# BASED AFFINE SPACE AND CLUSTER ALGEBRAS

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## Contents

## 1. Problem of tensor product multiplicities

Let $G$ be a group. A representation of $G$ is a vector space $V$ over the field of complex numbers $\mathbb{C}$, together with a group homomorphism

$$\rho : G \to GL(V)$$

The terms *$V$ is a $G$-module*, *$G$ acts linearly on $V$* are also used to mean that we have a representation of $G$ on $V$. If $V$ is a $G$-module such that every sub-$G$-module of $V$ is either $(0)$ or $V$, then we say $V$ is irreducible.

We will only consider the groups which satisfy an additional condition (called *linear reductivity*):

$$(*) : \text{ Every } G\text{-module is isomorphic to direct sum of irreducible } G\text{-modules}$$

Let $\Omega_G$ be set of isomorphism classes of irreducible $G$-modules. The problem of tensor product multiplicities (TPM for short) is to compute the numbers $c^U_{V,W}$ for every triple $V, W, U \in \Omega_G$ which is number of times $U$ occurs in the decomposition of $V \otimes W$ into irreducible representations.

$$V \otimes W = \bigoplus_{U \in \Omega_G} U^{\oplus c_{VW}^U}$$

The following is an application of Schur's lemma (which asserts that under the assumption (*), for every $V, W \in \Omega_G$, the space of homomorphisms $Hom_G(V, W)$ is either one dimensional (in case $V$ and $W$ are isomorphic) or zero dimensional (otherwise)):

$$c_{VW}^U = \dim\left(Hom_G(U, V \otimes W)\right)$$

Equivalently, under the canonical identification $Hom(M, N) = M^* \otimes N$ for vector spaces, we can rewrite

$$c_{VW}^U = \dim\left(U^* \otimes V \otimes W\right)^G$$

Here and henceforth, we will use the notation $V^G$ for the subspace of $V$ consisting of vectors which are fixed under $G$-action

$$V^G = \{v \in V | g.v = v \forall g \in G\}$$

From now on, we will only consider groups which are *defined by polynomials*, i.e, there exists $f_1, \ldots, f_r \in \mathbb{C}[x_{ij} : 1 \leq i, j \leq n]$ such that $G = \{x \in M_n(\mathbb{C}) | f_i(x) = 0, i = 1, \ldots, r\}$. Such groups are called *linear algebraic groups*. For such groups we restrict ourselves to representations which are given by polynomials.

## 2. Some easy examples

In this section we consider a few examples of linearly reductive groups where the problem posed in previous section is trivial

### 2.1. $G = \mathbb{C}^\times$. **Prof. Weyman's Invariant Theory: Homework 1, problem 1**

Let $\rho : G \to GL_n(\mathbb{C})$ be a representation of $G$. Consider the set

$$S = \{e^{2\pi\iota k/n} : k, n \in \mathbb{Z}\} \subset G$$

consisting of elements of finite order in $G$. Clearly under the map $\rho$ elements of $S$ are mapped to certain $n \times n$ matrices of finite order (hence diagonalizable) which pairwise commute. Thus we know that $\rho(S)$ can be simultaneously diagonalized. Assuming this done, $\rho$ maps $S$ to the subgroup of $GL_n$ consisting of only diagonal matrices $D_n$. The assumption of representation being regular implies that $\rho(G) \subset D_n$. Thus $\rho = \rho_1 \oplus \ldots \oplus \rho_n$ where for each $i$, $\rho_i(z) = z^{k_i}$ is a representation of dimension 1.

Thus if for each $m \in \mathbb{Z}$ we define $\rho_m : G \to GL_1(\mathbb{C}) = \mathbb{C}^\times$ by $z \longmapsto z^m$, then $\Omega_G = \{\rho_m : m \in \mathbb{Z}\}$ and

$$\rho_m \otimes \rho_n = \rho_{m+n}$$

One can consider the algebra $\mathcal{A} = \mathbb{C}[x, x^{-1}]$ together with $G$ acting on $x$ by multiplication. Then

$$\mathcal{A} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}x^n = \bigoplus_{n \in \mathbb{Z}} \rho_n$$

So the solution of the problem stated in last section is:

$$c_{m,n}^p = \begin{cases} 1 & \text{if} \quad p = m + n \\ 0 & \text{otherwise} \end{cases}$$

2.2. $G = SL_2(\mathbb{C})$. **Prof. Weyman's Invariant Theory: Homework 1, problem 5**Now take $G$ to be group of $2 \times 2$ matrices of determinant 1. One natural choice of representation of $G$ is $V = \mathbb{C}^2$ and $\rho : G \to GL_2(\mathbb{C})$ is just inclusion. Let $\mathcal{A} = Sym(V)$, i.e, if $x, y$ is a basis of $V$ then $\mathcal{A} = \mathbb{C}[x, y]$. Let $\mathcal{A}_d$ be the space of degree $d$ homogeneous polynomials from $\mathcal{A}$. It is known (and we will see a proof in next section) that in this case

$$\Omega_G = \{\mathcal{A}_d : d \in \mathbb{Z}_{\geq 0}\}$$

and we have following decomposition of tensor product (assuming $d \geq d'$)

$$\mathcal{A}_d \otimes \mathcal{A}_{d'} = \mathcal{A}_{d+d'} \oplus \mathcal{A}_{d+d'-2} \oplus \mathcal{A}_{d+d'-4} \oplus \cdots \oplus \mathcal{A}_{d-d'}$$

Therefore the problem TPM has solution (Clebsch-Gordon rule)

$$c_{d,d'}^{d''} = \begin{cases} 1 & \text{if} \quad d + d' - d'' \in 2\mathbb{Z}_{\geq 0} \text{ and } d'' \geq |d - d'| \\ 0 & \text{otherwise} \end{cases}$$

## 3. Review of the theory of simple Lie algebras

3.1. **Glossary.** For a simple Lie group $G$, $\mathfrak{g}$ will always denote its Lie algebra, $\mathfrak{h}$ will be the Cartan subalgebra of $\mathfrak{g}$ (CSA for short) $R \subset \mathfrak{h}^*$ will denote the root system, $\Pi$ a base of root system, $R_\pm$ positive/negative roots. We will always enumerate

$$\Pi = \{\alpha_1, \ldots, \alpha_n\}$$

The elements of $\Pi$ are called simple roots. $s_i = s_{\alpha_i} \in Aut(\mathfrak{h}^*)$ called simple reflections and $W \subset Aut(\mathfrak{h}^*)$, the group generated by $s_i's$ is called the Weyl group. $\mathfrak{g}$ is generated by elements $\{f_i, h_i, e_i\}_{i=1,\ldots,n}$ together with following relations

$$[h_i, h_j] = 0,\ [h_i, e_j] = a_{ij}e_j,\ [h_i, f_j] = -a_{ij}f_j,\ [e_i, f_j] = \delta_{ij}h_i$$

$$(ad(e_i))^{1-a_{ij}}(e_j) = 0$$

We choose $\omega_i \in \mathfrak{h}^*$ by requiring $\omega_i(h_j) = \delta_{ij}$ (called fundamental weights). Set $P = \oplus_i \mathbb{Z}\omega_i$ and $Q = \oplus_i \mathbb{Z}\alpha_i$ called weight lattice and root lattice respectively. For any representation $V$ of $\mathfrak{g}$ we denote

$$V(\gamma) = \{v \in V | h.v = \gamma(h)v \forall h \in \mathfrak{h}\}$$

called weight space of $V$ of weight $\gamma \in \mathfrak{h}^*$. It is easy to check that

$$e_i V(\gamma) \subset V(\gamma + \alpha_i),\ f_i V(\gamma) \subset V(\gamma - \alpha_i)$$

We define an ordering on $P$ by saying $\lambda > \mu$ if $\lambda - \mu \in Q_+$. This justifies the names "raising operators" for $e_i's$ and "lowering operators" for $f_i's$.

In this case the finite dimensional irreducible representations of $G$ are parametrized by $P_+$, i.e, $\Omega_G = P_+$. The correspondence maps every $\lambda \in P_+$ to unique finite dimensional irreducible representation $V_\lambda$ which is characterized by requiring $V_\lambda(\mu) \neq 0$ implies $\mu < \lambda$ (that is, $\lambda$ is *highest weight*).

3.2. **Rank 2 case.** Let us work out the rank 2 cases. Let $R$ be irreducible rank system of rank 2. Choose two linearly independent vectors $\alpha, \beta \in R$ at an acute angle with each other (exist since $R$ spans $\mathbb{R}^2$ and is invariant under reflections through $\alpha^\perp$). Set

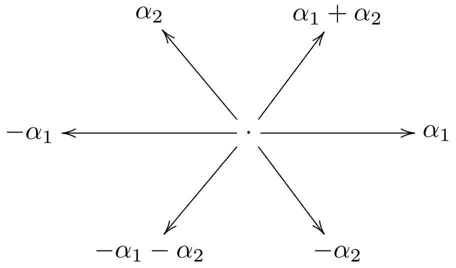
$$k = A_{\alpha,\beta} = \frac{2||\beta|| \cos\theta}{||\alpha||}, \; l = A_{\beta\alpha} \in \mathbb{Z}$$

Then $0 < kl = 4\cos^2\theta < 4$. Assuming $l < k$, we have following possibilities:

($A_2$) $l = 1, k = 1$ The Cartan matrix is

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

with root system



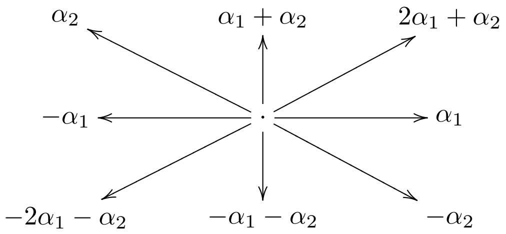
In this case the Weyl group $W$ is generated by two reflections $s_1$ and $s_2$ modulo relations: $s_1^2 = s_2^2 = (s_1 s_2)^3 = 1$. The corresponding Lie group is $SL_3$ and Lie algebra $\mathfrak{sl}_3$. A concrete realization of these basis elements is given in next section. In this case the fundamental weights are computed as:

$$\omega_1 = \frac{2\alpha_1 + \alpha_2}{3}, \quad \omega_2 = \frac{\alpha_1 + 2\alpha_2}{3}$$

($B_2$) $l = 1, k = 2$. The Cartan matrix is

$$A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$$

And root system is:



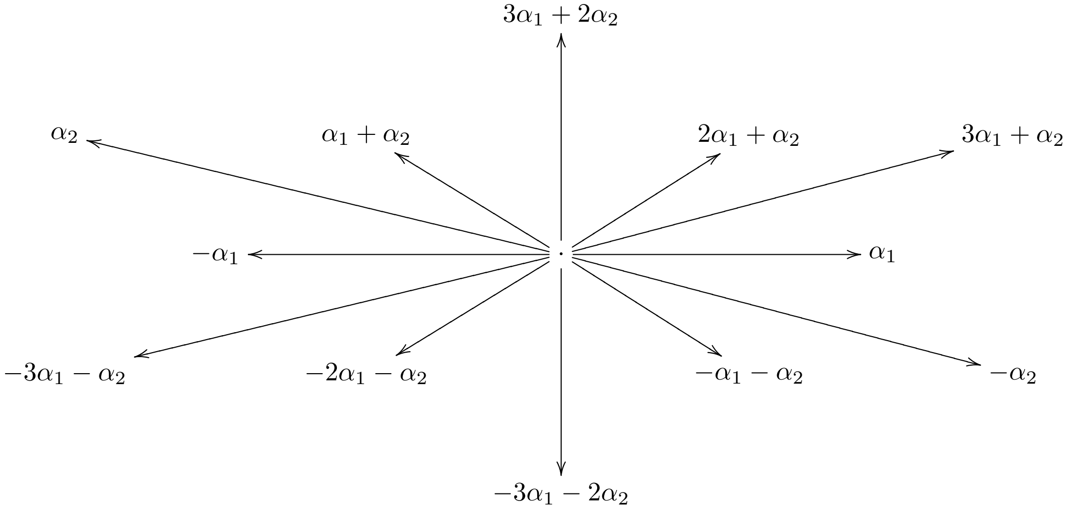
The Weyl group is generated by $s_1, s_2$ modulo the relations: $s_1^2 = s_2^2 = (s_1 s_2)^4 = 1$. The corresponding Lie group is $Sp(4)$ with Lie algebra $\mathfrak{sp}(4)$ A concrete realization in this case is given in next section. The fundamental weights are computed as:

$$\omega_1 = \alpha_1 + \frac{1}{2}\alpha_2, \quad \omega_2 = \alpha_1 + \alpha_2$$

($G_2$) $l = 1, k = 3$. The Cartan matrix in this case is:

$$A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

And the root system is:



The Weyl group is again generated by $s_1, s_2$ modulo relations: $s_1^2 = s_2^2 = (s_1s_2)^6 = 1$. The fundamental weights are computed as:

$$\omega_1 = 2\alpha_1 + \alpha_2, \quad \omega_2 = 3\alpha_1 + 2\alpha_2$$

## 4. Examples of $SL_2(\mathbb{C})$, $SL_3(\mathbb{C})$ and $Sp(4)$

First we finish the proof in the case of $G = SL_2$. Its Lie algebra is $\mathfrak{g} = \mathfrak{sl}_2$ consisting of $2 \times 2$ matrices of trace 0, which has basis

$$f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

together with relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h$$

If $V$ is a representation of $\mathfrak{g}$ then first we can choose a non-zero eigenvector $v_c$ of $h$ with eigenvalue $c \in \mathbb{C}$. One can easily check that $e.v_c = v_{c+2}$ and $f.v_c = v_{c-2}$. Thus the subspace of $V$ consisting of eigenvectors of $h$ is a subrepresentation of $V$. In case $V$ is irreducible, this means that $h$ acts by diagonal matrix on $V$. Moreover choose (maximal) part of basis of $V$ given by iterated action of $e$:

$$v_d \xleftarrow[e]{} v_{d-2} \qquad \cdots \xleftarrow[e]{} v_{d-2l}$$

It is easy to check that this is subrepresentation of $V$ and hence equal to $V$. Moreover $l + 1 = dim(V)$. Finally the commutation relations among $f, h, e$ yield that $d = l$, in particular $d \in \mathbb{Z}_{\geq 0}$.

In the example constructed in section 2.2, one can choose a basis $x, y$ of $\mathbb{C}^2$ such that

$$e.x = y, \ e.y = 0, \ f.x = 0, \ f.y = x, \ h.x = -x, h.y = y$$

and the action of $\mathfrak{g}$ is extended to $\mathcal{A} = \mathbb{C}[x, y]$ by Leibniz identity, making $\mathcal{A}_d$ an irreducible representation of dimension $d + 1$ as follows:

$$y^d \underset{e}{\overset{f/d}{\rightleftarrows}} xy^{d-1} \underset{e/2}{\overset{f/(d-1)}{\rightleftarrows}} x^2y^{d-2} \qquad \cdots \underset{e/d}{\overset{f}{\rightleftarrows}} x^d$$

Now the decomposition $\mathcal{A}_d \otimes \mathcal{A}_{d'} = \mathcal{A}_{d+d'} \oplus \mathcal{A}_{d+d'-2} \oplus \cdots \oplus \mathcal{A}_{d-d'}$ is easy to check.

4.1. **Model algebra: the based affine space.** To this end, it is clear that in order to attack the problem TCM, one must follow following steps (this approach was initiated by Gel'fand and Zelevinsky in "Canonical basis in irreducible representations of $\mathfrak{gl}_3$ and its applications" in Group theoretical methods in Physics - II (1983))

- Construct an algebra $\mathcal{A}$ together of $G$-action such that $\mathcal{A}$ breaks into direct sum of irreducible $G$ modules, each with multiplicity one (the term *model algebra* is used sometimes to describe such an algebra)
- Construct a "nice basis" of this algebra.

For simple Lie groups, it is well known that (as right $G$-modules)

$$\mathbb{C}[N^- \backslash G] = \bigoplus_{\lambda \in P_+} V_\lambda$$

That is, the algebra of regular functions on $G$ which are invariant under left multiplication by elements of $N^-$ is the required "model algebra". Thus we have to come up with a "nice basis" of $\mathbb{C}[G]^{N^-}$.

4.2. $G = SL_3(\mathbb{C})$**.** In this case $\mathfrak{g} = \mathfrak{sl}_3$ has basis

$$h_1 = E_{11} - E_{22}, h_2 = E_{22} - E_{33}, e_1 = E_{12}, e_2 = E_{23}, [e_1, e_2] = E_{13}$$

and $f_i = e_i^T$. Here $E_{ij}$ is matrix with 1 on $(i, j)$ position and zeros everywhere else.

$$N^- = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} | a, b, c \in \mathbb{C} \right\}$$

The algebra $\mathcal{A} = \mathbb{C}[G]^{N^-}$ in this case consists of polynomials in entries of matrices from $G$ which are unchanged by row operations which change a row by some linear combination of previous rows. Hence $\mathcal{A}$ is generated by following functions:

$$D_1(A) = a_{1i} (i = 1, 2, 3), \quad \Delta_{ij} = a_{1i}a_{2j} - a_{1j}a_{2i} (1 \leq i < j \leq 3)$$

with only one relation

$$D_1\Delta_{23} - D_2\Delta_{13} + D_3\Delta_{12} = 0$$

$\mathfrak{g}$ acts on $\mathcal{A}$ by following:

$$\begin{tikzcd} D_2 \arrow[rr, "e_1"] & & D_1 \\ & D_3 \arrow[ul, "e_2"] & \end{tikzcd}$$

$$\begin{tikzcd}
 & \Delta_{12} & \\
\Delta_{23} \arrow[rr, "e_1"'] & & \Delta_{13} \arrow[ul, "e_2"']
\end{tikzcd}$$

Hence the answer can be computed as: number of times $\lambda + \nu - \gamma$ appears in $V_\lambda \otimes V_\mu$ is number of positive integer solutions to following system (together with constraint that $n_2 m_{13} = 0$)

$$n_1 + n_2 + n_3 = \lambda_1, m_{12} + m_{13} + m_{23} = \lambda_2$$

$$n_2 + n_3 + m_{23} + (n_2 - m_{13}) = 2\gamma_1 - \gamma_2$$

$$n_3 + m_{13} + m_{23} - (n_2 - m_{13}) = 2\gamma_2 - \gamma_1$$

$$n_2 + m_{23} \leq \nu_1, \; m_{13} + n_3 \leq \nu_2$$

4.3. $G = Sp(4)$**. Prof. Weyman's Invariant Theory: Homework 1, problem 3 c)** Consider a vector space $V = \mathbb{C}^4$ together with bilinear form defined by matrix

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

The group $G = Sp(4)$ is defined to be the group of automorphisms of $V$ preserving the bilinear form, i.e

$$G = \{A \in GL(4) | A^T J A = J\}$$

More explicitly $G$ consists of $4 \times 4$ matrices $A$ satisfying

$$A^{ij}_{14} - A^{ij}_{23} = \begin{cases} 1 & \text{if} \quad (i,j) = (1,4) \\ -1 & \text{if} \quad (i,j) = (2,3) \\ 0 & \text{otherwise} \end{cases}$$

Therefore the Lie algebra $\mathfrak{g} = \mathfrak{sp}(4)$ consists of all matrices $X$ such that $X^T J = -JX$. The basis of this Lie algebra can be taken to be:

$$h_1 = E_{11} - E_{22} + E_{33} - E_{44}, h_2 = E_{22} - E_{33}$$

$$e_1 = E_{12} + E_{34}, e_2 = E_{23}, [e_1, e_2] = E_{13} - E_{24}, [e_1, [e_1, e_2]] = -2E_{14}$$

and $f_i = e_i^T$.

Similar to previous case, $N^-$ invariant polynomial functions on $G$ are generated by

$$D_i (1 \leq i \leq 4) \text{ and } \Delta_{ij} (1 \leq i < j \leq 4)$$

and we have $\Delta_{14} = \Delta_{23}$. $\mathfrak{g}$ acts on these functions by:

$$\begin{tikzcd}
D_2 \arrow[dr, "f_2"'] & D_1 \arrow[l, "f_1"'] \\
D_4 & D_3 \arrow[l, "f_1"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & \Delta_{12} \arrow[dr] & \\
\Delta_{24} \arrow[dr] & \Delta_{14} \arrow[l] & \Delta_{13} \arrow[l] \\
 & \Delta_{34} &
\end{tikzcd}$$

Moreover there are following relations among these functions

$$(\forall 1 \leq i < j < k \leq 4) \quad D_i\Delta_{jk} - D_j\Delta_{ik} + D_k\Delta_{ij} = 0$$

$$\Delta_{12}\Delta_{34} + \Delta_{14}^2 = \Delta_{13}\Delta_{24}$$

## 5. Cluster algebras

5.1. **Definition.** Let $m \geq n$ be two non-negative integers. Let $\mathcal{F} = \mathbb{Q}(u_1, \ldots, u_m)$ be field of rational functions in $m$ variables. A *seed* in $\mathcal{F}$ is a tuple $\Sigma = (\mathbf{x}, \widetilde{B})$ where

- $\widetilde{\mathbf{x}} = \{x_1, \ldots, x_m\}$ is a set of $m$ free generators of $\mathcal{F}$ over $\mathbb{Q}$.
- $\widetilde{B} = (b_{ij})$ is $m \times n$ matrix with integer entries, such that the square submatrix $B = (b_{ij})_{1\leq i,j\leq n}$ obtained by taking first $n$ rows (called principal part of $\widetilde{B}$) is skew symmetrizable (i.e, there exist $n$ positive integers $d_1, \ldots, d_n$ such that diagonal$(d_1, \ldots, d_n).B$ is skew symmetric).

For each $k$, $1 \leq k \leq n$, we define *mutation of* $\Sigma$ *in direction* $k$ as another seed $\Sigma' = \mu_k(\Sigma)$ if $\Sigma' = (\widetilde{\mathbf{x}'}, \widetilde{B}')$ where

- $\widetilde{\mathbf{x}'} = (x'_1, \ldots, x'_m)$ are given by

$$x'_i = x_i \text{ if } i \neq k$$

$$x_k x'_k = \Pi_{i=1}^m x_i^{[b_{ik}]_+} + \Pi_{i=1}^m x_i^{[-b_{ik}]_+}$$

  here we use the notation $[b]_+ = \max(0, b)$.
- The matrix $\widetilde{B}' = (b'_{ij})$ is given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if} \quad i = k \text{ or } j = k \\ b_{ij} + \operatorname{sgn}(b_{ik})[b_{ik}b_{kj}]_+ & \text{otherwise} \end{cases}$$

It is well known (and easy to check) that a) the principal part of $\widetilde{B}'$ is again skew-symmetrizable; and b) mutation is an involution (i.e, $\mu_k(\mu_k(\Sigma)) = \Sigma$). This allows us to define equivalence relation: two seeds $\Sigma$ and $\Sigma'$ in $\mathcal{F}$ are *mutation equivalent* if there exists a sequence $(k_1, \ldots, k_l)$ such that $\Sigma' = \mu_{k_1} \ldots \mu_{k_l}(\Sigma)$.

If $\Sigma = (\widetilde{\mathbf{x}}, \widetilde{B})$ is a seed of $\mathcal{F}$, then the set $\widetilde{\mathbf{x}}$ is called *extended cluster*, $\mathbf{x} = \{x_1, \ldots, x_n\}$ is called cluster and $x_i$, $1 \leq i \leq n$ are called *cluster variables.*

Note that by applying mutations we only change the functions $x_i$ $1 \leq i \leq n$. Therefore if $\Sigma$ is mutation equivalent to $\Sigma'$, then $x_j = x'_j$ for every $n < j \leq m$. We refer to $x_j$ with $n < j \leq m$ as *coefficients.* We fix the ground ring $R$ as some subring of $\mathbb{Z}[x_j^{\pm} : n < j \leq m]$ containing $\mathbb{Z}[x_j : n < j \leq m]$, i.e,

$$\mathbb{Z}[x_{n+1}, \ldots, x_m] \subseteq R \subseteq \mathbb{Z}[x_{n+1}^{\pm}, \ldots, x_m^{\pm}]$$

Define *the upper cluster algebra* $\overline{\mathcal{A}(\Sigma)}$ as subring of $\mathcal{F}$ consisting of functions $f \in \mathcal{F}$ such that for every $\Sigma' \sim \Sigma$, $f$ can be expressed as Laurent polynomial in cluster variables of $\Sigma'$ with coefficients from $R$.

We define cluster algebra associated to seed $\Sigma$, denoted by $\mathcal{A}(\Sigma)$ as $R$-subalgebra of $\mathcal{F}$ generated by all cluster variables belonging to seeds $\Sigma' \sim \Sigma$. The following result is known as *Laurent phenomenon*

$$\mathcal{A}(\Sigma) \subset \overline{\mathcal{A}(\Sigma)}$$

5.2. **Model algebra as cluster algebra.** In this section I will describe a combinatorial procedure to define cluster algebra structure on the model algebra $\mathbb{C}[N^- \backslash G]$. It is clear from previous part that we only need to define a matrix $\widetilde{B}$ to have a cluster algebra. This matrix is computed in following steps:

(1) Choose a reduced expression $\mathbf{i} = (i_1, \ldots, i_n)$ of the longest element $w_0 \in W$.
(2) Construct a valued graph $\Gamma$ whose vertices are $\{-r, -r+1, \ldots, -1, 1, 2, \ldots, n\}$. There are two types of arrows in $\Gamma$. To describe them we use the notation $[k]^+$ for the smallest index $l$ larger than $k$ such that $i_k = i_l$ (for $1 \leq k \leq n$). For $k < 0$ the notation $[k]^+$ means the smallest index $l$ ($1 \leq l \leq n$) such that $i_l = -k$.

(Horizontal arrows) $k \to l$ (no valuation) if $l = [k]^+$.
(Vertical arrows) $k \leftarrow l$ if $k < l < [k]^+ < [l]^+$; together with valuation $|b_{kl}| = |a_{kl}|$.

Let $\widetilde{B}$ be the adjacency matrix of this valued graph, with rows labeled by vertices of $\Gamma$ and columns labeled by vertices $\{i : 1 \leq i \leq n,\ i^+\text{exists}\}$

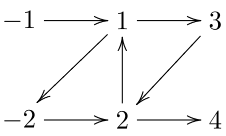
The description of $N^-$ invariant functions on $G$ corresponding to each cluster variable is slightly more involved and is skipped here for simplicity.

## 6. Example of $Sp(4)$ revisited

In this section we illustrate the constructions of previous part in the case of $G = Sp(4)$. In this case the Weyl group has following presentation

$$W =< s_1, s_2 | s_1^2 = s_2^2 = (s_1 s_2)^4 = 1 >$$

Thus the longest element has two reduced expressions: $\mathbf{i_1} = (1, 2, 1, 2)$ and $\mathbf{i_2} = (2, 1, 2, 1)$. The $\Gamma$ for the fist reduced expression is:



where the arrows between top and bottom rows are valued so as to have following exchange matrix

$$\widetilde{B} = \begin{bmatrix} 0 & -2 \\ 1 & 0 \\ 1 & 0 \\ -1 & 1 \\ -1 & 2 \\ 0 & -1 \end{bmatrix}$$

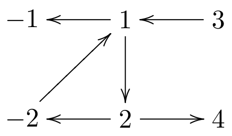
The isomorphism between cluster algebra with above exchange matrix and $\mathbb{C}[G]^{N^-}$ is given by

$$X_{-1} = D_4,\ X_1 = D_3,\ X_3 = D_1$$
$$X_{-2} = \Delta_{34},\ X_2 = \Delta_{13},\ X_4 = \Delta_{12}$$

Let us denote this seed in $\mathbb{C}[N^-\backslash G]$ by $\Sigma$

In this case one can compute all the clusters, by using the relations for $Sp(4)$:

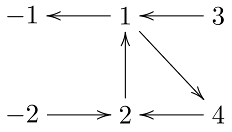
- $\mu_1(\Sigma)$: The exchange matrix is encoded in following graph

  and all the functions except for the one at 1 remain unchanged. We compute $X_1^{(1)}$ as:

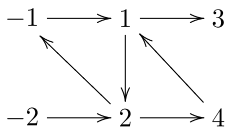
  $$X_1^{(1)} = \frac{X_2X_{-1} + X_{-2}X_3}{X_1} = \frac{D_4\Delta_{13} + D_1\Delta_{14}}{D_3} = \Delta_{14}$$

  where last equality follow from relations described in section 4.3
- $\mu_2\mu_1(\Sigma)$: The exchange graph is given by:

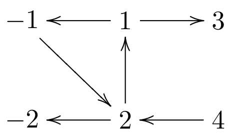
  And the function at vertex 2 is obtained by

  $$X_2^{(2)} = \frac{(X_1^{(1)})^2 + X_{-2}X_3}{X_2} = \frac{\Delta_{14}^2 + \Delta_{12}\Delta_{34}}{\Delta_{13}} = \Delta_{24}$$
- $\mu_1\mu_2\mu_1(\Sigma)$: again we write the exchange matrix in the following form:

  and the function $X_1^{(3)}$ is computed as:

  $$X_1^{(3)} = \frac{X_2^{(2)}X_3 + X_4X_{-1}}{X_1^{(1)}} = \frac{\Delta_{24}D_1 + \Delta_{12}D_4}{\Delta_{14}} = D_2$$
- $\mu_2\mu_1\mu_2\mu_1(\Sigma)$:

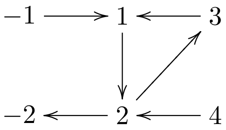
  

  The function $X_2^{(4)}$ ends up being equal to $\frac{D_2^2\Delta_{34}+D_4^2\Delta_{12}}{\Delta_{24}}$ which we compute as:

  $$D_2^2\Delta_{34} = D_2(D_3\Delta_{24} - D_4\Delta_{23})$$
  $$D_4^2\Delta_{12} = D_4(D_2\Delta_{14} - D_1\Delta_{24})$$

Therefore, $X_2^{(4)} = D_2 D_3 - D_1 D_4$.

- $\mu_1\mu_2\mu_1\mu_2\mu_1(\Sigma)$:



and the function can be easily computed to be $X_1^{(5)} = D_3$.

- Finally one can check that $(\mu_2\mu_1)^2(\Sigma) = \Sigma$.

So as canonical basis of $Sp(4)$, one can take monomials in $X_i^{(j)}$, such that no monomial contains functions not belonging to same cluster (called cluster monomials).

## 7. Appendix: review of the theory of simple Lie algebras

In this section I will describe (very roughly) the major constituents of classification theorem of simple Lie groups (simply connected). The classification takes place in following steps:

- Lie's Theorem
- The bijection between simple Lie algebras and root systems (irreducible)
- The bijection between root systems and Dynkin diagrams

### 7.1. **Lie's Theorem.**

For a Lie group $G$, one can define its Lie algebra $\mathfrak{g}$ as tangent space to $G$ at identity, or equivalently as space of left invariant vector fields on $G$. For our purposes (when $G$ is defined as set of matrices satisfying certain polynomial conditions), its Lie algebra can be computed by substituting $x_{ij} \to t.x_{ij} + \delta_{ij}$ in the polynomial conditions and taking coefficient of $t$ as defining equation. That is, if $f(x) = 0$ for every $x \in G$ then set

$$\underline{f}(y) = \frac{d}{dt}|_{t=0} f(I_n + ty + t^2y^2/2 + \cdots)$$

Thus, if $G$ is defined (as subset of $M_n(\mathbb{C})$) by equations $f_1 = \ldots = f_r = 0$, then $\mathfrak{g}$ is defined (again in $M_n(\mathbb{C})$) by $\underline{f_1} = \ldots = \underline{f_r} = 0$.

For instance $G = SL_n(\mathbb{C})$ is given by $X = (x_{ij}) \in SL_n$ if and only if

$$det(X) = \sum_{\sigma \in S_n} sgn(\sigma) x_{1\sigma(1)} x_{2\sigma(2)} \ldots x_{n\sigma(n)} = 1$$

replacing $x_{ij}$ by $tx_{ij} + \delta_{ij}$ in above expression yields

$$1 + t\sum_i x_{ii} + t^2(...) = 1$$

Thus its Lie algebra, denoted by $\mathfrak{sl}_n$ consists of matrices $X = (x_{ij})$ such that $\sum x_{ii} = Tr(X) = 0$.

The Lie algebra comes equipped with bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ denoted by $(x, y) \longmapsto [x, y]$ which satisfies:

- $[x, x] = 0$ for every $x \in \mathfrak{g}$
- $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for every $x, y, z \in \mathfrak{g}$.

Thus a finite dimensional vector space $\mathfrak{g}$ over $\mathbb{C}$ together with bilinear map $[., .]$ satisfying above conditions is defined to be a *Lie algebra.*

Lie's theorem asserts that, as long as we restrict ourselves to simply connected Lie groups, the whole group is completely determined by its Lie algebra. Moreover representation theory of Lie groups goes parallel to representation theory of $\mathfrak{g}$.

A representation of Lie algebra $\mathfrak{g}$ is a linear map $\rho : \mathfrak{g} \to End(V)$ such that

$$\rho([x, y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$$

We can obtain a representation of $\mathfrak{g}$ starting from a representation of the group $G$ as:

$$X.v = \frac{d}{dt} \exp tX.v|_{t=0}$$

7.2. **Simple Lie algebras and Root systems.** A root system $R$ of rank $n$ is a subset of $n$-dimensional Euclidean space $V$ (i.e, a real $n$-dimensional vector space together with inner product $(.,.)$) satisfying:

(R1) $0 \notin R$, $|R| < \infty$ and $R$ spans $V$.

(R2) For every $\alpha, \beta \in R$, we have

$$A_{\alpha,\beta} = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$$

(R3) For every $\alpha \in R$, define $s_\alpha \in Aut(V)$ as

$$s_\alpha(\gamma) = \gamma - \frac{2(\gamma,\alpha)}{(\alpha,\alpha)}\alpha$$

Then $s_\alpha(R) \subset R$.

We say $R$ is reducible if $V = V_1 \oplus V_2$ and $R_i = R \cap V_i$ gives $R = R_1 \sqcup R_2$. Otherwise $R$ is said to be irreducible.

Given an irreducible root system of rank $n$, one can define a Lie algebra $\mathfrak{g}$ as:

$$\mathfrak{g} = \mathfrak{h} \oplus \left( \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \right)$$

where $\mathfrak{h} = V^* \otimes_{\mathbb{R}} \mathbb{C}$ and each $\mathfrak{g}_\alpha$ is one dimensional complex vector space. The bracket on $\mathfrak{g}$ is defined as:

$$[h, \mathfrak{g}_\alpha] = < \alpha, h > \mathfrak{g}_\alpha,\ [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta},\ [\mathfrak{h}, \mathfrak{h}] = 0$$

One of the important results in theory of simple Lie algebras states that every simple Lie algebra is obtained this way.

7.3. **Root systems and Dynkin diagrams.** In this section I will outline the classification of irreducible root systems. Let $R$ be an irreducible root system of rank $n$. Firstly we can choose a base of $R$, denoted by $\Pi$, such that $\Pi$ is a basis of $V$ and every element of $R$ can be written as linear combination of elements of $\Pi$ with either all non-negative integer coefficients, or all non-positive integer coefficients. To prove that such a subset of $R$ exists, set $H_\alpha = \alpha^\perp$ and choose some $\gamma \notin \cup_{\alpha \in R} H_\alpha$. Choose one connected component $C$ of $R \setminus H_\gamma$ and define $R_+ = R \cap C$. $\Pi$ can be taken to be elements of $R_+$ which cannot be written as positive combination of other elements of $R_+$. Let

$$\Pi = \{\alpha_1, \ldots, \alpha_n\}$$

Set $A = (a_{ij})$ defined by

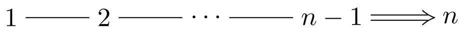
$$a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \in \mathbb{Z}$$

called the *Cartan matrix*. Define Dynkin diagram of $R$, $\Gamma$ as a valued graph with vertices $\{1, \ldots, n\}$ and $a_{ij}a_{ji}$ arrows connecting $i$ with $j$. The notation $i \Longleftarrow j$ indicates that $a_{ij}a_{ji} = 2$ and $\alpha_j$ is longer than $\alpha_i$. Thus the root system is completely determined by corresponding Dynkin diagram. Finally following is the list of all Dynkin diagrams:

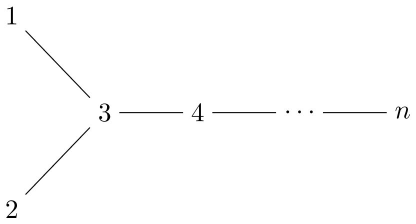
$$A_n (n \geq 1)\ 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow n$$

$$B_n (n \geq 2)\ 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-1 \Longleftarrow n$$

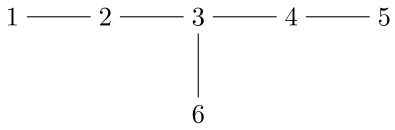
$C_n (n \geq 3)$



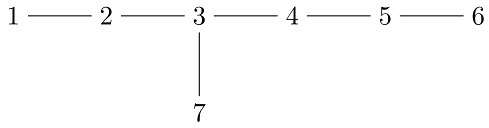
$D_n (n \geq 4)$



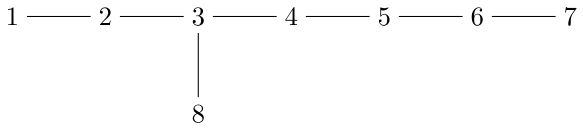
$E_6$



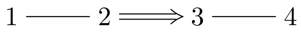
$E_7$



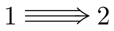
$E_8$



$F_4$



$G_2$



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