“Quivers with potentials and triangulated surfaces”

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Plan for this talk

1. Mutations of quivers with potentials.
2. Encoding some quiver mutations by flips.
3. Encoding some QP-mutations by flips.
4. Encoding some QP-representations by curves (the unpunctured case).
5. The punctured case (some work in progress).

Mutations of quivers with potentials

Quivers with potentials and their representations I: Mutations.
H. Derksen, J. Weyman and A. Zelevinsky.
To appear in Selecta Math.
arXiv:0704.0649

Quiver with potential; some types of QPs

A *quiver* is a finite directed graph $Q = (Q_0, Q_1, t, h)$ without loops. We write $t(a) \xrightarrow{a} h(a)$.

Definition 1. A *quiver with potential* is a pair (Q, S), where:

- Q is a quiver;
- $S \in R\langle\langle Q \rangle\rangle_{\mathrm{cyc}}$.

Definition 2 (DWZ)**.** (Q, S) is

- *reduced* if $S^{(2)} = 0$.
- *2-acyclic* if Q has no 2-cycles.
- *trivial* if $Q_1 = \{a_1, b_1, \dots, a_N, b_N\}$ with each $a_k b_k$ a 2-cycle and S is *essentially* of the form $a_1 b_1 + \dots + a_N b_N$.

Right-equivalence; trivial and reduced parts

Definition 3 (DWZ)**.** A *right-equivalence* between (Q,S) and (Q',S') is an R-algebra isomorphism $\varphi : R\langle\langle Q\rangle\rangle \to R\langle\langle Q'\rangle\rangle$ such that $\varphi(S)$ is cyclically equivalent to S'.

Theorem 1 (DWZ)**.** Every QP (Q,S) is right-equivalent to the direct sum of a trivial QP (called its *trivial part*) and a reduced QP (called its *reduced part*). The trivial and reduced parts of (Q,S) are determined up to right-equivalence.

QP-mutation

Definition 4 (DWZ)**.** Let (Q,S) be a QP and $i\in Q_0$ a vertex without 2-cycles incident to it. Denote by $\widetilde{Q}$ the quiver obtained after:

1. Introducing a new arrow $[ab]:j\rightarrow k$ for each 2-path $j\overset{b}{\rightarrow}i\overset{a}{\rightarrow}k$, and
2. replacing each arrow $a:i\rightarrow k$ with $a^*:k\rightarrow i$, and each arrow $b:j\rightarrow i$ with $b^*:i\rightarrow j$.

The *QP-premutation* $\widetilde{\mu}_i(Q,S)$ is $(\widetilde{Q},\widetilde{S})$, where

$$\widetilde{S}=[S]+\sum_{j\overset{b}{\rightarrow}i\overset{a}{\rightarrow}k}[ab]b^*a^*.$$

The *QP-mutation* $\mu_i(Q,S)$ is the reduced part of $\widetilde{\mu}_i(Q,S)$.

Non-degeneracy

Definition 5 (DWZ)**.** A QP is *non-degenerate* if, after applying any sequence of QP-mutations, one always gets a 2-acyclic QP.

Theorem 2 (DWZ)**.** If the base field is uncountable, then every 2-acyclic QP admits a non-degenerate QP on it.

Problem 1. Can we give explicit constructions of non-degenerate QPs?

Encoding some quiver mutations by flips

Cluster algebras and triangulated surfaces, part I: Cluster complexes.
S. Fomin, M. Shapiro and D. Thurston.
To appear in Acta Mathematica.
arXiv:math.RA/0608367

Marked bordered surface

Definition 6. A *marked bordered surface* is a pair (Σ, M), where:

- Σ compact connected oriented 2-dimensional surface with (possibly empty) boundary;
- $\varnothing \neq M = \{marked\ points\}$ finite set with at least one point from each boundary component of Σ.

$P = \{punctures\} = M \cap \operatorname{int} \Sigma$

Some excluded cases (eg, unpunctured monogon, digon or triangle).

Arc, Compatibility of arcs

Definition 7. An *arc* in (Σ, M) is a curve i in $\Sigma \setminus (M \cup \partial\Sigma)$ connecting marked points such that:

- i has no self-intersections (except possibly for endpoints);
- i is not contractible to a point in $\Sigma \setminus M$.
- i cannot be deformed to $\partial\Sigma \setminus M$ in $\Sigma \setminus M$.

Arcs are considered up to isotopy and orientation.

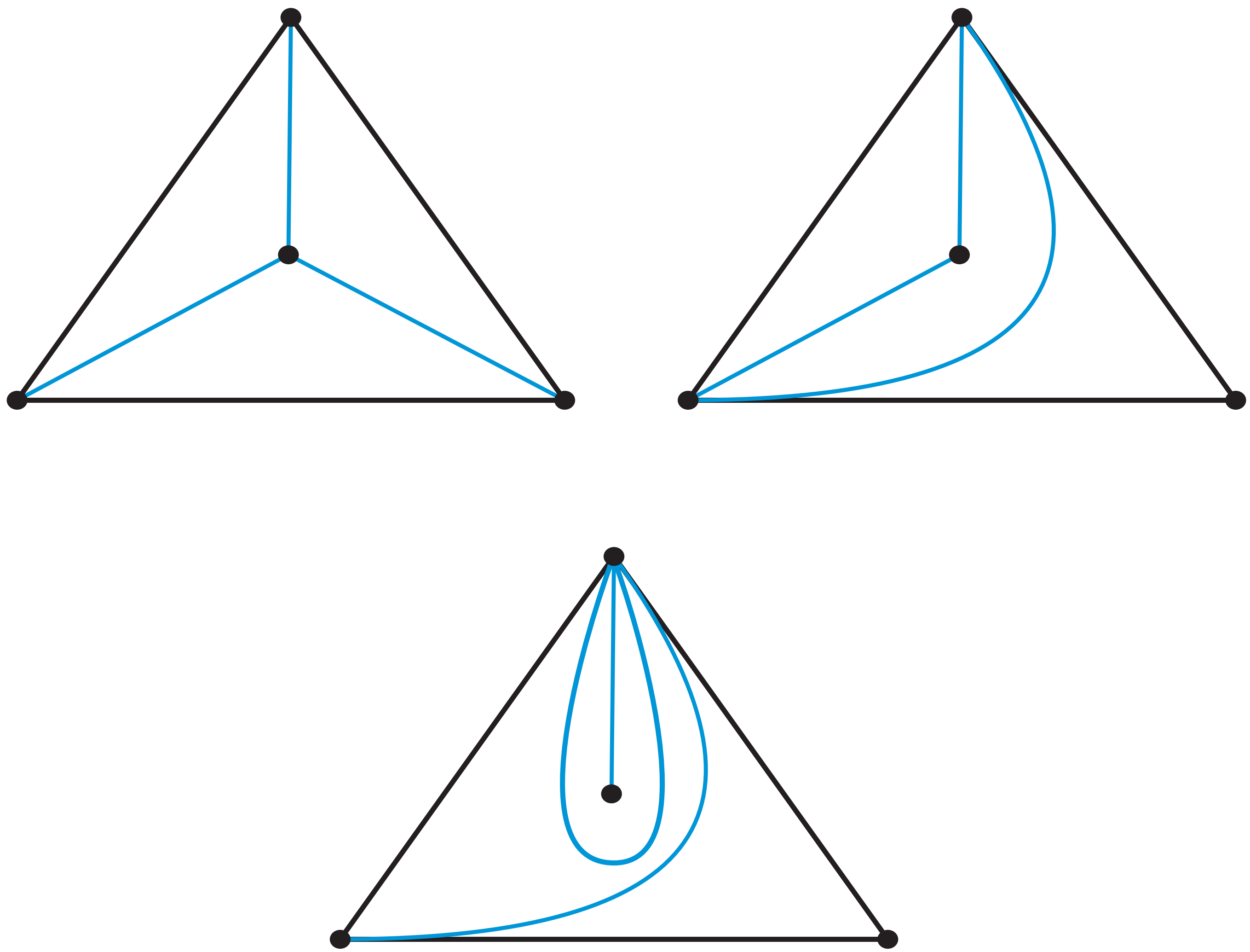
Two arcs are *compatible* if there are arcs in their respective isotopy classes that do not intersect (except for endpoints).

Ideal triangulation

Definition 8. An *ideal triangulation* of (Σ, M) is any maximal collection of pairwise compatible arcs.

The number n of arcs in an ideal triangulation of (Σ, M) is an invariant of (Σ, M), called the *rank* of (Σ, M) (because it coincides with the rank of the cluster algebra associated to (Σ, M) by FST).

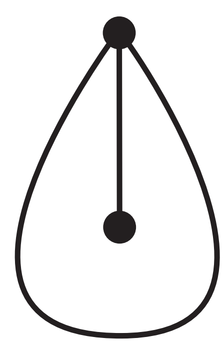
Figure 1: Some ideal triangulations of the once-punctured triangle



Flip

Let τ be an ideal triangulation of (Σ, M) and let $i \in \tau$ be an arc. If $i \neq$ folded side of a self-folded triangle, then we can *flip* it, ie, replace it with a well defined arc $i' \neq i$ and obtain another ideal triangulation $\sigma = f_i(\tau)$.

Figure 2: Self-folded triangle

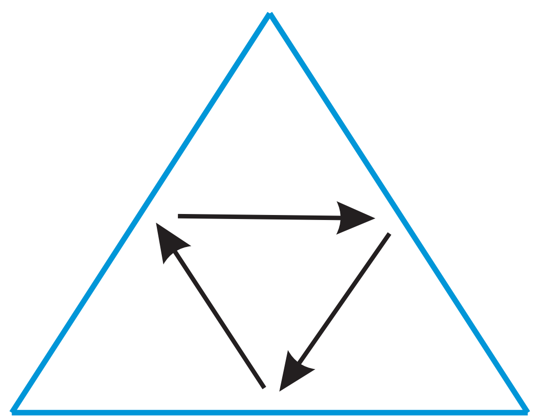


FST: *Tagged arcs* $\longrightarrow$ *tagged triangulations* $\longrightarrow$ folded sides are flippable.

Signed adjacency quiver

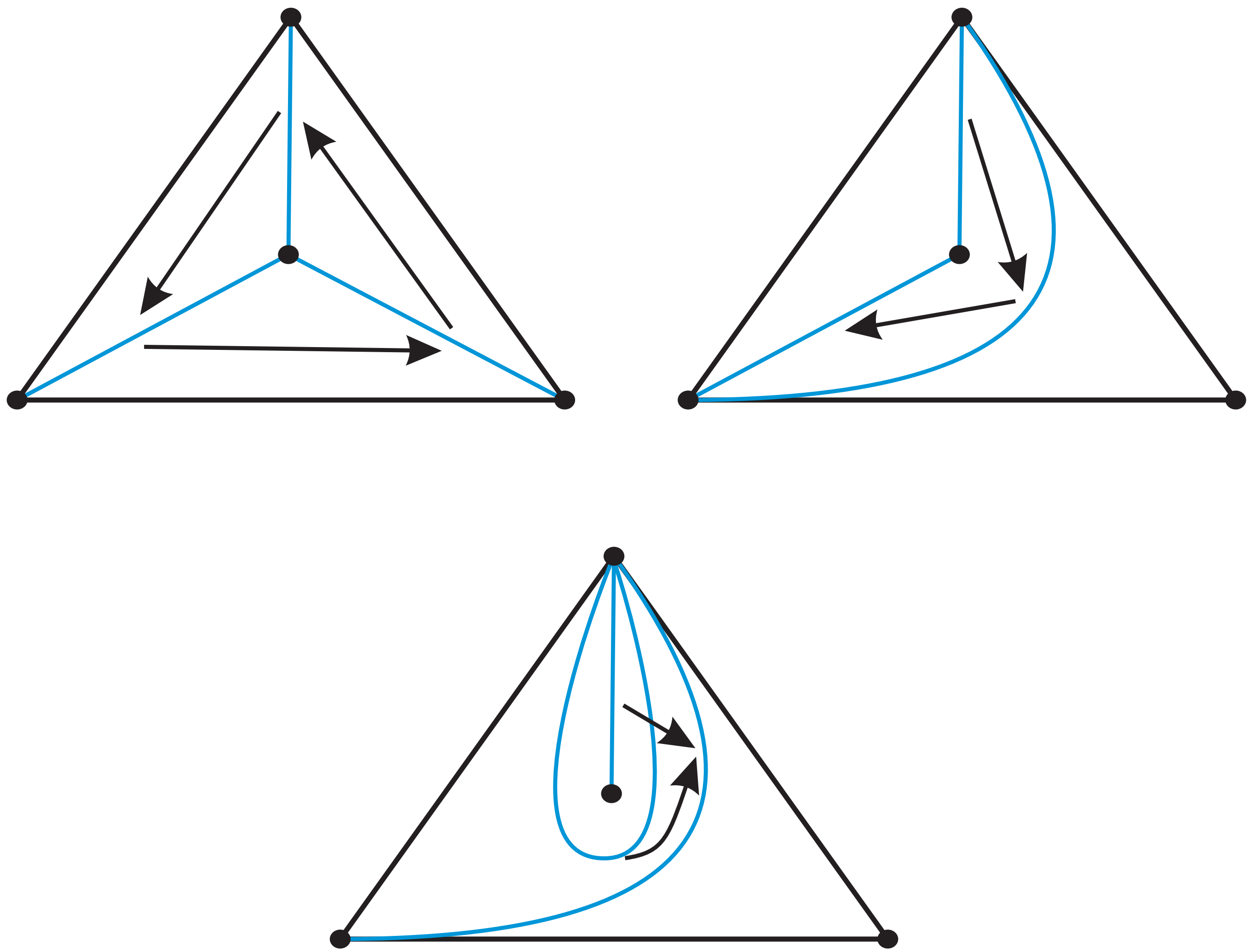
Definition 9 (FST)**.** Let τ be an ideal triangulation of (Σ, M); the *signed adjacency quiver* $Q(\tau)$ has τ as set of vertices, and the arrows are given by:

- for each non-self-folded triangle $\triangle$ of τ, draw arrows as in the following figure



- delete all 2-cycles;
- substitute each folded side i by the arc j enclosing it and reproduce for i the arrows incident to j.

Figure 3:



Flips $\leftrightarrow$ Mutations

Proposition 1 (FST). If two (tagged) triangulations are related by a flip, then their respective quivers are related by the corresponding mutation.

Encoding some QP-mutations by flips

arXiv:0803.1328

First aim

To each tagged triangulation τ of (Σ, M) associate a potential $S(\tau) \in R\langle\langle Q(\tau)\rangle\rangle$ in such a way that

- $S(\tau)$ can be combinatorially read in terms of τ;
- $\tau \stackrel{\text{flip}}{\longleftrightarrow} \sigma \implies (Q(\tau), S(\tau)) \stackrel{\text{QP-mut}}{\longleftrightarrow} (Q(\sigma), S(\sigma))$.

In this way we would have an explicit construction of non-degenerate potentials for quivers associated to surfaces.

Definition of $S(\tau)$ (particular case)

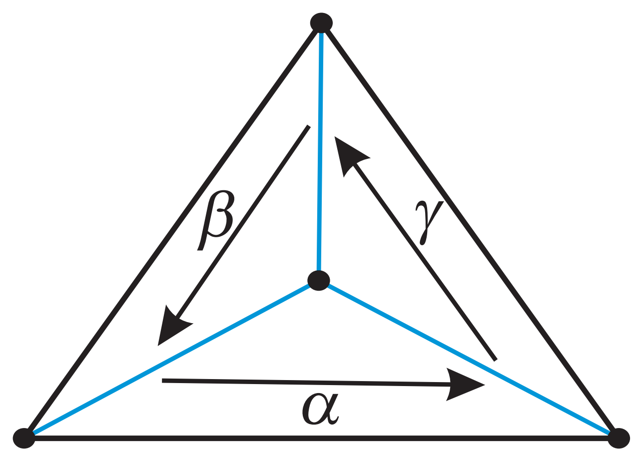
Fix a choice $(x_p)_{p\in P}$ of non-zero scalars (one scalar $x_p \in K$ for each puncture p). Let τ be a triangulation of (Σ, M) such that each puncture is incident to at least three arcs in τ.

- Interior triangle $\triangle$ of $\tau \longrightarrow$ oriented triangle $S^{\triangle} = abc$ in $Q(\tau)$;
- puncture $p \in P \longrightarrow$ oriented cycle S^p of $Q(\tau)$ around p.

$$S(\tau) := \sum_{\triangle} S^{\triangle} + \sum_{p\in P} x_p S^p.$$

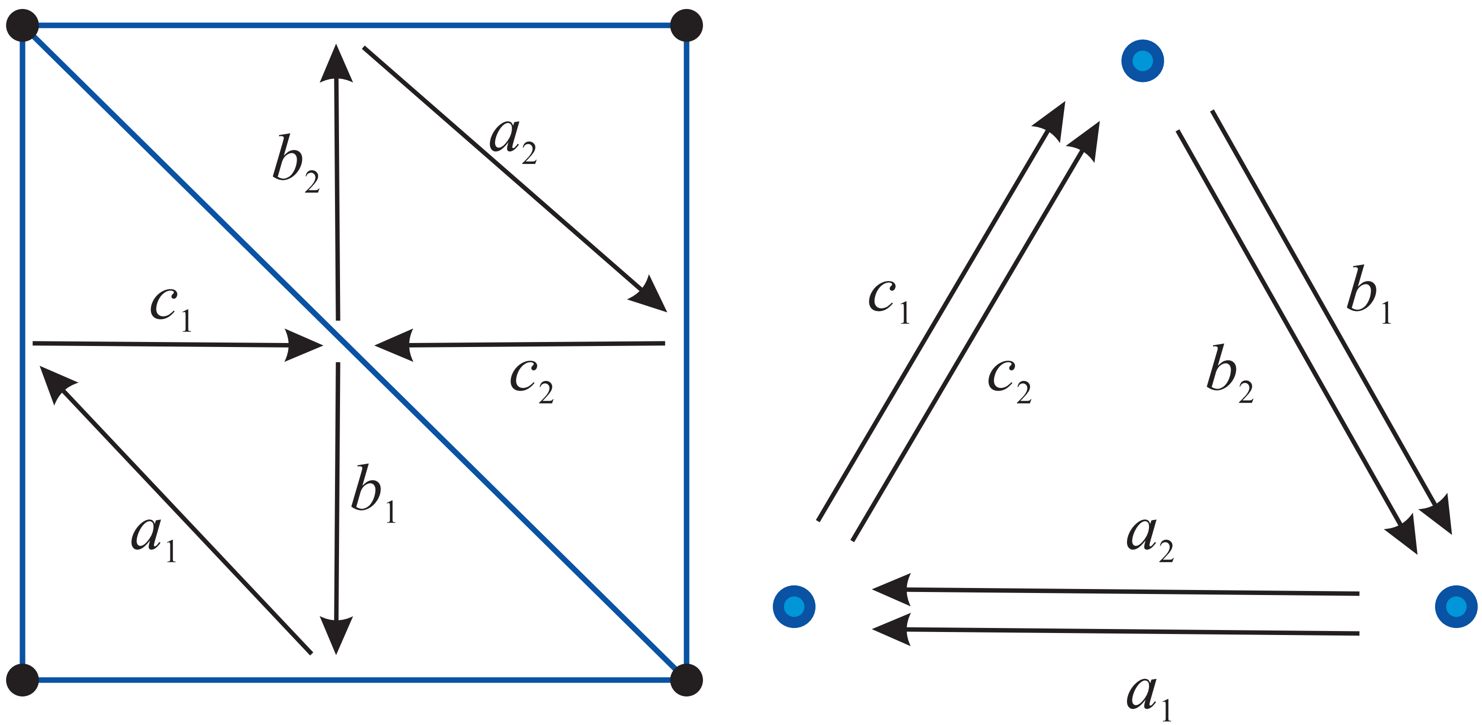
Example: the once-punctured triangle

Figure 4: $S(\tau) = x\alpha\beta\gamma$



Example: the once-punctured torus

Figure 5: $S(\tau) = a_1b_1c_1 + a_2b_2c_2 + xa_1b_2c_1a_2b_1c_2$



Theorem 3. If τ and σ are **ideal** triangulations of (Σ, M) such that $\sigma = f_i(\tau)$, then $(Q(\sigma), S(\sigma))$ is right-equivalent to $\mu_i(Q(\tau), S(\tau))$.

This theorem does not necessarily hold if one defines QPs of tagged triangulations "in the obvious way"(flips of folded sides are "badly behaved").

Nevertheless: non-empty boundary $\Longrightarrow$ non-degeneracy

Theorem 4. Let (Σ, M) be a marked bordered surface with $\partial\Sigma \neq \varnothing$. For any ideal triangulation τ of (Σ, M) the QP $(Q(\tau), S(\tau))$ is rigid and hence non-degenerate.

Problem 2. Establish non-degeneracy for surfaces with empty boundary (if it is indeed the case, of course!).

Problem 3. Extend the definition of $S(\tau)$ to tagged triangulations so that $S(\tau)$ can be combinatorially read in τ, in such a way that tagged triangulations related by a flip give rise to QPs related by the corresponding mutation.

Encoding some QP-representations by curves (the unpunctured case)

Quivers with relations arising from clusters (A_n case).
P. Caldero, F. Chapoton and R. Schiffler.
arXiv:math/0401316

Gentle algebras arising from surface triangulations.
I. Assem, T. Brüstle, G. Charbonneau-Jodoin and P-G. Plamondon.
Independent work in progress.

For unpunctured surfaces, ABCP define a set $\rho(\tau)$ of relations on $Q(\tau)$ such that:

- $Q(\tau)/\rho(\tau)$ is a gentle algebra;
- it generalizes a construction given by CCS for triangulations of unpunctured polygons.

It turns out that $\rho(\tau)=\{\partial_a(S(\tau)) \mid a\in Q_1(\tau)\}$, and the algebra $Q(\tau)/\rho(\tau)$ is the jacobian algebra $P(Q(\tau),S(\tau))$.

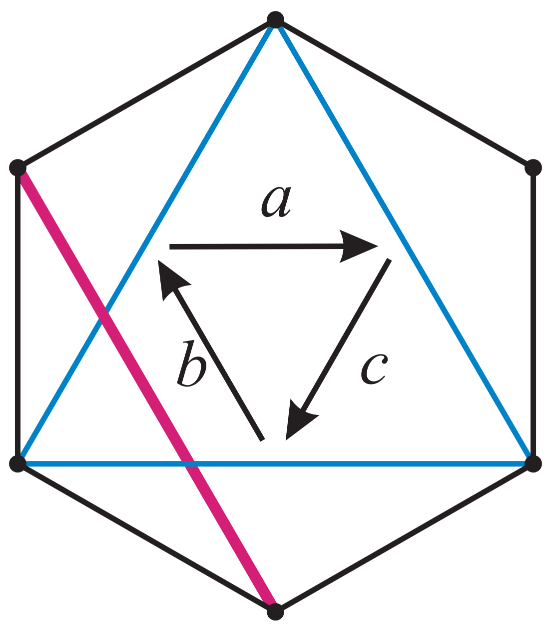
Traversing curves with 1_K

Definition 10 (ABCP)**.** Let i be an arc and τ an ideal triangulation of unpunctured (Σ, M). For each arc $j \in \tau$ place a copy of the field K at each point of intersection of i with j. Given an arrow $a : j \to k$, put the identity 1_K along a if and only if a is a segment of i. In this way we obtain a representation $M(\tau, i)$ of $(Q(\tau), S(\tau))$.

When (Σ, M) =unpunctured polygon, this representation was defined by CCS.

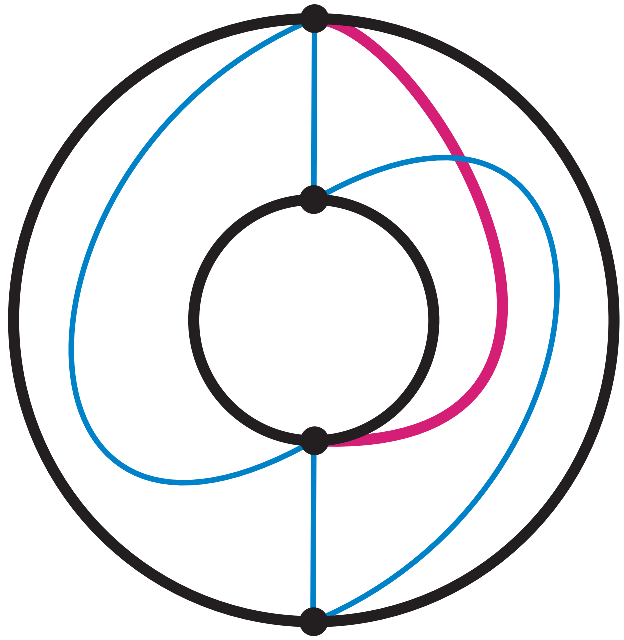
Example: the unpunctured hexagon

Figure 6: $S(\tau) = abc$



Example: the annulus

Figure 7:



DWZ $\longrightarrow$ mutation of QP-representations.

Theorem 5 (L)**.** Let (Σ, M) be an unpunctured surface and i an arc on (Σ, M). If the ideal triangulations τ and σ of (Σ, M) are related by $\sigma = f_j(\tau)$, then the QP-representations $M(\sigma, i)$ and $\mu_j(M(\tau, i))$ are right-equivalent.

The punctured case
(some work in progress)

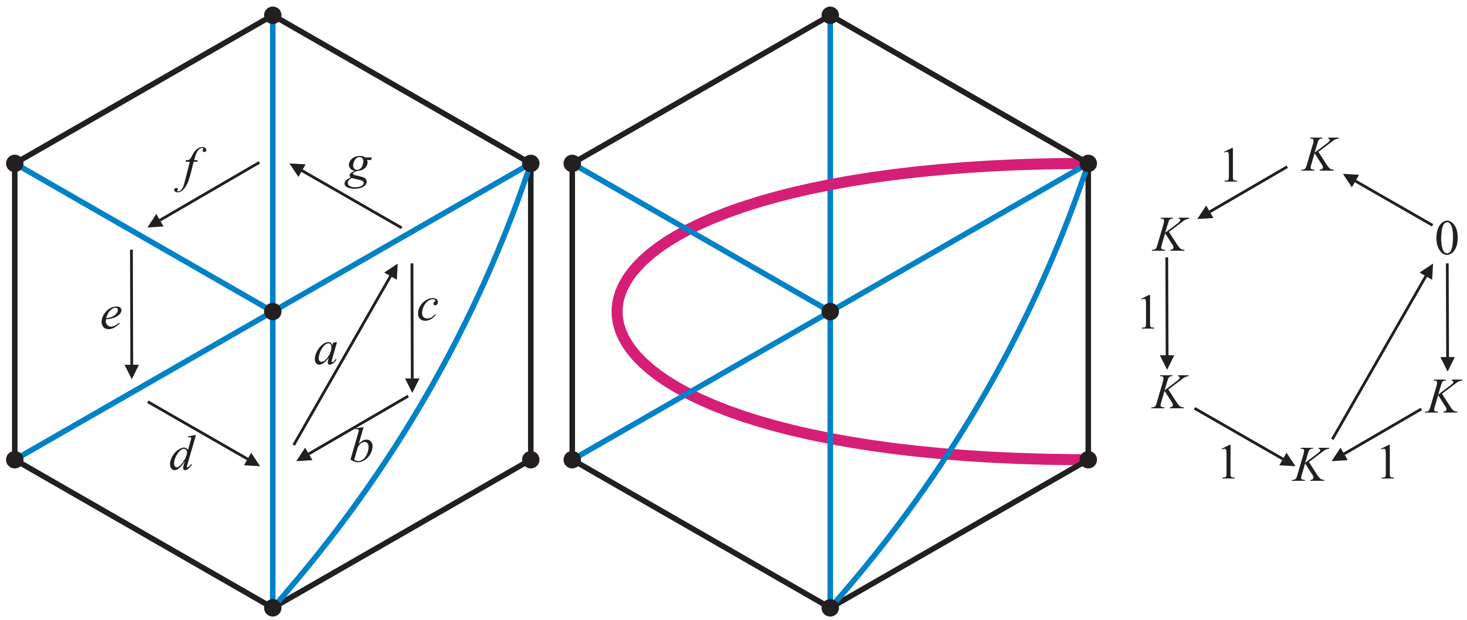
Current Aim

Generalize ABCP's construction to the case of surfaces with punctures, in such a way that:

- $M(\tau, i)$ satisfies the relations imposed by $S(\tau)$.
- $\tau \overset{\text{flip}}{\longleftrightarrow} \sigma \quad \Longrightarrow \quad M(\tau, i) \overset{\text{QP-mut}}{\longleftrightarrow} M(\sigma, i)$ when the arc i is fixed.

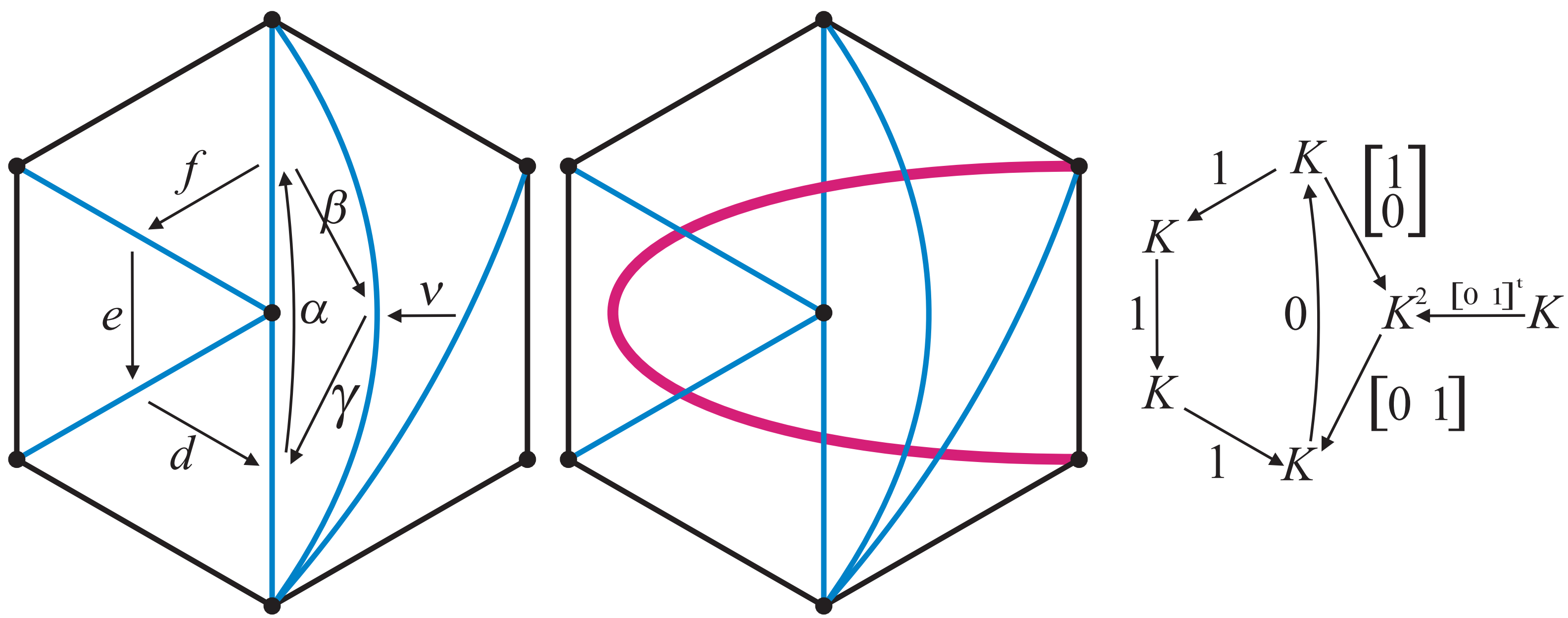
Example: the once-punctured hexagon

Figure 8: $S(\sigma) = abc + xadefg$



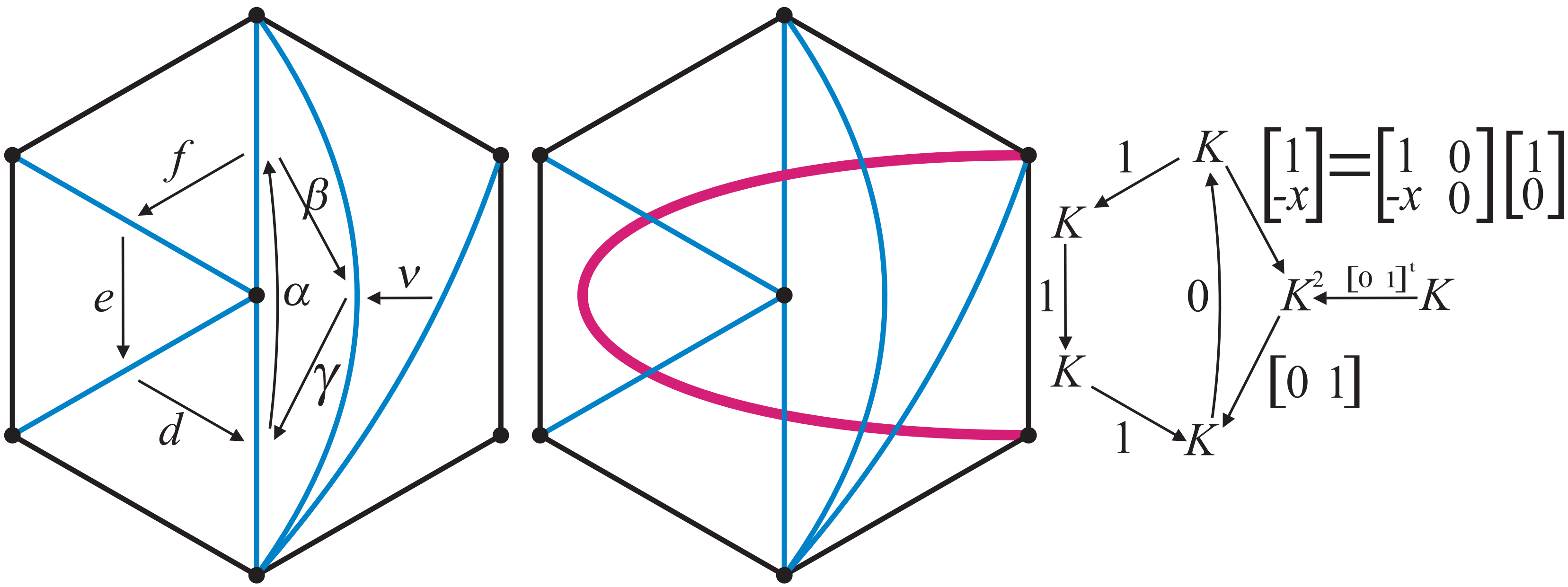
Non-example: the once-punctured hexagon

Figure 9: $S(\tau) = \alpha\beta\gamma + x\alpha def,\ \ \partial_\alpha(S(\tau)) = \beta\gamma + xdef$



Non-example $\longrightarrow$ example

Figure 10: $S(\tau) = \alpha\beta\gamma + x\alpha def, \;\; \partial_\alpha(S(\tau)) = \beta\gamma + xdef$



This way of modifying "non-examples" can be done in a systematic way.

Theorem 6. Let (Σ, M) have non-empty boundary and at most one puncture. If the arc i is not a loop then:

- $M(\tau, i)$ satisfies the relations imposed by $S(\tau)$;
- $\tau \overset{\text{flip}}{\longleftrightarrow} \sigma \quad \Longrightarrow \quad M(\tau, i) \overset{\text{QP-mut}}{\longleftrightarrow} M(\sigma, i)$ when the arc i is fixed. Here, the (possible) puncture of (Σ, M) is incident to at least three arcs of τ and at least three arcs of σ.

For more general situations, further modifications are often needed.

Possible further directions

- Describe combinatorially all "modifications of non-examples" for surfaces with more than one puncture.
- Relation between $\mathbf{dim}M(\tau, i)$ and denominator vector of the cluster variable i in terms of the cluster τ (cf. FST's work).
- Combinatorial description of all right-equivalence classes of representations of $(Q(\tau), S(\tau))$ in terms of curves on (Σ, M).
- Geometry (as in Fomin-Thurston's work)?
- Relation to DWZ's second paper?
- Categorification?

Thank you for your attention!