On the Structure of Additive Invariants for Cellular Automata

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May 15, 2008

Outline

- A bit of history and some fundamental results
- Motivations
- Noether's Theorem and a Least Action Principle for Cellular Automata?
- A conjecture on an open problem

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Self-Reproductive automata

- John von Neumann's theory of self reproduction (1950)
- Computing devices analogous to human brain
- Memory and the processing units are not separated from each other
- Parallel and capable of repairing and building themselves
- Discrete universe: An infinite 2-dim chessboard
- Each cell can assume only a finite number of *states*
- Dynamics: State of a cell x at time t depends on the states of the cells of a fixed-shape neighborhood of x at time $t - 1$

Cellular Automata

- $c : \mathbb{Z}^d \longrightarrow S$ where $S = \{0, 1, \dots, n-1\}$
- $S^{\mathbb{Z}^d} = \mathcal{C}(S, d)$
- $I = \{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_N\}$, $\vec{x}_i \in \mathbb{Z}^d \ \forall i = 0, 1, \dots, N-1$
- $f : S^N \longrightarrow S$
- Dynamics is defined by a “parallel” application of f:

$$c(\vec{x}, t+1) = f(c(\vec{x} + \vec{x}_1, t), c(\vec{x} + \vec{x}_2, t), \dots, c(\vec{x} + \vec{x}_N, t))$$

- The pair (f, I) induces a global map

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2 dim CA in action: Game of Life

- Invented by John Conway
- S={0=dead,1=alive}
- King-chess move neighborhood
- Any live cell with fewer than two live neighbours dies, as if by loneliness
- Any live cell with more than three live neighbours dies, as if by overcrowding
- Any live cell with two or three live neighbours lives, unchanged, to the next generation
- Any dead cell with exactly three live neighbours comes to life

Limit cycles

- Static configurations (*still lives*)
- Configurations repeating themselves (*oscillators*)
- Patterns that translate themselves (*gliders*)
- Patterns which produce gliders periodically (*glider guns*)

Coding 1-Dim CA

- $d = 1$,$S = \{0, 1\}$, $I = \{-1, 0, 1\}$ then $f : S^3 \longrightarrow S$ can be represented by the values taken on the ordered elements of its domain

$$111, 110, \ldots, 000$$

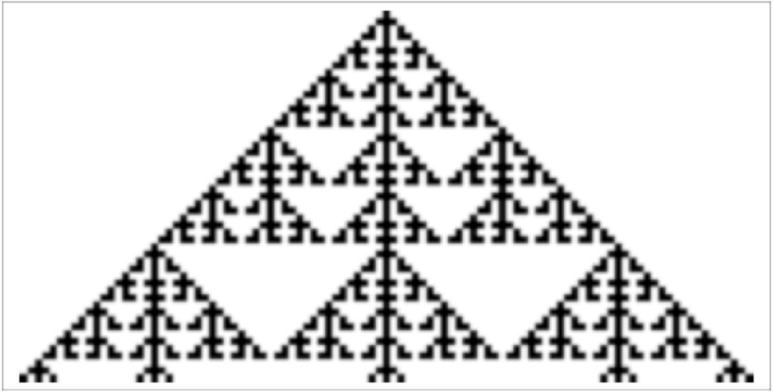
- $f(x_1, x_2, x_3) = x_1 + x_2 + x_2$ $(mod\ 2)$ is described by:

111	110	101	100	011	010	001	000
1	0	0	1	0	1	1	0

Therefore f it can be represented by the number 150, since the binary expansion of the latter is 10010110

Evolution of rule 150

$$x_0 = 1, x_i = 0 \ \forall i \in \mathbb{Z}$$



Finite and periodic configurations

- Periodic boundary conditions: replace $\mathbb{Z}$ with $\mathbb{Z}_N$
- If a configuration c is such that $|\{x \in \mathbb{Z} \mid c(x) \neq k\}| < \infty$ for some fixed state $k \in S$, then we say that c is a finite configuration.
- $\mathcal{C}_F =$ set of all finite configurations
- A finite configuration remains finite under the evolution induce by a CA
- They are used to study CA on computers

Topology

Let us consider 1-dim CA defined on $X(S)$. We can define a distance between two configurations $x, y \in X(S)$ by looking at the least $k > 0$ such that either $x_k \neq y_k$ or $x_{-k} \neq y_{-k}$ and setting

$$d(x,y) = \frac{1}{1+k}$$

Let $\sigma(x)$ be the shift map defined by

$$[\sigma(x)]_i = x_{i+1}$$

- *Theorem: A map* $\phi : X(S) \longrightarrow X(S)$ *such that* $\phi \circ \sigma = \sigma \circ \phi$ *is continuous if and only if it is induced by a CA*

References

- G. Hedlund, *Math. Systems Theory* **3** (1969) 320-375.

Decidability problems

A CA is called invertible if its global map ϕ has an inverse ϕ^{-1}
Some interesting results:

- ϕ^{-1} -when exists- is induced by a CA
- A CA which is injective on finite configuration is surjective and viceversa
- There is an algorithm to decide if a 1-dim CA is invertible
- There is no algorithm to decide if a 2-dim CA is invertible
- $n > 2$ open

References

- J. Kari, *Physica* **D 45** (1990) 379-385.
- S. Amoroso, Y. Patt, *J. Comput. System Sci.* **6** (1972) 448–464.
- Moore, E. F. (1963) *Proceedings of the American Mathematical Society* **14** (1963) 685–686
- Myhill, J. (1963) *Proceedings of the American Mathematical Society* **14** (1963) 17–33

CA as physical models

- Finitely defined
- Can be implemented in efficient way (periodic CA)
- Induce global evolutions that, under certain conditions, are good model models of physical processes
- Example: Lattice Gases

  -2-dim CA

  -Simple local updating rules

  -Chapman-Enskog expansion $\longrightarrow$ Navier-Stokes equations
- References
  - U. Frisch, B. Hasslacher, Y. Pomeau *Phys. Rev. Lett.* **56** (1986) 1505-1506
  - B.M. Boghosian and others *J. Stat. Phys.* **81** (1995) 105-128

Second-Order Reversible CA

The evolution of an important class of second-order CA is defined by a first-order CA automaton f and a reversible combiner γ:

- $\gamma : S^2 \longrightarrow S$
- $\gamma(y, \gamma(y, x)) = x$

  such that
- $\tilde{f} : S^{2\beta+1} \times S \longrightarrow S$

  Evolution:
- $(\hat{x}_i(t+1), x_i(t)) \mapsto \gamma(f(\hat{x}_i(t+1)), x_i(t)) = x_i(t+2)$

Example

- $x(t,i) = ((x(t-1,i-1)+x(t-1,i)+x(t-1,i+1))mod\ 2)\oplus x(t-2,i)$

x(4)=	0	1	0	1	0	0	0
x(3)=	1	0	1	1	1	0	1
x(2)=	0	1	0	0	0	0	0
x(1)=	0	1	0	1	1	0	1
x(0)=	0	0	0	0	0	0	1

CA as physical models

- Second order, reversible CA have interesting thermodynamical behavior
- Can we recover a least action principle for these systems?
- Are symmetries of these systems in correspondence with conserved quantities? (This can be thought as an analogue of Noether's Theorem for continuous systems)
- References
  - G.Caterina and B. Boghosian *Submitted to Physica A*

Additive conserved densities

Define a "density":

- $\epsilon : S^{2\alpha+1} \times S \longrightarrow \mathbb{R}$
- $(\hat{x}_i^{\alpha}(t), x_i(t-1)) \mapsto \epsilon(\hat{x}_i^{\alpha}(t), x_i(t-1))$
  with energy

$$E_N(x(t), x(t-1), \epsilon) = \sum_{i=0}^{N-1} \epsilon(\hat{x}_i(t), x_i(t-1))$$

- ϵ is a conserved density if, $\forall N \in \mathbb{N}^{+}$

$$E_N(x(t), x(t-1), \epsilon) = E_N(x(t+1), x(t), \epsilon)$$

- Hattori and Takesue proved that only a finite number of linear conditions are sufficient to find all the solutions to the above equations!!!

Example

- $I = \{-1, 0, 1\}$, $\mathcal{F} = \{f | f : \{0,1\}^3 \longrightarrow \{0,1\}\}$
- $\epsilon = \epsilon(x_{i-1}(t), x_i(t), x_{i-1}(t), x_i(t-1))$
- $\psi(f)$= set of all second-order additive conserved densities for $\tilde{f}$
- $f \sim g \iff \psi(f) = \psi(g)$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 1 & 2 & 2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & -2 & 0 & -2 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- $8 \mapsto (0,0,0,0,1,0,0,0)$, $64 \mapsto (0,0,0,1,0,0,0,0)$, $72 \mapsto (0,0,0,1,1,0,0,0)$
- This class is closed under bitwise inclusive OR

A closure property?

- Not every class is closed
- However it is always true that . . .

$$\psi(f) \subseteq \psi(f \vee g), \forall g \in \psi(f)$$

- Can we generalize this observation?

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A lemma

Generalized *XOR* :

1. $\gamma(y, \gamma(y, x)) = x$
2. $\gamma(x, \gamma(y, z)) = \gamma(y, \gamma(x, z))$,

then we have:

Lemma

If ϵ is an additive conserved quantity density for $\tilde{f}$ and $\tilde{g}$ then

$$\epsilon[\hat{x}_i(t), \gamma(f(\hat{x}_i(t)), x_i(t-1))] = \epsilon[\hat{x}_i(t), \gamma(g(\hat{x}_i(t)), x_i(t-1))] \quad (1)$$

for any $i \in \{0, 1, \ldots, N-1\}$, $t \geq 1$.

- What does "symmetry" have to do with this results?
- Noether's theorem: conserved quantities are in correspondence with continuous symmetries of the laws of nature
- Time translation symmetry is associated with conservation of energy
- Spatial translation symmetry with conservation of momentum
- Rotational symmetry is associated with conservation of angular momentum
- Look at the columns of $\psi(f)$ in the previous example. . .

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 1 & 2 & 2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & -2 & 0 & -2 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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A theorem

Theorem

Let $\tilde{f}$ and $\tilde{g}$ be a second-order reversible cellular automaton of size β and reversible combiner γ and suppose $\epsilon = \epsilon(\hat{x}_i(t), x_i(t-1))$ is an additive conserved density of size α for $\tilde{f}$ and $\tilde{g}$. Then ϵ is an additive conserved density also for $\tilde{h}$ with $h = f \vee g$.

Remark

Given an equivalence class C, let

$$< C >= \{g \vee h\}_{g,h \in C}$$

Since $\vee$ induces a partial order

$$f < g \iff f \vee g = g$$

then the theorem says that, for all elements of $< C >$, we have that

$$f < g \Rightarrow \psi(f) \subseteq \psi(g)$$

An Open Problem

- ϵ is said to be a *trivial conserved density* if

$$E(x(t), \epsilon) = k \ \forall \ x(t)$$

- Let the argument of ϵ vary
- *Given an automaton f, is there an algorithm which allows us to decide if f has any nontrivial conserved quantity?*
- $\psi(f, \alpha)$ is the set of the additive conserved quantities densities ϵ with $\alpha + 1$ arguments for the automaton f

Conjecture

There exists $\alpha >> \beta$ such that, for any $n \geq \alpha$

$$\mathcal{F}_I / \sim_n = \mathcal{F}_I / \sim_{n+1} .$$

Corollary

If the conjecture is true, than the problem above is decidable. Indeed, for α big enough, the class which all the automata with only trivial conserved quantities belong to will stabilize, and therefore no non-trivial conserved quantity can possibly emerge.

Numerical results

- $|S| = 2, I = \{0\}$

$$\mathcal{F}_I/\sim_1 = \mathcal{F}_I/\sim_2$$

$$\mathcal{F}_I/\sim_2 = \{\{0,3\},\{1\},\{2\}\}.$$

- $|S| = 4, I = \{0\}$

$$\mathcal{F}_I/\sim_1 = \mathcal{F}_I/\sim_2$$

  (See next slide)
- $|S| = 2, I = \{0,1\}$

$$\mathcal{F}_I/\sim_3 = \{\{0,6,8,9,14,15\},\{1\},\{7\},\{4,11\},\{2,13\},\{3,5\},\{10,12\}\}$$

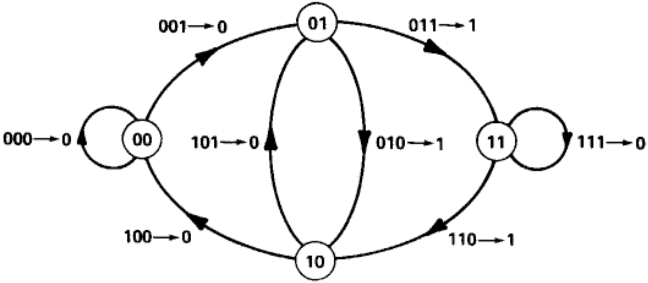
$$\mathcal{F}_I/\sim_2 = \mathcal{F}_I/\sim_3,$$

- $|S| = 2, I = \{-1,0,1\}$

Computation in progress

0	1	2	3	5	8	9	10	11	12	13	14	15	16	17	18
19	21	22	25	26	28	29	30	31	34	41	42	46	48	49	50
51	53	56	57	58	59	60	61	62	63	64	65	66	67	69	71
72	73	74	75	77	79	80	81	82	83	85	86	87	89	90	91
93	94	95	98	105	106	107	112	113	114	115	117	118	119	121	12
123	125	126	127	128	129	130	131	133	136	137	138	139	140	141	14
143	144	145	146	147	149	150	151	153	154	155	157	158	159	162	16
169	170	171	173	174	175	178	179	185	186	187	189	190	191	195	20
205	207	211	221	222	223	242	243	249	250	251	253	254	255		
160	161	165	176	177	181	240	241	245							
68	70	76	78	102	110	204	206	238							
84	88	92	104	116	120	124	148	152	156	168	172	184	188	220	24
252															
20	23	24	27	40	43	215	219	235							
4	6	7	38	52	54	55	132	134	135	166	167				
182	183	199	246	247											
32	33	35	37	44	45	47	96	97	99	101	103	109	111	227	23
239															
224	225	229													
212	216	232													
196	198	230													
164	180	244													
100	108	236													
36	39	231													
228															

De Bruijn Diagrams



- One-block automata respect the graph structure of the correspondent De Bruijn diagram (*graph homomorphisms*)
- Let f be a graph homomorphism.
- $\psi^{\mathcal{V}}(f)$ be the set of all the maps $\epsilon : \mathcal{V} \longrightarrow \mathbb{R}$ such that, for any closed path $e_1, e_2, \ldots, e_n$ and for any $n \in \mathbb{N}^+$ we have:

$$\sum_{j=1}^{n} \epsilon(i(e_j)) = \sum_{j=1}^{n} \epsilon(i(f(e_j))).$$

- $\psi^{\mathcal{E}}(f)$ be the set of all the maps $\tilde{\epsilon} : \mathcal{E} \longrightarrow \mathbb{R}$ such that, for any closed path $e_1, e_2, \dots, e_n$ and for any $n \in \mathbb{N}^+$ we have:

$$\sum_{j=1}^{n} \tilde{\epsilon}(e_j) = \sum_{j=1}^{n} \tilde{\epsilon}(f(e_j)).$$

Remark

$\psi^{\mathcal{V}}(f)$ *and* $\psi^{\mathcal{E}}(f)$ *do not depend on* n

Conjecture

If f *and* g *are graph homomorphism such that* $\psi^{\mathcal{V}}(f) = \psi^{\mathcal{V}}(g)$ *then*

$$\psi^{\mathcal{E}}(f) = \psi^{\mathcal{E}}(g).$$