CONFORMAL ALGEBRAS AND RELATED STRUCTURES

Vertex algebras first entered mathematics in the early 1980's as a central notion of modern mathematical physics. Since then, they have become "ubiquitous in the representation theory of infinite-dimensional Lie algebras" (to quote [4]), led to a deeper understanding of sporadic simple groups, in particular, the Monster in the works of Borcherds and Frenkel, Lepowsky, and Meurman, and provided rigorous foundations for conformal field theory. They also proved crucial in such diverse mathematical subjects as modular forms, mirror symmetry, and combinatorics. However, from a purely algebraic point of view vertex algebras remain rather mysterious objects. Many believe that they are just first examples of a new kind of "tensorial" mathematical structures (perhaps best understood in terms of pseudotensor categories of Beilinson and Drinfeld).

Roughly speaking, a vertex algebra is an infinite-dimensional vector space V such that its every element $v \in V$ gives rise to a map into the Laurent series over V, $Y(v,z) : V \to V((z))$ (this is known in physics as the state-field correspondence). The operators $Y(v,z)$ should satisfy a number of conditions, of which the major one is locality: the formal product $Y(v_1,z_1)Y(v_2,z_2)$ can be analytically continued to a meromorphic function on the (z_1,z_2)-plane with poles only on the lines $z_i = 0$ and the diagonal. In a more formal language, the bracket of $Y(v_1,z_1)$ and $Y(v_2,z_2)$ must be a finite linear combination of the delta-function on the diagonal of the (z_1,z_2)-plane. Even more formally, we obtain a map that sends a pair of mutually local series to a finite collection of series, i.e., a kind of a "generalized" multiplication.

The significance of such a map was noticed about a decade ago and led to the definition of a conformal algebra. A conformal algebra R is a space with a bilinear map $R \otimes R \to$ (polynomials over R) and an action of a derivation ∂. In a similar vein, one defines conformal modules, cohomology of conformal algebras, and so on. Every conformal algebra R can be represented as a space of series over an (ordinary) algebra A, hence gives an additional structure for A. In such a way, in the case of Lie algebras, conformal algebras provide an excellent formalism for working with algebras such as the Virasoro and affine Kac–Moody algebras or subalgebras of $\widehat{gl}_\infty$. It appears that the condition of being a coefficient algebra (resp. module) determines an important class of algebras (resp. modules) containing especially nice examples of infinite-dimensional such; however, the importance of this condition is not yet understood completely. Additional applications of conformal algebras include the already mentioned vertex algebras, superconformal algebras of superstring theory, and Hamiltonian formalism in the theory of non-linear evolution equations.

The plan of these lectures is roughly as follows:

1. Conformal algebras: examples, structure theory, representations.
2. Pseudotensor categories and pseudoalgebras.
3. Vertex algebras: examples and applications.

References

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