Matrix Reductions Methods and Differential Graded Algebras

In 1975, M. Kleiner and A.V. Roiter introduced Differential Graded Categories (dgc) with the purpose of study in a systematic way the different methods, developed in Kiev, for solving classification problems of matrices (see [5]). Later, in 1979, A.V. Roiter introduced the Bimodules over Categories with Structure of Coalgebra (bocs)(see [6]). The concepts of bocses and dgc are closely related.

The main feature of the theory of representations of bocses (and dgc) is the existence of procedures which allow to go from representations of a bocs to representations of smaller dimension of another bocs. This gives a powerful method of descent, which has been succesfully used in proving central results in representation theory of finite-dimensional algebras over algebraically closed fields (see [4], [3], [2]).

In this series of talks we consider a category of representations of certain differential k-algebras. Those categories are equivalent to the category of representations of a normal bocs in the sense of Roiter.

More specifically, let A be a k-algebra, V an $A-A$-bimodule and $T_A(V)$ the corresponding tensor algebra with the standard grading. A differential tensor algebra (ditalgebra) is a differential algebra of the form $(T_A(V), \delta)$ with δ a differential of $T_A(V)$ of degree one.

Given a ditalgebra $\mathcal{A} = (T_A(V), \delta)$, the category $\operatorname{Rep}\mathcal{A}$ of representations of $\mathcal{A}$ is defined as follows: the objects of $\operatorname{Rep}\mathcal{A}$ are left A-modules. For M, N left A-modules, a morphism from M to N in $\operatorname{Rep}\mathcal{A}$ are given by pairs $f = (f^0, f^1)$ with $f^0 \in \operatorname{Hom}_k(M,N)$ and $f^1 \in \operatorname{Hom}_{A-A}(V, \operatorname{Hom}_k(M,N))$ such that

$$f^0(am) = af^0(m) + f^1(\delta a)(m)$$

for all $m \in M$ and $a \in A$.

If $f = (f^0, f^1) : M \to N$ and $g = (g^0, g^1) : N \to L$ are morphisms, their composition is defined by $gf = (g^0 f^0, (gf)^1)$ with

$$(gf)^1(v) = g^0 f^1(v) + g^1(v) f^0 + \sum g^1(v_i^1) f^1(v_i^2),$$

where $v \in V$ and $\delta(v) = \sum v_i^1 v_i^2$ with $v_i^1, v_i^2 \in V$. We denote by $\operatorname{rep}\mathcal{A}$ the full subcategory of $\operatorname{Rep}\mathcal{A}$ whose objects are those M with $\dim_k M$ finite.

Our main interest is the study of layered ditalgebras (closely related to the layered bocses of W.W. Crawley-Boevey). A layer of $\mathcal{A} = (T_A(V), \delta)$ is a triple $L = (R, W_0, W_1)$, where R is a k-algebra, W_0 and W_1 are R-bimodules such that $A = T_R(W_0)$ and $V = A \otimes_R W_1 \otimes_R A$, plus some extra conditions on W_0 and W_1. We will prove that if $\mathcal{A}$ is a layered ditalgebra then $\operatorname{rep}\mathcal{A}$ is a Krull-Schmidt category and in case $\dim_k A$ is finite, $\operatorname{rep}\mathcal{A}$ has almost split sequences [1].

Let $\mathcal{A} = (T_A(V), \delta)$ be a bocs with layer $L = (R, W_0, W_1)$, suppose $X \in \operatorname{Mod} R$ is such that $\operatorname{End}_R(X)$ contains a k subalgebra S such that $\operatorname{End}_R(X) = S \oplus \mathcal{B}$ with $\mathcal{B}$ an ideal of $\operatorname{End}_R(X)$ such that it is a finitely generated projective as a right S-module. We will construct a new layered ditalgebra $\mathcal{A}^X$ and a full and faithful functor $F^X : \operatorname{Rep}\mathcal{A}^X \to \operatorname{Rep}\mathcal{A}$ such that for $N \in \operatorname{Rep}\mathcal{A}$ there is a

$M \in \operatorname{Rep} \mathcal{A}^X$ with $F^X(M) \cong N$ if and only if ${}_R N$ is a sum of direct summands of X.

In many cases of interest we may find X such that the category $\operatorname{Rep} \mathcal{A}^X$ is easy to describe. In such cases is easy to transport properties in $\operatorname{Rep} \mathcal{A}^X$ to properties in $\operatorname{Rep} \mathcal{A}$. The generic modules were discovered in this way (see [3]). We will study the functors F^X for specific X.

Finally, we will see how to study $\operatorname{Mod} \Lambda$ through the use of layered ditalgebras, for Λ a finite dimensional algebra over an algebraically closed field k.

References

[1] R. Bautista, M.Kleiner, *Almost split sequences for relatively projective modules.* J. Algebra **135** (1990) 19-56.

[2] W.W. Crawley-Boevey. *On tame algebras and bocses.* Proc. London Math. Soc. (3) **56** (1988) 451-483.

[3] W.W. Crawley-Boevey. *Tame algebras and generic modules.* Proc. London Math. Soc. **63** (1991) 241-264.

[4] Y.A. Drozd, *Tame and wild matrix problems.* Amer. Math. Soc. Transl. **128**(2) (1986) 31-55.

[5] M. Kleiner, A.V. Roiter, *Representations of differential graded categories*, Lecture Notes in Mathematics **488**, Springer-Verlag, Berlin (1975) 316-339.

[6] A.V. Roiter, *Matrix problems and representations of bocs's.* Lecture Notes in Mathematics **831**, Springer-Verlag, Berlin (1980) 288-324.