1-TORSION OF FINITE MODULES OVER SEMIPERFECT RINGS

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Abstract. We initiate the study of 1-torsion of finite modules over two-sided noetherian semiperfect rings. In particular, we give a criterion for determining when the 1-torsion submodule contains minimal generators of the module. We also provide an explicit construction for a projective cover of the submodule generated by the torsion elements in the top of the module. Some of the obtained results hold without the noetherian assumption. We also give several applications to local algebra.

1. Introduction

The goal of this paper is to study 1-torsion submodules (for definitions see below) of finite modules over semiperfect rings. In particular, we want to understand under what conditions such submodules contain minimal generators of the corresponding modules. The original motivation for this problem came from a question raised by Reiffen and Vetter ([17]) in their work on Pfaffian forms on complex spaces. An algebraic reformulation of it, due to G. Scheja, is discussed in detail in E. Platte's paper [16]. We quickly recall the basic facts. Let k be a valued field of characteristic zero and A a reduced equidimensional local analytic k-algebra with (universally finite) module of Kähler differentials $D_k(A)$. The torsion problem can be stated as follows: (if $k = \mathbb{C}$) is it possible for $D_k(A)$ to have common minimal generators with its torsion submodule? It was shown that this is impossible for the following classes of algebras:

(a) hypersurfaces (Reiffen - Vetter [17], Satz 4 and Scheja [18], Satz (9.11),
(b) homogeneous algebras (Scheja [18], p. 157),
(c) invariants of regular rings by finite groups (Platte [14], p. 9 and [15], Prop. (3.1), part 2).

The last result allowed Platte to improve necessary conditions, obtained earlier by V. Kac and K.-I. Watanabe, on rings of invariants of finite groups to be complete intersections. After mentioning the above three cases in [16], Platte constructs a class of examples showing that indeed the module of differentials can have torsion elements among its minimal generators. At the end of the paper, he mentions another question, raised by Scheja, whether the torsion submodule of the module of differentials can be a direct summand. He then quotes a result of Scheja that for hypersurface rings the new problem is equivalent to the original Reiffen - Vetter problem, quickly elevates the question to a conjecture (i.e., the torsion submodule is never a direct summand) and remarks that, if true, it would provide a quick proof of Grothendieck's version of the purity of the branch locus for complete intersections

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([6]). Platte concludes his paper with a remark that "[u]nfortunately, a proof of the weakened torsion conjecture seems to be [methodologically] remote".

Undoubtedly, Platte's pessimistic assessment of the problem reflects serious gaps in the existing techniques. This is corroborated, in particular, by an *ad hoc* nature of the counterexamples. But, while acknowledging the fundamental technical difficulties facing differential calculus in algebra, we believe that some parts of it can be approached in a constructive way by module-theoretic methods. Although such techniques do not replace the core calculus, they allow to recognize and treat algebraically some aspects which are *a priori* considered as pertaining to calculus. As examples, we mention the connections between almost split sequences and Zariski differentials over two-dimensional singularities ([8]), and the study of vectors fields on quasihomogeneous complete intersections with isolated singularities ([7]). In the present paper, we shall show how methods of stable module theory can be used to provide new insights and perspectives on the problems of Reiffen - Vetter and Platte - Scheja. One may begin, for example, by asking a natural question: given the module of differentials of an algebra, how does one determine whether or not this *specific* module has torsion elements among its minimal generators? In fact, properties of the torsion submodule of *any* finitely presented module is a topic of interest in its own right and the same question can be posed for any finitely generated module over a commutative noetherian local ring. Moreover, there is no reason not to pose this question in the utmost generality, for any finitely generated module over a two-sided noetherian semiperfect ring. In that setup, the torsion submodule should be replaced by the more general concept of 1-torsion (for definitions, see below).

The main result of this paper (Th. 8) provides a verifiable module-theoretic criterion for an *arbitrary* finitely generated module over an *arbitrary* noetherian semiperfect ring to have 1-torsion elements among its minimal generators. More precisely, this happens exactly when the first syzygy module of the Auslander transpose of the module has a projective summand. This has immediate applications in local algebra. First, it is known that, in the degrees beyond the codepth of the module, the syzygy modules never have projective (i.e., free) summands. This leads to an amazing consequence for finite modules over artinian commutative local rings: the 1-torsion submodule can never contain minimal generators of the module. Secondly, the non-existence of projective summands in the syzygy modules can be deduced from the vanishing of the ξ-invariants of the module (see below for details). Roughly speaking, those invariants measure the difference between the cohomology and the Tate - Vogel cohomology of the module. The latter is an example of an abstract stable homotopy theory, based on the Eckmann - Hilton homotopy groups of modules.

We also remark that, in equationally defined situations, the obtained criterion allows explicit calculations with a minimum of computing power: to determine whether or not the first syzygy module of the transpose has a free summand, one needs a presentation matrix for the syzygy module and a procedure to check whether or not one of the rows of the matrix is a linear combination of the remaining rows.[1]

In section 7 we give a criterion for the 1-torsion submodule to be a direct summand. This is done in a greater generality: the ring is two-sided noetherian but not

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[1] A careful reader may add that one needs a presentation matrix of the original module to begin with.

necessarily semiperfect. Our methods do not impose any significant restrictions on the rings in question: there is no assumption on the characteristic, the ring does not have to be commutative or a domain, nilpotent elements are allowed, etc. For that reason, it is to be hoped that a proof of the Platte - Scheja conjecture, if at all possible, can be obtained by some sort of a dimension-reduction procedure. Notice that, as we mentioned above, in dimension zero the 1-torsion submodule cannot be a direct summand!

While we hope that the methods developed here would be useful for the study of torsion in modules of differentials, in this paper we concentrate on purely module-theoretic aspects of the problem. However, for the benefit of the reader less familiar with module theory, a special effort has been made to present a self-contained and concise treatment of the relevant module-theoretic techniques.

As we mentioned in the beginning, the proper context for the present paper is the study of the 1-torsion of finite modules over semiperfect rings. The concept of a *trivial* k-torsion, for any natural k, was introduced by Auslander and Bridger ([3]). The case of a trivial 1-torsion was studied in detail in [13], where it was shown, among other things, that the operation $\lambda := \Omega\mathrm{Tr}$ (this is the same operation mentioned in the criterion above) gives rise to an involution on the isoclasses of 1-torsionfree (also called torsionless) modules and that that involution subsumes the classical concept of linkage of algebraic varieties. It turns out that all of those results can be extended much further, in a functorial way, to finite modules over arbitrary noetherian rings which are not necessarily semiperfect ([12]).

While at the present time there is no definition of higher *nonzero* k-torsion, $k \geq 2$, the case of a nonzero 1-torsion has a long prehistory, at least in the classical setting: when the ring is a commutative domain, the 1-torsion submodule of a finite module coincides with the classical torsion submodule. Notice also that the condition that the torsion of any finite module over a commutative domain be a direct summand is one of several equivalent characterizations of Prüfer rings, with the ring of all algebraic integers being an important example.

2. Module-theoretic preliminaries

Throughout this paper all rings will be assumed to be associative with identity and all modules to be unital. In this section we recall some basic facts from module theory. Most of this material, in one form or another, can be found in [1], [2], and [3]. Let Λ be a ring and M a (left) Λ-module with a *finite* projective presentation

$$P_1 \xrightarrow{\partial} P_0 \longrightarrow M \longrightarrow 0.$$

If finite Λ-modules admit projective covers (i.e., Λ is semiperfect, see below), we shall automatically assume that the presentation above is minimal. The first syzygy module ΩM of M is defined as the kernel of the map $P_0 \to M$. The transpose $\mathrm{Tr}\, M$ of M is defined by the exact sequence

$$0 \longrightarrow M^* \longrightarrow P_0^* \xrightarrow{\partial^*} P_1^* \xrightarrow{\omega} \mathrm{Tr}\, M \longrightarrow 0,$$

where $(-)^*$ stands for the functor $\mathrm{Hom}_\Lambda(-, \Lambda)$.[2] The finiteness assumption on the projective presentation of M implies that the beginning of the above sequence is a

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[2] It is sometimes convenient to extend this definition of the transpose to the case when the module is not finitely presented – see Prop. 5, where we use the operation $\Omega\mathrm{Tr}$.

finite projective presentation of $\operatorname{Tr} M$. Notice that while ΩM is still a left module, the transpose of M is a right module, i.e., a left module over the opposite ring Λ^{op}. If finite Λ-modules admit projective covers, then both ΩM and $\operatorname{Tr} M$ are defined uniquely up to isomorphism, because of our convention that projective presentations be minimal. In general, however, both ΩM and $\operatorname{Tr} M$ are only defined up to projective equivalence, i.e., different choices would lead to modules that become isomorphic only after adding projective summands. For the syzygy modules this is the content of Schanuel's lemma; for the transpose this can be seen, for example, by arguments similar to the one used in the proof of Schanuel's lemma.

The following operation on Λ-modules will be of fundamental importance to us.

Definition 1. $\lambda M := \Omega \operatorname{Tr} M$.[3]

In the notation of the defining sequence for the transpose,

$$\lambda M = \operatorname{Ker}\, \omega = \operatorname{Im} \partial^*,$$

which shows that while λM is still defined up to projective equivalence, it does not depend on the choice of P_1.

Assume now that Λ is a semiperfect ring with Jacobson radical J.[4] By a theorem of Bass, a ring is semiperfect if and only if every finitely generated module over it has a projective cover. If M is a finitely generated (left) Λ-module, then, by Nakayama's lemma, any submodule of JM is superfluous in M. (This is true without any assumption on the ring.)

Lemma 2. *Let Λ be a semiperfect ring and N a submodule of a finitely generated projective Λ-module P. Then N is superfluous in P if and only if N and P have no common nonzero projective summands.*

Proof. Since Λ/J is semisimple, P/JP is a finite direct sum $\coprod \overline{P_i}$ of simple Λ/J-modules. If P_i is a projective cover of $\overline{P_i}$ over Λ, then P is isomorphic to $\coprod P_i$. Suppose that N is not superfluous in P. Then the reduction $N/JN \to P/JP \simeq \coprod \overline{P_i}$ of the inclusion map $\iota : N \to P \simeq \coprod P_i$ is not zero. In other words, if the inclusion map has components $\alpha_1, \alpha_2, \ldots$, then, say, $\alpha_1 \otimes \Lambda/J$ is not zero. By Schur's lemma, $\alpha_1 \otimes \Lambda/J$ is surjective. The commutative diagram

$$\begin{tikzcd}
N \arrow[r, tail, "\iota"] \arrow[d, two heads] & \coprod P_i \arrow[r, two heads, "\pi_1"] \arrow[d, two heads] & P_1 \arrow[d, two heads] \\
N/JN \arrow[r] & \coprod \overline{P_i} \arrow[r, two heads] & \overline{P_1}
\end{tikzcd}$$

where all three vertical maps and the two horizontal maps on the right are the canonical projections, now shows that the composition $N \overset{\pi_1 \iota}{\to} P_1 \to \overline{P_1}$ is surjective. Therefore, $P_1 = \operatorname{Im}(\pi_1 \iota) + JP_1$. Since JP_1 is superfluous in P_1, we have $\operatorname{Im}(\pi_1 \iota) = P_1$ and therefore P_1 is a direct summand of N. Conversely, suppose that N is superfluous in P. Since a direct summand of a finite module cannot be superfluous, N cannot have a common nonzero projective summand with P. $\square$

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[3] This operation was denoted D_1 in [3]. We feel justified in abandoning the original notation because nowadays the symbol D has a variety of other established meanings.

[4] Recall that Λ is is said to be semiperfect if Λ/J is semisimple and idempotents of Λ/J can be lifted to Λ. In particular, this property is left-right symmetric.

The following consequence of the just proved result is of main interest to us. Before we state it we need to fix some terminology. We continue to assume that Λ is semiperfect. Recall that a module is said to be *stable* if it has no nonzero projective (direct) summands. We shall say that a projective summand P of a module M is *maximal* if $M \simeq \underline{M} \coprod P$, where $\underline{M}$ is stable.[5] This notion can also be defined in a relative context. Given a morphism $f : M \to N$ of modules and a common, under f, projective summand Q of M and N (i.e., restricting and corestricting f to the two copies of Q we have an isomorphism), we shall say that Q is a maximal common projective summand if the induced map between some (and then all) complementary to Q submodules provides no nonzero common projective summands.

Proposition 3. *Let Λ be a semiperfect ring, M a finitely presented Λ-module with* ***minimal*** *projective presentation*

$$\begin{tikzcd} P_1 \arrow[r, "\partial"] & P_0 \arrow[r] & M \arrow[r] & 0, \end{tikzcd}$$

and

$$\begin{tikzcd} 0 \arrow[r] & M^* \arrow[r] & P_0^* \arrow[r, "\partial^*"] & P_1^* \arrow[r, "\omega"] & \operatorname{Tr} M \arrow[r] & 0 \end{tikzcd}$$

the corresponding (*augmented on the left*) *presentation for* $\operatorname{Tr} M$*. Then:*

- a) *The map $\omega : P_1^* \to \operatorname{Tr} M$ is a projective cover.*
- b) *M is stable if and only if the above presentation of $\operatorname{Tr} M$ is minimal.*
- c) *M is stable if and only if $P_0^* \to \lambda M$ is a projective cover.*
- d) *If Q is a maximal projective direct summand of M, then Q^* is a maximal common direct summand of M^* and P_0^*.*
- e) $\operatorname{Tr} M$ *is stable.*
- f) $\operatorname{Tr} M$ *is zero if and only if M is projective.*

Proof. a) The finiteness of the presentation in question is immediate. Suppose that $\omega : P_1^* \to \operatorname{Tr} M$ is not a projective cover. Then $\operatorname{Ker} \omega$ is not superfluous in P_1^* and, by Lemma 2, it has a common nonzero projective summand with P_1^*. Therefore, P_0^* and P_1^* have a common nonzero finite projective summand. Since finitely generated projectives are reflexive and since a nonzero finite direct summand is never a superfluous submodule, the map $P_0 \to M$ is not a projective cover, contrary to the assumption. Thus ω is a projective cover.

b) Suppose that M is stable and $P_0^* \to \operatorname{Ker} \omega$ is not a projective cover. Then, by Lemma 2, M^* and P_0^* have a common nonzero projective summand and therefore the same is true for P_0^{**} and M^{**}. Composing now the two commuting paths in the diagram

$$\begin{tikzcd} P_0 \arrow[r, two heads] \arrow[d, "\cong"] & M \arrow[d] \\ P_0^{**} \arrow[r] & M^{**} \end{tikzcd}$$

with the projection of M^{**} onto that projective summand shows that M also has a projective summand, contrary to the assumption. To prove the "if" part, it suffices

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[5] Using Nakayama's lemma, one can easily show that for a finitely generated module over a semiperfect ring the summands in such a decomposition are unique up to isomorphism.

to remark that if M has a nonzero projective summand, then M^* and P_0^* have a common nonzero projective summand.

c) Follows from a) and b).

d) This part follows from b) and the obvious fact that the transpose respects (finite) direct sums.

e) If $\operatorname{Tr} M$ has a nonzero projective summand, then it is a common summand with P_1^* under ω. Dualizing the presentation of $\operatorname{Tr} M$, we have, taking into account that finite projectives are reflexive, that $\operatorname{Ker} \partial \simeq (\operatorname{Tr} M)^*$ has a common projective summand with P_1. In view of Lemma 2, this contradicts the minimality of the chosen presentation of M.

f) It is clear that if M is projective, then $\operatorname{Tr} M$ is zero. Conversely, suppose that $\operatorname{Tr} M$ is zero. Then ∂^* is a split projection and therefore ∂ is a split injection. Since we started with a minimal presentation of M, Lemma 2 implies that $P_1 = 0$, i.e., M is projective. □

Let Λ be any ring. For any Λ^{op}-module N and any *finitely presented* Λ-module M, there is a (bi)functorial exact sequence

$$0 \to \operatorname{Ext}^1_{\Lambda^{op}}(\operatorname{Tr} M, N) \to N \otimes_\Lambda M \to \operatorname{Hom}_{\Lambda^{op}}(M^*, N) \to \operatorname{Ext}^2_{\Lambda^{op}}(\operatorname{Tr} M, N) \to 0,$$

where the canonical middle map sends $n \otimes m$ to the map $F_{n,m} : M^* \to N$ defined by $F_{n,m}(f) := f(m)n$. Specializing to the case $N := \Lambda$, we have that the canonical map $e_M : M \to M^{**}$ is part of the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ext}^1_{\Lambda^{op}}(\operatorname{Tr} M, \Lambda) \arrow[r] & M \arrow[r, "e_M"] & M^{**} \arrow[r] & \operatorname{Ext}^2_{\Lambda^{op}}(\operatorname{Tr} M, \Lambda) \arrow[r] & 0
\end{tikzcd}$$

In accordance with [3], the kernel $\operatorname{Ext}^1_{\Lambda^{op}}(\operatorname{Tr} M, \Lambda)$ of e_M will be called the 1-torsion submodule of M; it will be denoted $t(M)$. If Λ is a commutative domain, then the kernel of e_M is just the usual torsion submodule of M.[6] (For this it is enough to assume that M is only finitely generated.)

In the next lemma we recall the construction of a map from M to $\lambda^2 M$ ([3], Appendix) and compute the image of the canonical evaluation map from M to M^{**}.

Lemma 4. *Let Λ be a semiperfect ring and M a finitely presented Λ-module such that M^* is finitely generated. If M is stable, then the image of the canonical map $e_M : M \to M^{**}$ is isomorphic to $\lambda^2 M$. If $M \simeq \underline{M} \coprod Q$, where $\underline{M}$ is stable and Q is projective, then the image of e_M is isomorphic to $\lambda^2 M \coprod Q \simeq \lambda^2 \underline{M} \coprod Q$.*

Proof. The second part of the lemma immediately follows from the first. To prove the first part, we start with a minimal presentation $P_1 \to P_0 \xrightarrow{\varphi} M \to 0$. The commutative diagram

$$\begin{tikzcd}
P_0 \arrow[r, two heads, "\varphi"] \arrow[d, "\cong"] & M \arrow[d, "e_M"] \\
P_0^{**} \arrow[r, "\varphi^{**}"] & M^{**}
\end{tikzcd}$$

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[6] The reader is cautioned not to draw a hasty analogy between 1-torsion over general rings and torsion over commutative domains. The operation $t(-)$ is still a subfunctor of the identity functor, but it does not give rise, at least immediately, to a torsion theory. For example, it is not true in general that $M \subseteq t(N)$ implies that $t(M) = M$. In fact, it follows from the results of this paper that the following weaker property $t(t(M)) = t(M)$ *never* holds, assuming $t(M) \neq 0$, for finitely generated modules M over a commutative artinian local ring.

shows that $\mathrm{Im}(e_M) = \mathrm{Im}(\varphi^{**})$. Applying $\mathrm{Hom}(-, \Lambda)$ to the minimal presentation above, we have an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & M^* \arrow[r, "\varphi^*"] & P_0^* \arrow[r] & P_1^* \arrow[r] & \mathrm{Tr}\, M \arrow[r] & 0 \end{tikzcd}$$

and, therefore, an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & M^* \arrow[r, "\varphi^*"] & P_0^* \arrow[r, "\pi"] & \lambda M \arrow[r] & 0\ . \end{tikzcd} \tag{A}$$

Let $\psi : Q \to M^*$ be a projective cover (it exists since M^* is finitely generated) and $\alpha := \varphi^* \psi$. Assuming that M is stable, we have, by Prop. 3, a minimal presentation

$$\begin{tikzcd} Q \arrow[r, "\alpha"] & P_0^* \arrow[r, "\pi"] & \lambda M \arrow[r] & 0 \end{tikzcd}$$

Applying to it the functor $\mathrm{Hom}(-, \Lambda)$ we have a commutative diagram with an exact top row:

$$\begin{tikzcd} 0 \arrow[r] & (\lambda M)^* \arrow[r, "\pi^*"] & P_0^{**} \arrow[rr, "\alpha^*"] \arrow[dr, "\varphi^{**}"] & & Q^* \arrow[r] & \mathrm{Tr}\,(\lambda M) \arrow[r] & 0 \\ & & & M^{**} \arrow[ur, "\psi^*"] & & & \end{tikzcd} \tag{B}$$

By the left-exactness of the Hom-functor, the map ψ^* is a monomorphism and therefore $\mathrm{Im}(\alpha^*) \simeq \mathrm{Im}(\varphi^{**})$. By the second part of Prop. 3, the map $Q^* \to \mathrm{Tr}\,(\lambda M)$ is a projective cover and thus $\mathrm{Im}(\alpha^*) \simeq \Omega(\mathrm{Tr}\,(\lambda M)) = \lambda^2 M$. This finishes the proof of the lemma.[7] □

3. 1-TORSION OF FINITE MODULES OVER NOT NECESSARILY NOETHERIAN RINGS

Throughout this section Λ is an associative (and not necessarily noetherian) ring with identity. We begin with a simple observation characterizing finite modules M such that $M = t(M)$.

Proposition 5. *Let Λ be an arbitrary (not necessarily noetherian) ring and M a finite (left) Λ-module. Then $t(M) = M$ if and only if M is stable and λM is projective.*[8]

Proof. Let $P_1 \to P_0 \overset{\varphi}{\to} M$ be a projective presentation of M with a finitely generated P_0. Assume first that $t(M) = M$. Since $t(M)$ is the kernel of the canonical map $e_M : M \to M^{**}$, each functional on M vanishes at each element of M, showing that $M^* = 0$. This implies that M is stable and, as is seen from sequence (A), that $\lambda M \simeq P_0^*$ is projective.[9] This proves the "only if" part of the lemma. Assume now that M is stable and that λM is projective. The latter assumption implies that sequence (A) splits, making $\varphi : M^* \to P_0^*$ a split injection. Suppose M^* is nontrivial, then M^{**} is a nontrivial direct summand of P_0^{**} and, as in the proof of Prop. 3, b), M would have a projective summand, contrary to the assumption. We conclude that $M^* = 0$. Therefore, $M^{**} = 0$ and $t(M) = M$. □

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[7] If Λ is not semiperfect, then the above proof still produces a map from M to $\lambda^2 M$, but in that case the codomain is only defined up to projective equivalence.

[8] Since λM is defined up to projective equivalence, the condition that λM be projective is independent of the chosen presentation of M.

[9] Recall that, over a general ring, λM is defined only up to projective equivalence. Thus the preceding isomorphism should be understood as one of many possible choices.

The just proved proposition has an interesting consequence. Recall that a ring is said to be *right semihereditary* if every finitely generated right ideal is projective. This is equivalent to saying that every finitely generated submodule of a projective right module is projective.

Proposition 6. *Let Λ be a (not necessarily noetherian)* ***right*** *semihereditary ring and M a finitely presented* ***left*** *Λ-module. Then the 1-torsion submodule $t(M)$ of M is a direct summand of M. If M is 1-torsionfree, then it is projective.*

Proof. Let $P_1 \to P_0 \xrightarrow{\varphi} M$ be a finite projective presentation of M. As P_0^* is a finite projective, sequence (A) shows that λM is finitely generated. As P_1^* is projective and Λ is right semihereditary, λM is also projective. Assume now that M is stable. By Prop. 5, we have $t(M) = M$. Thus, in general, we have an isomorphism $M \simeq t(M) \coprod Q$, where Q is a finitely generated projective. This proves both parts of the proposition. □

Under additional assumptions that Λ is semiperfect and M is finitely presented, Prop. 5 can be made a bit more precise.

Proposition 7. *Suppose Λ is a semiperfect ring and M a finitely presented Λ-module. Then the following are equivalent:*

1. *$M = t(M)$.*
2. *The map $\pi : P_0^* \to \lambda M$ is an isomorphism.*
3. *λM is isomorphic to P_0^*.*

If M^* is finitely generated, then the above conditions are equivalent to

4. *The map $\pi^* : (\lambda M)^* \to P_0^{**}$ is an isomorphism.*

Proof. The implication (1) ⇒ (2) was already shown. The implication (2) ⇒ (3) is trivial. Assume (3) holds. Then sequence (A) shows that π is a surjective endomorphism of a finite projective module; hence its kernel is a direct summand. The reduction of π modulo the Jacobson radical, being a surjective endomorphism of a module of finite length, is an isomorphism. Thus the reduction of the kernel of π is zero and therefore the kernel is trivial. Hence π is an isomorphism. Sequence (A) now shows that $M^* = 0$. This implies (1). Finally, assuming that M^* is finitely generated, we shall show the equivalence (2) ⇔ (4). The implication (2) ⇒ (4) is trivial. Assume (4) holds. Since M^* is finitely generated, λM is finitely presented. Exact sequence (B) shows that $\mathrm{Tr}\,(\lambda M)$ is projective. By Prop. 3, e), $\mathrm{Tr}\,(\lambda M)$ is stable. Thus $\mathrm{Tr}\,(\lambda M) = 0$. By Prop. 3, f), λM is projective. Dualizing π^* and using the fact that finite projectives are reflexive, we have (2). □

4. The main theorem and first applications

Our goal in this section is to give a necessary and sufficient condition for the 1-torsion submodule to contain a minimal generator of the ambient module. The ring Λ will be semiperfect and two-sided noetherian.

Theorem 8. *Let Λ be a two-sided noetherian semiperfect ring and M a finitely generated Λ-module. Then the 1-torsion submodule $t(M)$ contains a minimal generator of M if and only if λM has a nonzero projective summand.*

Proof. The constructions (and notation) used above are collected in the following commutative diagram:

(C)
$$\begin{tikzcd}
P_1^{**} \arrow[rr, "\partial^{**}"] \arrow[dr] & & P_0^{**} \arrow[rr, dotted, "\alpha^*"] \arrow[dr, dotted, two heads] \arrow[drr, "\varphi^{**}"] & & Q^* \arrow[r, dotted, two heads] & \mathrm{Tr}\,(\lambda M) \\
& (\lambda M)^* \arrow[ur, dotted, hook, "\pi^*"] & & \mathrm{Im}\, e_M \arrow[r, hook] & M^{**} \arrow[u, "\psi^*"'] & \\
P_1 \arrow[uu, "\cong"', "e_{P_1}"] \arrow[rr, "\partial"] & & P_0 \arrow[uu, "\cong"', "e_{P_0}"] \arrow[rr, two heads, "\varphi"] & & M \arrow[u, "e_M"'] \arrow[ul, dotted, two heads] & \\
& & & t(M) \arrow[ur, dotted, hook, "\iota"] & &
\end{tikzcd}$$

where the complexes consisting of dotted arrows are exact (assuming that the epimorphisms are followed by maps to the zero module and the monomorphisms are preceded by maps from the zero module). The two shorter complexes of such type give rise to the following commutative diagram with exact rows and columns:

(D)
$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & & \\
& \Omega M \arrow[r, equal] \arrow[d] & \Omega M \arrow[d] & & \\
0 \arrow[r] & (\lambda M)^* \arrow[r, "\pi^*"] \arrow[d] & P_0^{**} \arrow[r] \arrow[d, "\varphi e_{P_0}^{-1}"] & \mathrm{Im}\, e_M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & t(M) \arrow[r, "\iota"] \arrow[d] & M \arrow[r] \arrow[d] & \mathrm{Im}\, e_M \arrow[r] & 0 \\
& 0 & 0 & &
\end{tikzcd}$$

The map $\varphi e_{P_0}^{-1}$, being the composition of an isomorphism and a projective cover, is an isomorphism modulo the Jacobson radical J of Λ. By Nakayama's lemma, $t(M)$ contains a minimal generator of M if and only if $t(M)$ is not contained in JM. Reducing the south-west square modulo J, we see that this happens precisely when$(\lambda M)^*$ is not contained in JP_0^{**}. In view of Prop. 3, a), this is equivalent to saying that the map α^* in diagram (C) is not a minimal presentation of $\mathrm{Tr}\,(\lambda M)$. By Prop. 3, c), (with λM in place of M) this is equivalent to saying that λM has a nonzero projective summand. □

Corollary 9. *Under the assumptions of Th. 8, the* 1*-torsion submodule* $t(M)$ *contains a minimal generator of* M *if and only if* $(\lambda M)^*$ *and* P_0^{**} *have a common nonzero projective summand under the map* π^*.

Proof. This is just a reformulation of the theorem. In view of Lemma 2, the "only if" part was already shown at the end of the proof of the theorem. Suppose now that there is a common nonzero projective summand. Then sequence (B) is not a minimal presentation of $\operatorname{Tr}(\lambda M)$ and we are done by Prop. 3, b). □

Remark. Diagram (C) and the snake lemma show that the maps in the leftmost column of diagram (D) coincide with the maps in the leftmost column of the diagram

$$\begin{tikzcd}
 & & & t(M) \arrow[d] & \\
0 \arrow[r] & \Omega M \arrow[r] \arrow[d, tail] & P_0 \arrow[r, "\varphi"] \arrow[d, "\cong"', "e_{P_0}"] & M \arrow[r] \arrow[d, "e_M"] & 0 \\
0 \arrow[r] & (\lambda M)^* \arrow[r, "\pi^*"] \arrow[d, two heads] & P_0^{**} \arrow[r, "\varphi^{**}"] & M^{**} & \\
 & t(M) & & &
\end{tikzcd}$$

Since in the stable category the images of the maps going into the southeast corner are isomorphic to $\lambda^2 M$, in that category the map $\Omega M \to (\lambda M)^*$ is the result of applying the functor Ω to the map $M \to \lambda^2 M$ (equivalently, $M \to \operatorname{Im} e_M$ or $M \to M^{**}$). The short exact sequence $0 \to \Omega M \to (\lambda M)^* \to t(M) \to 0$ yields the following.

Corollary 10. *Under the assumptions of Th. 8, if $t(M) = 0$, then ΩM and $(\lambda M)^*$ are isomorphic. If Λ is artin, then the converse is true.*

Proof. The first assertion is immediate. The second follows from the fact than an injective endomorphism of a module of finite length is an isomorphism. □

As a consequence of the proof of Th. 8, we can now quantify the extent to which the 1-torsion submodule $t(M)$ "penetrates" the top of M (i.e., M/JM).

Definition 11. *Let $\mathsf{T}(M)$ be the submodule of M generated by the elements of $t(M)$ not contained in JM.*

Proposition 12. *Suppose $\lambda M \simeq \underline{\lambda M} \coprod S$, where $\underline{\lambda M}$ is stable and S is projective. Then, in the notation of Th. 8, $\varphi e_{P_0}^{-1}|_{S^*} : S^* \to \mathsf{T}(M)$ is a projective cover.*

Proof. Prop. 3, d) shows that S^* is a maximal common projective summand of $(\lambda M)^*$ and P_0^{**}. Reducing modulo J the south-west commutative square in diagram (D), we see that the statement of this corollary correctly identifies $\mathsf{T}(M)$ as the image of the restriction map in question. That this map is a projective cover follows from the fact that it is an isomorphism modulo the radical. □

Suppose Λ is a local ring. Then the top of any finitely generated projective Λ-module becomes a vector space over the residue skew field $\Lambda/J\Lambda$ and we have

Proposition 13. *If Λ is a two-sided noetherian local ring, then the dimension of the vector subspace of the top of M generated by the image of $\mathsf{T}(M)$ (equivalently, by the image of $t(M)$) equals the rank of a maximal projective summand of λM.*

Remarks. a) The last proposition can be quickly proved by an argument which does not appeal to Th. 8. Let f-rank M denote the rank of a maximal projective (i.e., free) summand of M and $b(M)$ the minimal number of generators of M. Using the definition of the operator λ and Prop. 3, d), we have

$$b(\lambda M) = b(M) - \text{f-rank}\, M.$$

Applying this formula twice, we have

$$b(\lambda^2 M) = b(M) - \text{f-rank}\, \lambda M - \text{f-rank}\, M.$$

Lemma 4 gives rise to a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & t(M) \arrow[r] & M \arrow[r] & \lambda^2 M \coprod \Lambda^{\text{f-rank}\, M} \arrow[r] & 0, \end{tikzcd}$$

which shows that $b(\mathsf{T}(M)) = b(M) - b(\lambda^2 M) - \text{f-rank}\, M$. In view of the previous formula, this equals f-rank λM.

b) For any ring Λ and any superfluous epimorphism $f : M \to N$ of finite Λ-modules we have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & t(M) \arrow[r, "i_M"] \arrow[d] & M \arrow[r, "e_M"] \arrow[d, "f"] & M^{**} \arrow[d, "f^{**}"] \\ 0 \arrow[r] & t(N) \arrow[r, "i_N"] & N \arrow[r, "e_N"] & N^{**} \end{tikzcd}$$

Let J be the radical of Λ and suppose that $i_M \otimes \Lambda/J \neq 0$. Since $f \otimes \Lambda/J$ is an isomorphism, we have that $i_N \otimes \Lambda/J \neq 0$.

5. Applications to local algebra

We can now offer another perspective on the results of Reiffen-Vetter and Scheja on hypersurface algebras.

Proposition 14. *Let R be a commutative noetherian local ring, $a_1, \dots, a_n$, where $n \geq 1$, elements of R generating a nonzero proper ideal $\mathfrak{a} \subsetneq R$, and M an R-module with presentation*

$$\begin{tikzcd} R \arrow[r, "{[a_1 a_2 \dots a_n]^T}"] & R^n \arrow[r] & M \arrow[r] & 0. \end{tikzcd}$$

Then $\mathsf{T}(M)$ is nonzero if and only if $\mathfrak{a}$ is a principal ideal generated by a nonzerodivisor. In that case, the 1-torsion submodule $t(M)$ is a direct summand of M.[10]

Proof. If $\mathfrak{a}$ is a principal ideal generated by $a \in R$, then for $i = 1, \dots, n$ there are elements $b_i \in R$ and $c_i \in R$ such that $a_i = b_i a$ and $a = \sum_{i=1}^n c_i a_i$. Therefore $(1 - \sum_{i=1}^n c_i b_i)a = 0$. Since R is local and $(a) = \mathfrak{a} \neq 0$, one of the b_i (and the corresponding c_i) must be a unit. Thus one of the a_i generates $\mathfrak{a}$ and M has a presentation

$$\begin{tikzcd} R \arrow[r, "{[a\, 0 \dots 0]^T}"] & R^n \arrow[r] & M \arrow[r] & 0. \end{tikzcd}$$

This shows that $M \simeq R/(a) \coprod R^{n-1}$. By assumption, a is neither the zero element nor a unit. Therefore the obtained presentation is minimal and $\lambda M \simeq \mathfrak{a} = (a)$. When a is a nonzerodivisor, this module is isomorphic to R, showing that $\mathsf{T}(M) \neq (0)$. In that case $(R/(a))^* \simeq \operatorname{Ann} a = (0)$ and thus $t(M) = R/(a) = \mathsf{T}(M)$.

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[10] Under an additional assumption that the ideal $\mathfrak{a}$ contains a nonzerodivisor, this result was also proved in [18], Hilfsatz (9.10).

To prove the other implication, we may assume that $\mathfrak{a}$ is minimally generated by $a_1, \dots, a_n$, thus making the defining presentation of M minimal. By Th. 8, $\lambda M \simeq \mathfrak{a}$ has a nonzero projective summand, say, $\mathfrak{a} \simeq \mathfrak{a}_1 \coprod \mathfrak{a}_2$ with $\mathfrak{a}_2$ isomorphic to R. Since R is commutative, $\mathfrak{a}_1\mathfrak{a}_2$ is contained in both $\mathfrak{a}_1$ and $\mathfrak{a}_2$ and is therefore zero. Since a nonzero element cannot annihilate the identity of R, we must have $\mathfrak{a}_1 = (0)$ and therefore $\mathfrak{a} \simeq R$. This shows that $\mathfrak{a}$ is principal and generated by a nonzerodivisor. □

Example. Let $R := k[|x, y|]/(x^6 - x^2y^3 - y^5)$, where k is a field of characteristic 0. The extension of the Jacobian ideal $(6x^5 - 2xy^3, -3x^2y^2 - 5y^4)$ of this curve to R is nonprincipal, and therefore there are no torsion elements among minimal generators of the module of differentials, i.e., $\mathsf{T}(D_k(R)) = 0$. Assume now that $\operatorname{char} k = 2$. Then the extension of the Jacobian ideal $(x^2y^2 + y^4)$ is generated by a nonzerodivisor and, therefore, the torsion submodule of the module of differentials reaches the top of the module. In this case, $\lambda D_k(R)$ is free of rank one, since it is isomorphic to the ideal generated by the image of the nonzero partial derivative. Consequently, $\mathsf{T}(D_k(R))$ is a nontrivial cyclic module.

Assume once again that R is a commutative noetherian local ring. As another application of Th. 8, we shall show that if the transpose of the module M is of large enough depth, then $\mathsf{T}(M) = 0$. First we recall an auxiliary result ([4], Lemma 4.7; see also [11], Prop. 3 for a proof inspired by the present paper.)

Lemma 15. *Let N be a finitely generated R-module. Then $\Omega^i N$ has no nonzero free summands for $i > \max(\operatorname{depth} R - \operatorname{depth} M, 0)$*

Combining this with Th. 8 and recalling that $\lambda M = \Omega \operatorname{Tr} M$, we have

Proposition 16. *Let M be a finitely generated R-module such that $\operatorname{depth} \operatorname{Tr} M \geq \operatorname{depth} R$. Then the 1-torsion submodule of M contains no minimal generators of M.*

As a consequence, we have a surprising result in dimension zero.

Proposition 17. *Let R be a zero-dimensional commutative noetherian local ring and M an arbitrary finitely generated R-module. Then the 1-torsion submodule of M contains no minimal generators of M. In particular, the 1-torsion submodule of M, if it is nonzero, is not a direct summand of M.*

6. Further applications to local algebra: 1-torsion and Tate-Vogel cohomology

For finite modules over a commutative local ring, the absence of free summands can be detected by the vanishing of the ξ-invariant, introduced by the author in [9]. This nonnegative integer is the dimension of the kernel of the natural transformation from the cohomology of the module with coefficients in the residue field to the Tate-Vogel cohomology of the same pair. The details of the construction can be found in the above reference. Since in this paper we are only interested in applications, we provide a very simple equivalent definition of the ξ-invariant.

Definition 18. *Let R be a commutative noetherian local ring and M a finite R-module. Let $V(M, k)$ denote the vector subspace of $\operatorname{Hom}_R(M, k)$ consisting of bounded maps, i.e., the maps that admit a lifting with only finitely many nonzero components to some projective resolutions of M and k. We set $\xi(M) := \dim V(M, k)$ and $\xi^n(M) := \dim V(\Omega^n M, k)$. We also set $\xi^0(M) := \xi(M)$.*

Immediately from the definition we deduce that ξ is additive on direct sums and, for a module of *finite* projective dimension, coincides with the zeroth betti number. In particular, $\xi(R^n) = n$. Consequently, if $\xi(M) = 0$, then M cannot have a nonzero projective summand. Taking account of Th. 8, we have

Proposition 19. *Let R be a commutative noetherian local ring and M a finite R-module. If $\xi(\lambda M) = 0$, then the 1-torsion submodule $t(M)$ does not contain minimal generators of M.*

In order to make this useful we need to be able to compute the ξ-invariant. In general this is difficult. But in some situations ([9, 10]) this invariant has been computed. A case of interest to us is provided by the following result (Th. 3.1, [9]).

Theorem 20. *Let $(S, \mathfrak{m}, k)$ be a commutative noetherian local ring, $x \in \mathfrak{m}$ an S-regular element, $R := S/(x)$, and M a finite R-module. If $x \in \mathfrak{m}\mathrm{Ann}_S M$, then $\xi^i(M) = 0$ for all i.*

Remark. If $x \in \mathfrak{m}\mathrm{Ann}_S M$, then, clearly, the same condition holds if M is replaced by any of its quotient modules.

Proposition 21. *Let $(S, \mathfrak{m}, k)$ be a commutative noetherian local ring, $x \in \mathfrak{m}$ an S-regular element, $R := S/(x)$, and M a finite R-module. If $x \in \mathfrak{m}\mathrm{Ann}_S(\lambda_R M)$, then the 1-torsion submodule $t(M)$ does not contain minimal generators of M.*

7. 1-TORSION AS A DIRECT SUMMAND

In this section we shall give a necessary and sufficient condition for a finitely generated module to have its 1-torsion submodule as a direct summand. This will be done in a more general context than we have been working in so far: the ring will be two-sided noetherian but not necessarily semiperfect.

As a motivating example, we consider first the "hypersurface" module of Prop. 14. Let R be a commutative *domain*, $\mathfrak{a} := (a_1, \dots, a_n)$ a *nonzero* ideal of R, and M a module with presentation

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "{[a_1 \dots a_n]^T}"] & R^n \arrow[r, "\varphi"] & M \arrow[r] & 0 \end{tikzcd}$$

Problem 1. Describe the torsion submodule $t(M)$ of M.

Solution. Suppose $x \in M$ is torsion: there is $\mu \neq 0$ in R such that $\mu x = 0$. Choose $x_1, \dots, x_n \in R$ such that $\varphi((x_1 \dots x_n)^T) = x$; then $\mu x_i = \lambda a_i$, $i = 1, \dots, n$ for some $\lambda \in R$. In other words, $x_i = (\lambda/\mu) a_i$, $i = 1, \dots, n$ in the field of quotients K of R. Since each x_i is in R, we must have $(\lambda/\mu) \in (R : \mathfrak{a})$. As a result, $(x_1 \dots x_n)^T$ is in the image of the R-linear map

$$f : (R : \mathfrak{a}) \to R^n : \lambda/\mu \mapsto (\lambda/\mu)(a_1 \dots a_n)^T$$

Conversely, it is immediate that any element in the image of f gives rise, after applying φ, to a torsion element of M. The canonical inclusions $R \to (R : \mathfrak{a})$ and $\iota : t(M) \to M$ now become parts of a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & R \arrow[r] \arrow[d, equal] & (R:\mathfrak{a}) \arrow[r, dashed] \arrow[d, "f"] & t(M) \arrow[r] \arrow[d, "\iota"] & 0 \\
0 \arrow[r] & R \arrow[r, "{[a_1 \ldots a_n]^T}"] & R^n \arrow[r, "\varphi"] & M \arrow[r] & 0
\end{tikzcd}$$

This diagram describes both the torsion submodule $t(M)$ and its embedding in M.

Corollary 22. *M is torsion-free if and only if $(R:\mathfrak{a}) = R$.*

Completing the columns of the above diagram we also have the following description of $\operatorname{Im} e_M \simeq \operatorname{Coker} \iota$:

Corollary 23. *The sequence*

$$0 \longrightarrow (R:\mathfrak{a}) \xrightarrow{f} R^n \xrightarrow{e_M\varphi} \operatorname{Im} e_M \longrightarrow 0$$

is exact.

The next problem appears as an exercise in ([5], Ch. VII, §1, Ex. 32).

Problem 2. Under the above assumptions, show that $t(M)$ is a direct summand of M if and only if

$$\mathfrak{a}(R:\mathfrak{a}) + (R:(R:\mathfrak{a})) = R.$$

Lemma 24. *$(R:(R:\mathfrak{a})) \subseteq R$.*

Proof. Since $(R:\mathfrak{a}) \supseteq R$, we have $(R:(R:\mathfrak{a})) \subseteq (R:R) = R$. □

The lemma shows that the left-hand side of the desired equality is contained in R. Thus we have to show that the torsion is a direct summand if and only if the identity of R belongs to the left-hand side.

First, assume that the embedding $\iota : t(M) \to M$ admits a splitting $p : M \to t(M)$. Using the lifting property of the projective resolution of M, we obtain a commutative diagram of R-linear maps with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r] \arrow[d, equal] & (R:\mathfrak{a}) \arrow[r] \arrow[d, "f"] & t(M) \arrow[r] \arrow[d, "\iota"] & 0 \\
0 \arrow[r] & R \arrow[r, "{[a_1 \ldots a_n]^T}"] \arrow[d, dashed, "\sigma"] & R^n \arrow[r, "\varphi"] \arrow[d, dashed, "g"] & M \arrow[r] \arrow[d, "p"] & 0 \\
0 \arrow[r] & R \arrow[r] & (R:\mathfrak{a}) \arrow[r] & t(M) \arrow[r] & 0
\end{tikzcd}$$

We now examine the maps g and σ. Let $g(e_i) := b_i \in (R:\mathfrak{a})$, $i = 1, \ldots, n$, where e_i is the ith standard basis vector. The commutativity of the south-west square implies then that $\sigma = \sum a_i b_i \in \mathfrak{a}(R:\mathfrak{a})$. Since $\mathrm{Id}_{t(M)} - p\iota$ is the zero map, there exists an R-linear map $h : (R:\mathfrak{a}) \to R$ such that $(1-\sigma) \cdot r = h(r)$ for any $r \in R$. The map h can be computed explicitly. Indeed, if $\lambda/\mu \in (R:\mathfrak{a})$, then $\mu \cdot h(\lambda/\mu) = h(\lambda) = (1-\sigma) \cdot \lambda$ and, therefore, $h(\lambda/\mu) = (1-\sigma) \cdot \lambda/\mu$. Since the image of h is in R, we must have $(1-\sigma) \in (R : (R : \mathfrak{a}))$ thus obtaining the desired decomposition of the identity: $1 = \sigma + (1-\sigma)$.

Conversely, suppose $1 = \sigma + (1-\sigma)$, where $\sigma \in \mathfrak{a}(R : \mathfrak{a})$ and $1-\sigma \in (R : (R : \mathfrak{a}))$. Writing σ as $\sum a_i b_i$ with all $b_i \in (R : \mathfrak{a})$, and setting $g(e_i) := b_i$ for each i, we recover the above diagram. By construction, $\mathrm{Id}_{(R:\mathfrak{a})} - gf$ is multiplication by $1-\sigma$, the latter being an element of $(R : (R : \mathfrak{a}))$. Therefore, $\mathrm{Id}_{(R:\mathfrak{a})} - gf$ factors through R, showing that $\mathrm{Id}_{t(M)} - p\iota = 0$. This solves Problem 2.

Remark. Using Cor. 23 we can provide an alternative solution to Problem 2. The 1-torsion submodule of M is a direct summand if and only if the canonical map $e_M : M \to \operatorname{Im} e_M$ is a split epimorphism. Suppose there is a map $i : \operatorname{Im} e_M \to M$ such that $e_M i = \mathrm{Id}_{\operatorname{Im} e_M}$. Lifting i by maps k and l, we have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & (R : \mathfrak{a}) \arrow[r, "f"] \arrow[d, "l"] & R^n \arrow[r, "e_M\varphi"] \arrow[d, "k"] & \operatorname{Im} e_M \arrow[r] \arrow[d, "i"] & 0 \\
0 \arrow[r] & R \arrow[r, "{[a_1 \ldots a_n]^T}"] \arrow[d, tail, "j"] & R^n \arrow[r, "\varphi"] \arrow[d, equal] & M \arrow[r] \arrow[d, "e_M"] & 0 \\
0 \arrow[r] & (R : \mathfrak{a}) \arrow[r, "f"] & R^n \arrow[r, "e_M\varphi"] & \operatorname{Im} e_M \arrow[r] & 0
\end{tikzcd}$$

Here j is the canonical inclusion. As $\mathrm{Id}_{\operatorname{Im} e_M} = e_M i$, there is a map $h : R^n \to (R : \mathfrak{a})$ such that $fh = \mathrm{Id}_{R^n} - k$. This is equivalent to saying that $hf = \mathrm{Id}_{(R:\mathfrak{a})} - jl$. Let $h(e_i) := b_i \in (R : \mathfrak{a})$, where e_i is the ith standard basis vector. Then hf is just multiplication by $\sigma := \sum a_i b_i \in \mathfrak{a}(R : \mathfrak{a})$ and therefore $\mathrm{Id}_{(R:\mathfrak{a})} - hf$ is multiplication by $1-\sigma$. On the other hand, since the latter factors through R as the composition jl, the image of this map must be in R. Consequently, $1 - \sigma \in (R : (R : \mathfrak{a}))$.

Conversely, suppose there is $\sigma \in \mathfrak{a}(R : \mathfrak{a})$ such that $1 - \sigma \in (R : (R : \mathfrak{a}))$. Our immediate goal is to recover the triple-decker diagram above. Let $\sigma = \sum_1^n a_i b_i$, with each b_i in $(R : \mathfrak{a})$. We first define a map $h : R^n \to (R : \mathfrak{a})$ by setting $h(e_i) := b_i$ for each i. This allows to define a map $k : R^n \to R^n$ by setting $k := \mathrm{Id}_{R^n} - fh$, and a map $c : (R : \mathfrak{a}) \to (R : \mathfrak{a})$ by setting $c := \mathrm{Id}_{(R:\mathfrak{a})} - hf$. Since $hf = \sigma$, the last map is just multiplication by $1 - \sigma \in (R : (R : \mathfrak{a}))$, and therefore its image must be in R. In other words, c factors through R, i.e., $c = jl$, where j is the canonical inclusion $R \to (R : \mathfrak{a})$ and l is a map $(R : \mathfrak{a}) \to R$. It is now straightforward to check that the pair l, k gives rise to a map $i : \operatorname{Im} e_M \to M$ and that $e_M i = \mathrm{Id}_{\operatorname{Im} e_M}$.

We now switch to a general context: Λ is a two-sided noetherian ring and M a finitely generated (left) Λ-module. Diagram (D) of Sec. 3 provides the following lifting of the canonical inclusion $\iota : t(M) \to M$:

$$\begin{tikzcd}
0 \arrow[r] & \Omega M \arrow[r] \arrow[d, equal] & (\lambda M)^* \arrow[r] \arrow[d, tail, "e_{P_0}^{-1}\pi^*"] & t(M) \arrow[r] \arrow[d, tail, "\iota"] & 0 \\
0 \arrow[r] & \Omega M \arrow[r] & P_0 \arrow[r, "\varphi"] & M \arrow[r] & 0
\end{tikzcd}$$

Suppose now that ι admits a splitting $p : M \to t(M)$. Lifting p, we have a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \Omega M \arrow[r] \arrow[d, equal] & (\lambda M)^* \arrow[r] \arrow[d, tail, "e_{P_0}^{-1}\pi^*"] & t(M) \arrow[r] \arrow[d, tail, "\iota"] & 0 \\
0 \arrow[r] & \Omega M \arrow[r] \arrow[d, "\sigma"] & P_0 \arrow[r, "\varphi"] \arrow[d, "g"] & M \arrow[r] \arrow[d, "p"] & 0 \\
0 \arrow[r] & \Omega M \arrow[r] & (\lambda M)^* \arrow[r] & t(M) \arrow[r] & 0
\end{tikzcd}$$

Since $\mathrm{Id}_{t(M)} - p\iota$ is the zero map, there exists a Λ-linear map $h : (\lambda M)^* \to \Omega M$ such that $\mathrm{Id}_{(\lambda M)^*} - ge_{P_0}^{-1}\pi^*$ factors through h.

Conversely, given Λ-linear maps $g : P_0 \to (\lambda M)^*$ and $h : (\lambda M)^* \to \Omega M$ such that $\mathrm{Id}_{(\lambda M)^*} - ge_{P_0}^{-1}\pi^*$ factors through h, define $\sigma : \Omega\, M \to \Omega\, M$ by setting $\sigma := \mathrm{Id}_{\Omega M} - h\,|_{\Omega M}$. It is then clear that g and σ are part of a commutative square as above, and therefore they give rise to a map $p : M \to t(M)$. It is also clear that $p\iota = \mathrm{Id}_{t(M)}$. Thus we have

Proposition 25. *Let Λ be a two-sided noetherian ring and M a finitely generated (left) Λ-module. Then the 1-torsion submodule $t(M)$ is a direct summand of M if and only if there is a Λ-linear map $g : P_0 \to (\lambda M)^*$ such that $\mathrm{Id}_{(\lambda M)^*} - ge_{P_0^{-1}}\pi^*$ admits a lifting $h : (\lambda M)^* \to \Omega M$.*

Similar to the remark on p. 15, we can give an alternative criterion for 1-torsion being a direct summand. Suppose the canonical map $M \to \operatorname{Im} e_M$ admits a splitting $i : \operatorname{Im} e_M \to M$. Augmenting the notation of Th. 8, we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & (\lambda M)^* \arrow[r, "e_{P_0}^{-1}\pi^*"] \arrow[d, "l"] & P_0 \arrow[r, "e_M\varphi"] \arrow[d, "k"] & \operatorname{Im} e_M \arrow[r] \arrow[d, "i"] & 0 \\
0 \arrow[r] & \Omega M \arrow[r, "\nu"] \arrow[d, "j"] & P_0 \arrow[r, "\varphi"] \arrow[d, equal] & M \arrow[r] \arrow[d, "e_M"] & 0 \\
0 \arrow[r] & (\lambda M)^* \arrow[r, "e_{P_0}^{-1}\pi^*"] & P_0 \arrow[r, "e_M\varphi"] & \operatorname{Im} e_M \arrow[r] & 0
\end{tikzcd}$$

where the maps l and k are some liftings of the map i. (Once again, for the sake of simplicity, we have slightly abused the notation for the maps going into the southeast corner.) Since $\mathrm{Id}_{\operatorname{Im} e_M} - e_M i = 0$, there is a Λ-linear map $h : P_0 \to (\lambda\, M)^*$ such that $\mathrm{Id}_{P_0} - k = e_{P_0}^{-1}\pi^* h$. This implies that $\mathrm{Id}_{(\lambda\, M)^*} - jl = he_{P_0}^{-1}\pi^*$.

Conversely, suppose there are Λ-linear maps $l : (\lambda M)^* \to \Omega M$ and $h : P_0 \to (\lambda\, M)^*$ such that $\mathrm{Id}_{(\lambda\, M)^*} - jl = he_{P_0}^{-1}\pi^*$. Define $k := \mathrm{Id}_{P_0} - e_{P_0}^{-1}\pi^* h$. It is then clear that k and l are part of a commutative square as above and therefore they give rise to a map $i : \operatorname{Im} e_M \to M$. It is also clear that $e_M i = \mathrm{Id}_{\operatorname{Im} e_M}$. Thus we have proved

Proposition 26. *Let Λ be a two-sided noetherian ring and M a finitely generated (left) Λ-module. Then the 1-torsion submodule $t(M)$ is a direct summand of M if* *and only if there is a* Λ*-linear map* $l : (\lambda M)^* \to \Omega M$ *such that* $\mathrm{Id}_{(\lambda M)^*} - jl$ *admits an extension* $h : P_0 \to (\lambda M)^*$.

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