

## Chiral Polytopes – Tapas 2009

Today I'll be talking about chiral polytopes, which are "half-regular" in the sense that they have full rotational symmetry but not full reflectional symmetry. In order to make this idea precise, let us briefly recap what Mark talked about on Tuesday with Abstract Regular Polytopes.

Briefly, an *abstract polytope* is a ranked poset  $\mathcal{P}$  such that:

- (i)  $\mathcal{P}$  has a single maximal element (face) and a single minimal element (face),
- (ii) All flags of  $\mathcal{P}$  have the same length,
- (iii) All sections of  $\mathcal{P}$  are connected, and
- (iv) Given a face  $F$  of rank  $i$  and a face  $H$  of rank  $i+2$  such that  $F < H$ , there are precisely two faces  $G_1$  and  $G_2$  (of rank  $i+1$ ) such that  $F < G_j < H$  for  $j = 1, 2$ .

An abstract polytope  $\mathcal{P}$  is regular if the action of its (combinatorial) automorphism group  $\Gamma(\mathcal{P})$  on the flags is transitive.

What we saw last time is that for a regular polytope  $\mathcal{P}$ , its automorphism group is always a certain nice quotient of a string Coxeter group; we call these groups *string C-groups*. In other words, a string C-group is generated by involutions  $\rho_0, \dots, \rho_n$  such that nonadjacent generators commute; the extra property we require is that

$$\langle \rho_j \mid j \in J \rangle \cap \langle \rho_j \mid j \in K \rangle = \langle \rho_j \mid j \in J \cap K \rangle,$$

and we call this *the intersection property with respect to the distinguished generators of  $\Gamma$* . It turns out that from a string C-group  $\Gamma$  we can build a regular polytope  $\mathcal{P}(\Gamma)$  such that  $\Gamma(\mathcal{P}(\Gamma)) = \mathcal{P}$ . In particular, letting  $\Gamma_j = \langle \rho_i \mid i \neq j \rangle$ , we define the  $j$ -faces of  $\mathcal{P}(\Gamma)$  to be the right cosets  $\Gamma_j \varphi$  with  $\varphi \in \Gamma$ .

Now, given a string C-group  $\Gamma$ , we can look at its *rotation subgroup*  $\Gamma^+$ , generated by  $\sigma_1, \dots, \sigma_n$ , where  $\sigma_i = \rho_{i-1} \rho_i$ . Now, if we let  $\rho_0$  act on the set of cosets  $\Gamma/\Gamma^+$  in the obvious way, we see that  $\rho_0 \Gamma^+$  contains all the generators  $\rho_i$ , so that  $\rho_i \Gamma^+ = \rho_j \Gamma^+$  for all  $i, j$ , and so  $\rho_i \rho_j \Gamma^+ = \Gamma^+$  for all  $i, j$ . Thus  $\Gamma^+$  has index at most 2 in  $\Gamma$ .

Given a regular polytope  $\mathcal{P}$ , we say that it is *directly regular* if the index of  $\Gamma^+$  in  $\Gamma$  is exactly 2. It follows from a result of Coxeter and Moser that  $\mathcal{P}$  is directly regular if and only if it corresponds to a regular map on an orientable surface. As an example of a polytope that is not directly regular, consider the hemicube, with group

$$\langle \rho_0, \rho_1, \rho_2 \mid \rho_i^2, (\rho_0 \rho_1)^4, (\rho_0 \rho_2)^2, (\rho_1 \rho_2)^3, (\rho_0 \rho_1 \rho_2)^3 \rangle.$$

Since  $\Gamma^+$  is of index at most 2 in  $\Gamma$ , then  $\Gamma/\Gamma^+$  is actually a group, so what happens when we kill the rotations  $\sigma_i$ ? We can rewrite the last relation as

$$\rho_0(\rho_1\rho_2)(\rho_0\rho_1)(\rho_2\rho_0)(\rho_1\rho_2) = \rho_0\sigma_2\sigma_1(\sigma_1\sigma_2)\sigma_2,$$

so that since this word is killed, and so are the rotations  $\sigma_i$ , we conclude that  $\rho_0$  is killed, and it follows that the whole group is killed. So  $\Gamma^+ = \Gamma$ . And when we sketch the hemicube, we see that it has 3 square faces, 6 edges, and 4 vertices, so that its Euler characteristic is odd, so it cannot be a regular map on an orientable surface. Indeed, we obtain the hemicube by identifying opposite faces, edges, and vertices, and so it naturally corresponds to a regular map on the real projective plane.

We are now almost ready to define chiral polytopes. The rotation group of a directly regular polytope inherits many nice properties from the full automorphism group. In particular, whenever  $i < j$  we have  $(\sigma_i \cdots \sigma_j)^2 = 1$ , and an analogous intersection property is satisfied. It is natural, then, to wonder what happens if you start with a group with these properties. Does it always correspond to the rotation group of a directly regular polytope?

Let us call a group  $G$  generated by  $\sigma_1, \dots, \sigma_n$  with the above properties a *string  $\chi$ -group*. Each such group  $G$  is a quotient of  $C^+ = [p_1, p_2, \dots, p_n]^+$ , the rotation subgroup of a string Coxeter group. In other words, we have  $G \cong C^+/N$ , where  $N \triangleleft C^+$ . Now, there is only one possible choice for  $\overline{G}$ ; we adjoin to  $G$  an external involutory automorphism  $\rho_0$  that acts on  $G$  by conjugation in the obvious way. Then it's not hard to see that we must have  $\overline{G} \cong C/N$ , so that  $N \triangleleft C$ . Now we see how this can fail; if we pick  $N \triangleleft C^+$  such that  $N$  is not normal in  $C$ , then when we adjoin  $\rho_0$  to  $G$ , part of the group collapses.

Now we are ready to define chiral polytopes.

**Definition ("global"):** A polytope  $\mathcal{P}$  is chiral if its automorphism group  $G(\mathcal{P})$  has two orbits on the flags, and such that adjacent flags are in different orbits.

We can formulate the definition in another way:

**Definition ("local"):** A polytope  $\mathcal{P}$  with base flag  $\Phi$  is chiral if it is not regular, and there exist automorphisms  $\sigma_1, \dots, \sigma_n$  such that  $\sigma_i$  fixes all faces in  $\Phi/\{F_{i-1}, F_i\}$  and cyclically permutes consecutive  $i$ -faces of  $\mathcal{P}$  in  $F_{i+1}/F_{i-2}$ .

Let's look at a simple example, again due to Coxeter and Moser: the chiral polytope  $\{4, 4\}_{(1,2)}$ . We start with the tessellation of the plane by squares, and we consider the translation group  $T$  generated by  $(1, 2)$  and  $(-2, 1)$ . Then by construction,  $T$  is normal in  $[4, 4]^+$  but not in  $[4, 4]$ . So modding out by  $T$  can't yield a regular polytope, since it would have  $[4, 4]/T$  as its automorphism group. So we get a chiral polytope (what Coxeter called an "irreflexible map"). Note that  $\{4, 4\}_{(1,2)} \cong \{4, 4\}_{(2,1)}$ ; these are the two *enantiomorphic forms*

of this polytope.

As in the case of string C-groups, it is possible to build polytopes directly from a string  $\chi$ -group  $G$ . Schulte and Asia Weiss show that the polytope  $\mathcal{P}(G)$  is either chiral or directly regular, and that the rotation group of  $\mathcal{P}(G)$  is  $G$ . Furthermore,  $\mathcal{P}(G)$  is directly regular if and only if we can lift  $G$  to  $\overline{G}$  as described earlier.

Given a chiral polytope  $\mathcal{P}$ , all of its sections are either chiral or directly regular. In particular, its facets and vertex figures are chiral or directly regular. What is surprising is that its  $(n - 2)$ -faces and edge figures must be directly regular! Why is this important? Given a regular polytope  $\mathcal{P}$ , it is possible to extend it to a regular polytope  $\mathcal{Q}$  having  $\mathcal{P}$  as facets, and to repeat this process. What the above remark says is that we can't repeatedly extend a chiral polytope to larger and larger chiral polytopes. So in general, in order to find chiral polytopes in a given rank, we pick a string Coxeter group  $C$  and look for subgroups  $N$  that are normal in  $C^+$  but not in  $C$ .

Just as with regular polytopes, we are interested in the amalgamation problem: given chiral and/or directly regular polytopes  $\mathcal{P}_1$  and  $\mathcal{P}_2$ , are there any chiral polytopes having facets isomorphic to  $\mathcal{P}_1$  and vertex figures isomorphic to  $\mathcal{P}_2$ ? And just as with regular polytopes, if there is at least one such polytope, there is a universal such polytope that covers every such polytope. There is an added subtlety here; it can matter which enantiomorphic form of a polytope you take. For instance, the universal polytope  $\{\{4, 4\}_{(1,3)}, \{4, 4\}_{(1,3)}\}^{ch}$  has an automorphism group of order 960, while  $\{\{4, 4\}_{(3,1)}, \{4, 4\}_{(1,3)}\}^{ch}$  has an automorphism group of order 2000. So we need to specify not just an isomorphism class, but an orientation.

For the remainder of the talk, I'll tell you a bit about my own research. As we saw above, there is a chiral polytope  $\{4, 4\}_{(1,2)}$  that corresponds to a square tessellation of the torus (i.e. a regular irreflexible map). This generalizes to polytopes  $\{4, 4\}_{(b,c)}$ , which are chiral when  $bc(b - c) \neq 0$ . It's then possible to build polytopes having these as facets and cubes as vertex figures; these polytopes  $\{\{4, 4\}_{(b,c)}, \{4, 3\}\}$  are chiral as long as the facets are. When are these universal polytopes finite? In other words, when is the following finitely presented group finite:

$$\Gamma_{b,c} := \langle \sigma_1, \sigma_2 \sigma_3 \mid \sigma_1^4, \sigma_2^4, \sigma_3^3, (\sigma_1 \sigma_2)^2, (\sigma_2 \sigma_3)^2, (\sigma_1 \sigma_2 \sigma_3)^2, (\sigma_1^{-1} \sigma_2)^b (\sigma_1 \sigma_2^{-1})^c \rangle?$$

In the book *Abstract Regular Polytopes*, McMullen and Schulte show that  $\Gamma_{b,0}$  is finite if and only if  $b \leq 3$ . Furthermore, by using GAP, I've found a few finite chiral groups: the pairs  $(1, 2)$ ,  $(1, 3)$ ,  $(1, 4)$ ,  $(2, 3)$ , and  $(2, 4)$  all yield finite groups. What sort of tools do we have to help analyze this problem?

It turns out that there is this nice covering relation induced by factorization in the Gaussian integers. Writing  $X$  for  $\sigma_1^{-1} \sigma_2$  and  $Y$  for  $\sigma_1 \sigma_2^{-1}$ , we see that if we conjugate the relation  $X^b Y^c = 1$  by  $\sigma_1^{-1}$ , we get  $X^{-c} Y^b = 1$  as well. Now, the translations  $X$  and  $Y$  commute, so

we can also conclude  $X^{br}Y^{cr} = 1$  and  $X^{-cs}Y^{bs} = 1$ , and therefore  $X^{br-cs}Y^{cr+bs} = 1$ . Thus, when we mod out by  $X^bY^c$ , we are also modding out by  $X^{br-cs}Y^{cr+bs}$ , which means that the latter normal subgroup is smaller, so there is a surjective map from  $\Gamma_{br-cs, cr+bs} \rightarrow \Gamma_{b,c}$ . By reversing the roles of  $b$  and  $r$  and the roles of  $c$  and  $s$  above, we see that there is also a surjective map from  $\Gamma_{br-cs, cr+bs} \rightarrow \Gamma_{r,s}$ . Now, if we associate the Gaussian integer  $u+vi$  with any group  $\Gamma_{u,v}$ , then we note that  $(br-cs) + (cr+bs)i = (b+ci)(r+si)$ , so that in general, there is a surjection from  $\Gamma_{u,v} \rightarrow \Gamma_{b,c}$  if  $b+ci$  is a divisor of  $u+vi$ . In particular, if  $\Gamma_{b,c}$  is infinite, then so is  $\Gamma_{u,v}$ . This provides our first example of chiral polytopes in this class that are known to be infinite; if  $g := \gcd(b,c) \geq 4$ , then the covering from  $\Gamma_{b,c} \rightarrow \Gamma_{g,0}$  shows that  $\Gamma_{b,c}$  is infinite. In general, this covering relation means that when  $b+ci$  is a Gaussian prime, the group  $\Gamma_{b,c}$  is especially important. Nevertheless, it's not sufficient to know the finiteness (or not) of the prime groups; even when one is finite, that certainly doesn't imply that the groups covering it are all finite!