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Cohomology with non smooth forms

<u>Lemma Poincaré for $\Omega^*_{\infty, loc}(\mathbb{R}^n)$</u>

Λ^k - k-forms on $U \subset \mathbb{R}^n$, U open

$$\omega = \omega_{\vec{i}}\, dx^{\vec{i}} \quad , \quad \vec{i} = (i_1, i_2, \ldots, i_m), \quad i_1 < i_2 < \ldots < i_n .$$

d - operator $d\omega = \dfrac{\partial \omega_{\vec{i}}}{\partial x^\alpha} dx^\alpha \wedge dx^{\vec{i}}$, $\quad d \circ d = 0$.

$\omega_1 \sim \omega_2$ cohomologous $\Rightarrow$ we define $H^k = Z^k / B^k$ (closed forms/exact forms)

<u>Poincaré Lemma</u> : $\forall \omega \in Z^k \ \exists \theta$ s.t. $\omega = d\theta$ (on $\mathbb{R}^n$).

This lemma is a basis for calculating cohomology of more complicated manifolds than $\mathbb{R}^n$. It is also a basis for de Rham theorem; in layman's terms, this says that the smooth structure is irrelevant. This theorem gives a topological invariant of manifolds that has many uses: for example, there are no diffeomorphisms between S^1 and $\mathbb{R}$ because there is a form on S^1

$$d\theta = \frac{-y\,dx + x\,dy}{x^2 + y^2}$$

which has no primitive.

We also get $H^1(D^2 \setminus \{0\}) = H^1(S^1)$.

What if the forms are not smooth?

$\mathcal{F}^k(U)$:= k-forms with almost everywhere defined coefficients on $U \subseteq \mathbb{R}^n$

$$\mathcal{F}^*(U) = \bigcup_{k=0}^{n} \mathcal{F}^k(U)$$

with addition and external multiplication is a graded ring.

$\mathcal{L}^k_p(U)$:= k-forms with p-summable coefficients:

$$\|\omega\|_p = \sum_{\vec{i}} \|\omega_{\vec{i}}\|_p$$

We also introduce:

$$\mathcal{L}^k_\infty(U) = \{\omega \mid \underset{x \in U}{\mathrm{ess.sup}} \, |\omega_{\vec{i}}(x)| < \infty\}$$

which is the set of forms that are bounded almost everywhere.

To define an operator d we need to take derivative which may not exist, so we need to take them in a generalized sense.

$\mathcal{D}^*(U)$ = smooth forms with compact support

de Rham currents: $\mathcal{E}^k(U) : \mathcal{D}^{n-k}(U) \longrightarrow \mathbb{R}$ linear continuous functionals.

For $\omega \in \mathcal{E}^k$: $d\omega$ is defined like this: for $\alpha \in \mathcal{D}^{n-k-1}$

$$\langle d\omega, \alpha \rangle = (-1)^{k+1} \langle \omega, d\alpha \rangle$$

Now we can define the class we will work with:

$$\mathcal{L}^*_{\infty,\infty} = \Omega^*_\infty(U) = \{\omega \mid \omega \in \mathcal{L}^*_\infty(U),\ d\omega \in \mathcal{L}^*_\infty(U)\}$$

This implies $\exists \omega' \in \mathcal{L}^*_\infty(U)$ s.t. $\langle d\omega, \alpha\rangle = \int_U \omega' \wedge \alpha$.

By analogy:

$$\mathcal{L}^*_{p,q} = \{\omega \mid \omega \in \mathcal{L}^*_p(U),\ d\omega \in \mathcal{L}^*_q(U)\}.$$

Now we can talk about cohomology.

Notation for the seminorm: $\forall K$ compact, $K \subset U$:

$$|\omega|_{K,\infty} = \max_{\vec{i}} \text{ ess.sup } |\omega_{\vec{i}}|$$

$$|\omega|_{K,\infty,\infty} = |\omega|_{K,\infty} + |d\omega|_{K,\infty}.$$

We define cohomology as usual:

$$H^*_{p,q} = Z_{p,q} / B_{p,q}$$

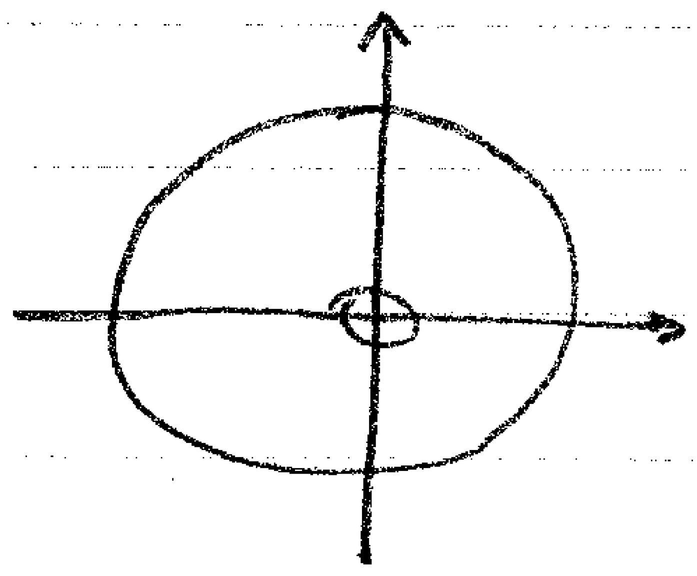
Because of a functional analysis problem, $B_{p,q}$ may not be closed so $H^*_{p,q}$ may be complicated. So we introduce:

$$\overline{H}^*_{p,q} = Z_{p,q} / \overline{B}_{p,q}$$

the reduced cohomology.

For compact manifolds, this invariant is the same as de Rham cohomology. It is useful for distinguishing more general manifolds.

<u>example</u> : punctured disk

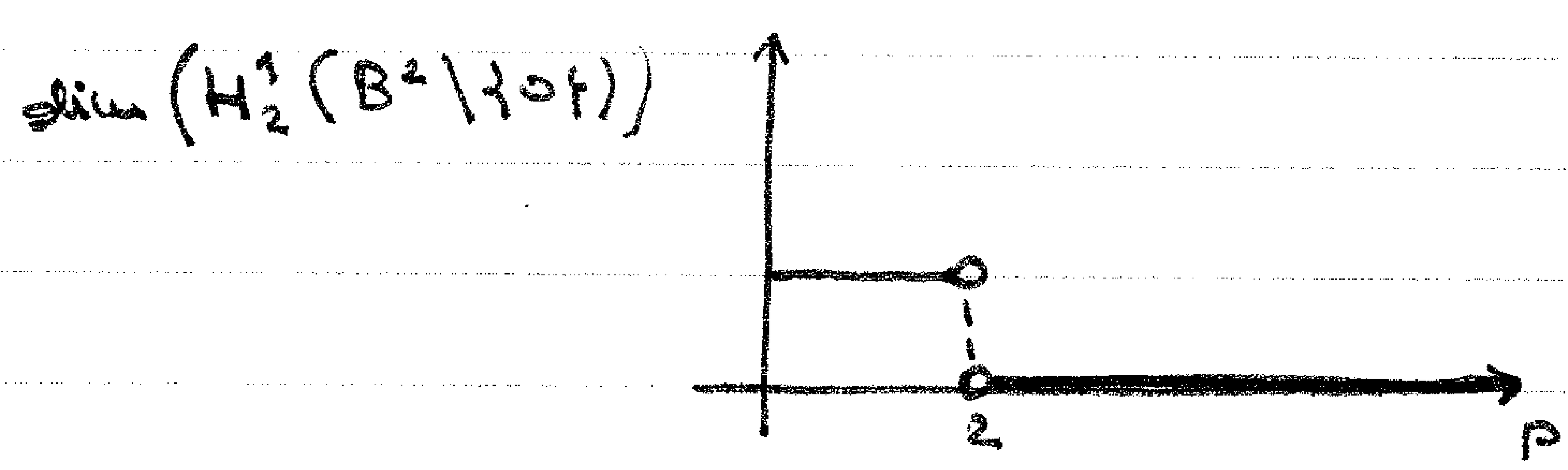


$$d\theta = \frac{-y\,dx + x\,dy}{x^2+y^2}$$

The puncture will be noticed if $d\theta$ is summable; then the cohomology will be non trivial.

$$\|d\theta\|_2 = \int_{B^2\setminus\{0\}} |d\theta|^2 = \iint \underbrace{|d\theta|^2}_{\sim 1/r^2}\, r\,dr\,d\theta \sim \int_0^1 \frac{1}{r}\,dr$$

So we get

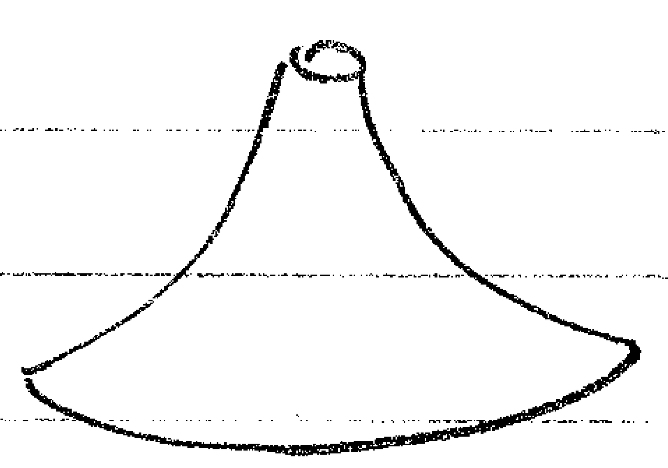


If U has a compact closure, all these cohomologies are called local and we use the notation: $\mathcal{L}^k_{\infty,loc}(U)$. In this case, we can prove Poincaré Lemma.

For $\mathcal{L}^*_{p,q,loc}(\mathbb{R}^n)$, $p,q<\infty$, Poincaré lemma was proven in the 1980s by Gol'dshtein, Kuz'minov, Shvedov who also introduced $H^*_{p,q}$.
For $H^*_{p,p}$, $p\neq 2$ and applications see survey of Pansu.

Poincaré lemma is proven by ordering approximations using smooth forms and some clever estimates on the norm. But in $\mathcal{L}^*_{\infty,\infty,loc}$ strong convergence may also be proven.
There is also Whitney's geometric theory of integration for flat forms, which are exactly $\mathcal{L}_{\infty,\infty,loc}$.

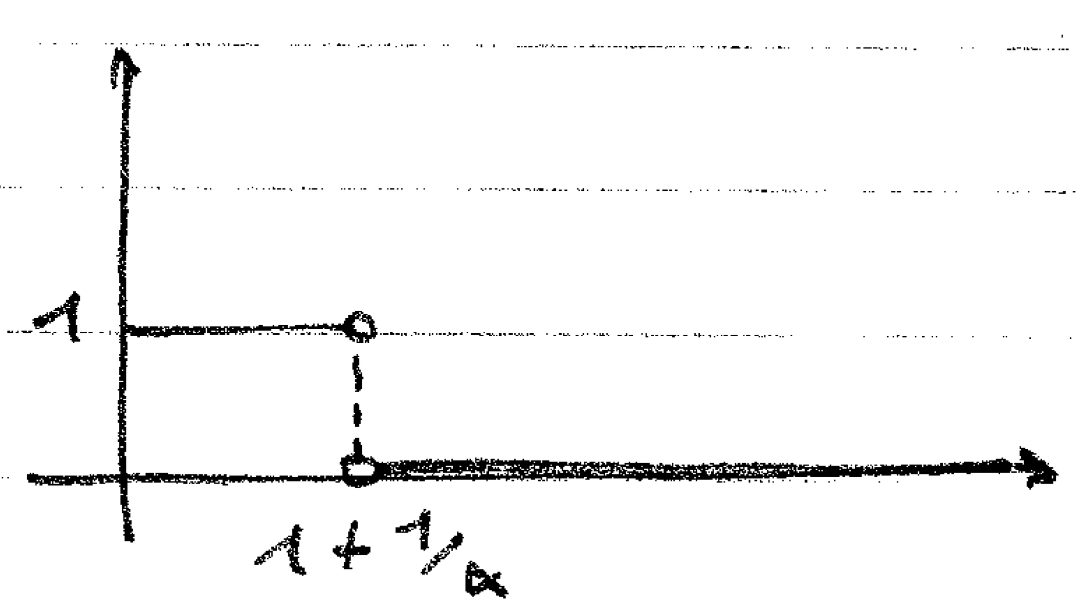
<u>Example</u>: of non compact manifold where the new cohomology is different from de Rham: $H^*_{p,loc}(M) \not\cong H^*(M)$.



$$dg^2 = dr^2 + r^2 d\theta^2$$

$$dg_\alpha^2 = dr^2 + r^{2\alpha} d\theta^2$$

The dimension graph gives:



In general, this cohomology will depend on the metric so we can say it "feels" the metric.

__Theorem__ For $M = X \times_f (a,b]$ with metric $g_M = dt^2 + f^2(t)\, g_X$,

we have: $H^k_{p,p}(M) = 0 \iff \int_a^b f^{(m-kp)}(r)\,dr$.

For example, if f is monotonic there will be a single jump in the dimension graph (like in the previous example).

Goal: develop integration of forms in Lipschitz category.