} \Phi(P) \longrightarrow \Phi(\Omega M) \overset{\delta(\Phi(r))}{\cdots\cdots\!\!>} \Phi(M)(-1)$$

in $\mathcal{D}^b(\mathrm{gr}\,\Gamma)$. Explicitly ([3], Prop. IV, 2.8), the exact sequence

$$0 \longrightarrow \Phi(M) \xrightarrow{\Phi(r)} \Phi(P) \longrightarrow \Phi(\Omega M) \longrightarrow 0$$

is embedded as the bottom row in the corresponding cone-cylinder diagram. The morphism $\delta(\Phi(r))$ is then defined as the composition of the inverse of the vertical quasi-isomorphism from the cone to the rightmost complex in the exact sequence and the horizontal morphism from the cone to the shift of the leftmost complex.

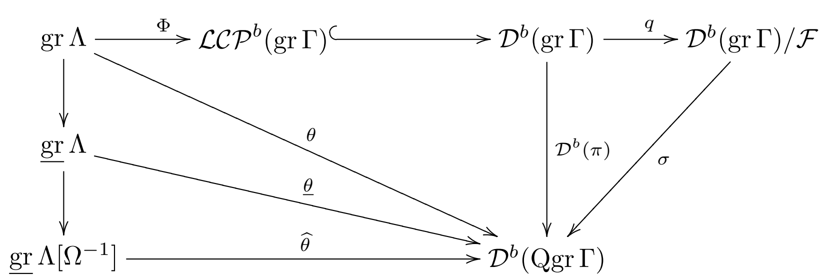
Given now a module homomorphism $f : M \to N$, we lift it to the corresponding projective covers and syzygy modules and apply the functor Φ. Direct verification shows that the corresponding maps δ form a commutative square with $\Phi(\Omega f)$ and $\Phi(f)(-1)$. As the same is true in $\mathcal{D}^b(\mathrm{Qgr}\,\Gamma)$, this implies that

$$\delta : \underline{\theta}(\Omega(-)) \longrightarrow \underline{\theta}(-)(-1)$$

is a natural transformation. Moreover, since $\Phi(P)$ is a zero object in $\mathcal{D}^b(\mathrm{Qgr}\,\Gamma)$, we have that δ is a natural isomorphism or, in other words, that the linearization functor $\underline{\theta}$ inverts Ω. Formally inverting Ω and using the corresponding universal property, we have a uniquely defined functor

$$\widehat{\theta} : \underline{\mathrm{gr}}\,\Lambda[\Omega^{-1}] \longrightarrow \mathcal{D}^b(\mathrm{Qgr}\,\Gamma),$$

which is part of the following commutative diagram of functors:



Again abusing the language, we shall call $\widehat{\theta}$ the linearization functor. Taking into account that Ω becomes an autoequivalence on $\underline{\mathrm{gr}}\,\Lambda[\Omega^{-1}]$, we deduce from the foregoing discussion the following result.

Lemma 35. *For any integer n the natural transformation*

$$\delta : \widehat{\theta}(\Omega^n(-)) \longrightarrow \widehat{\theta}(-)(-n)$$

is an isomorphism in $\mathcal{D}^b(\operatorname{Qgr}\Gamma)$. Moreover, it is in fact a morphism in $\mathcal{D}^b(\operatorname{gr}\Gamma)$ (although it need not be an isomorphism there).

We now state the main result of the paper.

Theorem 36. *Let Λ be a finite-dimensional Koszul algebra over a field $\Bbbk$. Assume that Λ_0 is a finite product of copies of $\Bbbk$ and that the Yoneda algebra Γ of Λ is noetherian. Then the linearization functor*

$$\widehat{\theta} : \underline{\operatorname{gr}}\,\Lambda[\Omega^{-1}] \longrightarrow \mathcal{D}^b(\operatorname{Qgr}\Gamma)$$

is a triangulated duality.

9. Proof of the main theorem

9.1. **The linearization functor is triangulated.** Applying the functor Φ to the push-out diagram on p. 22 we have a commutative diagram with exact rows in the category of complexes of graded Γ-modules:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Phi(\Omega^{-1}M) & \xrightarrow{\Phi(p)} & \Phi(X) & \longrightarrow & \Phi(N) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle \Phi(f)} & & \\
0 & \longrightarrow & \Phi(\Omega^{-1}M) & \longrightarrow & \Phi(P) & \longrightarrow & \Phi(M) & \longrightarrow & 0
\end{array}$$

We want to show that

$$\Phi(\Omega^{-1}M) \xrightarrow{\Phi(p)} \Phi(X) \longrightarrow \Phi(N) \xrightarrow{\Phi(f)} \Phi(M)$$

is a distinguished triangle in $\mathcal{D}^b(\operatorname{Qgr}\Gamma)$. Each of the two exact sequences of complexes above can be embedded in a distinguished triangle in $\mathcal{D}^b(\operatorname{gr}\Gamma)$ as was done at the beginning of Sec. 8. That construction shows that the two exact sequences are embedded in distinguished triangles together with a morphism between those triangles such that its first three terms coincide with the morphisms in the original diagram above:

$$\begin{array}{ccccccc}
\Phi(\Omega^{-1}M) & \xrightarrow{\Phi(p)} & \Phi(X) & \longrightarrow & \Phi(N) & \longrightarrow & \Phi(\Omega^{-1}M)(-1) \\
\| & & \downarrow & & \downarrow{\scriptstyle \Phi(f)} & & \| \\
\Phi(\Omega^{-1}M) & \longrightarrow & \Phi(P) & \longrightarrow & \Phi(M) & \xrightarrow{\delta} & \Phi(\Omega^{-1}M)(-1)
\end{array}$$

We now view this diagram as a morphism of triangles in the quotient category $\mathcal{D}^b(\operatorname{Qgr}\Gamma)$. In that category $\Phi(P)$ is isomorphic to a zero object and therefore δ becomes an isomorphism. This yields an isomorphism of triangles

$$\begin{array}{ccccccc}
\Phi(\Omega^{-1}M) & \xrightarrow{\Phi(p)} & \Phi(X) & \longrightarrow & \Phi(N) & \longrightarrow & \Phi(\Omega^{-1}M)(-1) \\
\| & & \| & & \| & & \uparrow{\scriptstyle \delta} \\
\Phi(\Omega^{-1}M) & \xrightarrow{\Phi(p)} & \Phi(X) & \longrightarrow & \Phi(N) & \xrightarrow{\Phi(f)} & \Phi(M)
\end{array}$$

As the top triangle is distinguished, we conclude that so is the bottom one. Thus $\widehat{\theta}$ is indeed triangulated. It remains to show that $\widehat{\theta}$ is full, faithful, and dense.

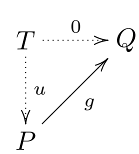
9.2. **The linearization functor is dense.** Choose any bounded complex B of finitely generated graded Γ-modules. Since the Koszul algebra Λ is finite-dimensional, its Yoneda algebra Γ is of finite global dimension. By Prop. 25, B is isomorphic, as an object of $\mathcal{D}^b(\operatorname{Qgr}\Gamma)$, to a vertical shift of a totally linear complex. But then it is isomorphic to a *horizontal* shift of a totally linear complex, say, $B \simeq L(p)$. By Prop. 22, L is quasi-isomorphic to a linear complex of projectives. By Prop 4, we then have $B \simeq \widehat{\theta}(M)(p)$ for some M in $\operatorname{gr}\Lambda$. By Lemma 35, B is isomorphic to $\widehat{\theta}(\Omega^{-p}M)$, which shows that $\widehat{\theta}$ is dense. It remains to show that $\widehat{\theta}$ is full and faithful. First we establish two properties of this functor.

9.3. **The linearization functor reflects zero objects and isomorphisms.** Suppose $f : M \to N$ is a homomorphism of modules in $\operatorname{gr}\Lambda$ and let $g := \Phi(f) : \Phi(N) \to \Phi(M)$.

Lemma 37. *The morphism g is zero in $\mathcal{D}^b(\operatorname{Qgr}\Gamma)$ if and only if it factors, as a chain map, through a bounded complex, say, Y of Γ-modules with finite length homology. Moreover, Y can be chosen to consist of projectives and be sub-diagonal.*

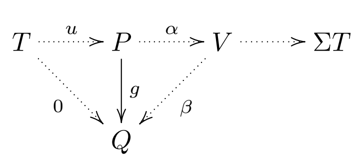
Proof. The "if" part. Suppose g factors through a complex Y with finite length homology. For a large enough integer n the complex $Y_{\geq n}$ is acyclic. Thus the complexes Y and $Y/Y_{\geq n}$ are quasi-isomorphic. But the latter is a bounded complex of modules of finite length, which is a zero object in the derived tails.

Conversely, suppose $g = 0$ in $\mathcal{D}^b(\operatorname{Qgr}\Gamma)$. Setting $P := \Phi(N)$ and $Q := \Phi(M)$, we have a commutative diagram in $\mathcal{D}^b(\operatorname{gr}\Gamma)$:

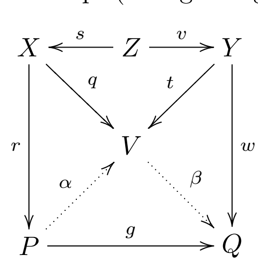


where the dotted arrows are morphisms in $\mathcal{D}^b(\operatorname{gr}\Gamma)$ and the third vertex of the distinguished triangle containing u is isomorphic to a bounded complex of modules of finite length.

The fact that $gu = 0$ places the just mentioned triangle in a commutative diagram



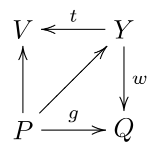
Thus g factors in $\mathcal{D}^b(\operatorname{gr}\Gamma)$ through a bounded complex V of modules of finite length. This results in a diagram of chain maps (disregarding, of course, the dotted arrows)



which is commutative in the homotopy category of complexes over $\mathrm{gr}\,\Gamma$. In this diagram r, s and t are quasi-isomorphisms. Using Cor. 18, we clear the denominator rs of the morphism

$$P \xleftarrow{rs} Z \xrightarrow{v} Y$$

This results in a diagram of chain maps



which is commutative in the homotopy category of complexes. By Lemma 14, the south-east part of this diagram yields the desired factorization of g. Finally, truncating Y in the vertical degrees above the diagonal and using Props. 24 and 15, we may assume that Y is a sub-diagonal complex of projectives. □

Proposition 38. *$\widehat{\theta}$ reflects zero objects.*

Proof. Let M be an object in $\underline{\mathrm{gr}}\,\Lambda[\Omega^{-1}]$ such that $\widehat{\theta}(M)$ is a zero object in $\mathcal{D}^b(\mathrm{Qgr}\,\Gamma)$. Since Ω is an autoequivalence on the domain category, without loss of generality we may assume that M is a module. Moreover, by Th. 27, we may assume that M is weakly Koszul. Let $P := \Phi(M)$. By Lemma 37, the identity map on P factors through a complex with finite length homology. Assume first that M is Koszul. By Th. 5, we have that P is a projective resolution of the Koszul-dual $K(M)$ of M. Thus the identity map on $K(M)$ factors through a module of finite length. Hence $K(M)$ itself is of finite length and therefore M is of finite projective dimension and hence a zero object in $\underline{\mathrm{gr}}\,\Lambda[\Omega^{-1}]$. In the general case we proceed by induction on $w(M)$ (see Sec. 5). The case $w(M) = 1$ has just been established. Thus assume that the desired assertion has been proved for all weakly Koszul modules whose w-invariant does not exceed $w(M) - 1$. We now examine the canonical relative extension

$$0 \to \langle M_0 \rangle \xrightarrow{\iota} M \to M/\langle M_0 \rangle \to 0$$

associated with M (see Cor. 31). By Th. 28, $\langle M_0 \rangle$ is a Koszul module. Our assumptions together with Lemma 34 imply that the degree zero homology of $\Phi(\langle M_0 \rangle)$ is of finite length. As we have just seen, this implies that $\langle M_0 \rangle$ is of finite projective dimension. Applying Prop. 32 to the relative extension above we have that high enough syzygy modules of M and of $M/\langle M_0 \rangle$ become isomorphic. Therefore M and $M/\langle M_0 \rangle$ are isomorphic as objects of $\underline{\mathrm{gr}}\,\Lambda[\Omega^{-1}]$. Hence $\Phi(M/\langle M_0 \rangle)$ is a zero object in $\mathcal{D}^b(\mathrm{Qgr}\,\Gamma)$. By Cor. 29 and the induction hypothesis, $M/\langle M_0 \rangle$ is a zero object in $\underline{\mathrm{gr}}\,\Lambda[\Omega^{-1}]$ and therefore the same is true for M. □

Corollary 39. *$\widehat{\theta}$ reflects isomorphisms.*

Proof. Let f be a morphism in $\underline{\mathrm{gr}}\,\Lambda[\Omega^{-1}]$ such that $\widehat{\theta}(f)$ is an isomorphism. Being triangulated, $\widehat{\theta}$ sends the third vertex Z of a triangle containing f to (the shift of) the third vertex of a triangle containing $\widehat{\theta}(f)$, which is a zero object. By Prop. 38, Z must be a zero object. But then f must be an isomorphism ([13], Cor. 1.2.6). □

9.4. **The linearization functor is full.** Let M and N be objects of $\underline{\mathrm{gr}}\,\Lambda[\Omega^{-1}]$ and $h' : \Phi(N) \to \Phi(M)$ a morphism in $\mathcal{D}^b(\mathrm{Qgr}\,\Gamma)$. We need to find a morphism h in $\underline{\mathrm{gr}}\,\Lambda[\Omega^{-1}]$ such that $h' = \Phi(h)$. Since Ω is an autoequivalence and $\widehat{\theta}$ is a morphism of categories with endofunctors, we may assume for simplicity that M and N are modules. The morphism h' is represented by a diagram

$$\Phi(N) \xleftarrow{\;s\;} B \xrightarrow{\;f\;} \Phi(M)$$

in $\mathcal{D}^b(\mathrm{gr}\,\Gamma)$, where the third vertex of the triangle containing s is isomorphic in $\mathcal{D}^b(\mathrm{gr}\,\Gamma)$ to a complex of modules of finite length. By Prop. 25, B contains a subcomplex B' of finite colength which is a vertical shift of a totally linear complex.

The octahedron diagram

$$\begin{array}{ccccccc}
B' & \xrightarrow{\iota} & B & \longrightarrow & C_\iota & \longrightarrow & \Sigma B' \\
\| & & \downarrow & & \downarrow & & \| \\
B' & \xrightarrow{s\iota} & \Phi(N) & \longrightarrow & C_{s\iota} & \longrightarrow & \Sigma B' \\
\downarrow{\scriptstyle \iota} & & \| & & \downarrow & & \downarrow \\
B & \xrightarrow{s} & \Phi(N) & \longrightarrow & C_s & \longrightarrow & \Sigma B \\
\downarrow & & \downarrow & & \| & & \downarrow \\
C_\iota & \longrightarrow & C_{s\iota} & \longrightarrow & C_s & \longrightarrow & \Sigma C_\iota
\end{array}$$

where $\iota : B' \to B$ is the inclusion map, shows that $C_{s\iota}$, being the middle vertex of the triangle containing C_ι and C_s, is isomorphic in $\mathcal{D}^b(\mathrm{gr}\,\Gamma)$ to a complex of modules of finite length. In other words, the third vertex of the triangle containing $s\iota$ is isomorphic to a complex of modules of finite length. The commutative diagram

$$\begin{array}{ccccc}
 & & B & & \\
 & \swarrow{\scriptstyle s} & \uparrow{\scriptstyle \iota} & \searrow{\scriptstyle f} & \\
\Phi(N) & \xleftarrow{s|_{B'}} & B' & \longrightarrow & \Phi(M)
\end{array}$$

where the solid arrow denotes the inclusion chain map, shows now that B may be assumed to be a downward shift, say $B = L'[-n]$ with $n \geq 0$, of a totally linear complex. But then B is also a horizontal shift to the right, say $B = L(n)$, of a totally linear complex L. The construction of Prop. 22 and Prop. 4 yield a quasi-isomorphism $\pi : \Phi(X)(n) \to L(n)$, where X is in $\mathrm{gr}\,\Lambda$. As π is an isomorphism in the derived category, the commutative diagram

$$\begin{array}{ccccc}
 & & L(n) & & \\
 & \swarrow{\scriptstyle s} & \uparrow{\scriptstyle \pi} & \searrow{\scriptstyle f} & \\
\Phi(N) & \longleftarrow & \Phi(X)(n) & \longrightarrow & \Phi(M)
\end{array}$$

shows that B can be replaced by $\Phi(X)(-n)$. Replacing M and N by their nth syzygy module and using Lemma 35, we may assume without loss of generality that $n = 0$. Thus the original morphism h' can be represented by a diagram in

$\mathcal{D}^b(\operatorname{gr}\Gamma)$:

$$\Phi(N) \underset{s}{\longleftarrow} \Phi(X) \underset{f}{\longrightarrow} \Phi(M)$$

By Cor. 18, each of the two morphisms can be represented by a chain map. Therefore h' can be represented by a diagram of chain maps:

$$\Phi(N) \underset{s}{\longleftarrow} \Phi(X) \underset{f}{\longrightarrow} \Phi(M)$$

Since Φ is a duality, there are homomorphisms $\sigma : N \to X$ and $\phi : M \to X$ such that $s = \Phi(\sigma)$ and $f = \Phi(\phi)$. Since s is an isomorphism in derived tails, Cor. 39 shows that σ is an isomorphism in $\underline{\operatorname{gr}}\,\Lambda[\Omega^{-1}]$. But then $h' = fs^{-1} = \widehat{\theta}(\phi)\widehat{\theta}(\sigma^{-1}) = \widehat{\theta}(\sigma^{-1}\phi)$, which is the desired assertion. It remains to show that $\widehat{\theta}$ is faithful.

9.5. **The linearization functor is faithful.** This is a consequence of the following result.

Proposition 40. *A full triangulated functor reflecting isomorphisms is faithful.*

Proof. We give a proof in the case when the functor, say Ψ, is contravariant. The covariant case is argued similarly. Let $f : M \to N$ be a morphism in the domain category such that $\Psi(f) : \Psi(N) \to \Psi(M)$ is a zero morphism. Embedding f in a distinguished triangle

$$M \xrightarrow{f} N \xrightarrow{\delta} X \longrightarrow \Sigma M$$

and applying Ψ, we have a distinguished triangle

$$\Psi(\Sigma M) \longrightarrow \Psi(X) \xrightarrow{\Psi(\delta)} \Psi(N) \xrightarrow{\Psi(f)} \Psi(M)$$

in the target category. The fact that $\Psi(f)$ is zero results in a morphism of distinguished triangles

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Psi(N) & = & \Psi(N) & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \gamma'} & & \| & & \downarrow \\
\Psi(\Sigma M) & \longrightarrow & \Psi(X) & \xrightarrow{\Psi(\delta)} & \Psi(N) & \xrightarrow{\Psi(f)} & \Psi(M)
\end{array}$$

where γ' exists by TR3. Since Ψ is full, $\gamma' = \Psi(\gamma)$ for some morphism $\gamma : X \to N$. Since Ψ reflects isomorphisms, we have that $\gamma\delta$ is an isomorphism in the domain category. In that category we also have a morphism of distinguished triangles

$$\begin{array}{ccccccc}
M & \xrightarrow{f} & N & \xrightarrow{\delta} & X & \longrightarrow & \Sigma M \\
\downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} & & \downarrow \\
0 & \longrightarrow & N & = & N & \longrightarrow & 0
\end{array}$$

where α exists by TR3. Since $\gamma\delta$ is an isomorphism, so is α. The commutativity of the left-hand square now shows that f is a zero morphism. Thus Ψ is faithful. $\square$

In view of Prop. 40, the fact that the linearization functor is faithful now follows from Cor. 39 and 9.4. This finishes the proof of the theorem.

Example. Let Λ be the exterior algebra on $n+1$ letters. By a theorem of Heller, the functor Ω is an autoequivalence on the category of finitely generated (graded) Λ-modules modulo projectives. Therefore $\underline{\operatorname{gr}}\,\Lambda[\Omega^{-1}] = \underline{\operatorname{gr}}\,\Lambda$. On the other hand, the Yoneda algebra Γ of Λ is the polynomial algebra on $n+1$ letters. By a theorem of Serre, the category of finitely generated graded Γ-modules modulo modules of finite length is equivalent to the category of coherent sheaves on the n-dimensional projective space. The just proved theorem now specializes to (a contravariant analog of) the Bernstein - Gelfand - Gelfand correspondence.

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