Convex Polytopes
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<u>Def</u> The convex hull of a set S is the smallest convex set containing S

<u>note:</u> This is well-defined as the intersection of all convex sets containing S

<u>Notation</u> $\operatorname{conv} S$

<u>Def</u> A convex polytope is the convex hull of a finite set

<u>note:</u> If these points are in $\mathbb{R}^n$, but the affine hull of the points is a copy of $\mathbb{R}^d$ $(d \leq n)$ we view the points as points of $\mathbb{R}^d$ and call d the dimension of the polytope.

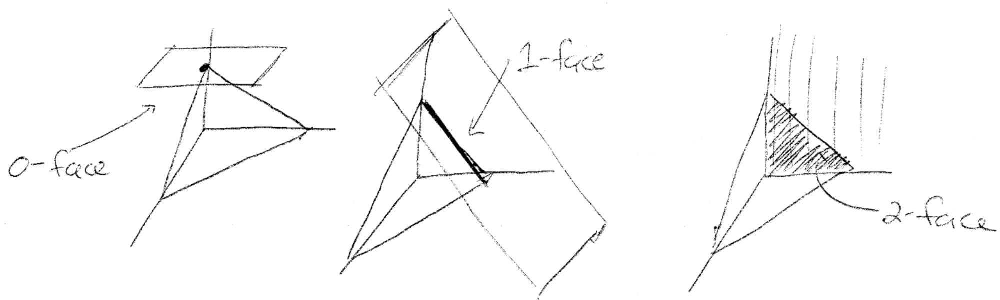
<u>Def</u> A hyperplane, H, <u>cuts</u> a set A provided that both open half-spaces determined by H contain points of A

H <u>supports</u> A if H does not cut A, but $\delta(H, A) = 0$

<u>Def</u> A <u>face</u> of a polytope P is either $\emptyset$, P itself or the intersection of P with

a supporting hyperplane. (If the face is not $\emptyset$ or P, it is called a <u>proper face</u>).

<u>Example</u>



A face is called of dim k or a k-face if the affine hull of the face has dim k

0-faces are called vertices.
1-faces are called edges.
maximal proper faces are called facets.

The set of all faces is denoted $\mathcal{F}(P)$.

<u>Examples</u>
2 polytopes are polygons: $\triangle$, $\square$, ⌂, ...

There are many ways to build up from d-dimensional polytopes to $d+1$ dimensional polytopes, though not all polytopes can be derived in one of these ways.

<u>d-simplex</u> is the convex hull of $(d+1)$ affinely independant points (this is the fewest number of vertices a d-polytope can have)

it can be built inductively:

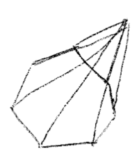
a 1-simplex is a segment

To build a d-simplex, take a $d-1$ simplex, add a point not in the affine hull of the $d-1$ simplex, and take the convex hull.

• Each facet of a d-simplex is a $(d-1)$ simplex.

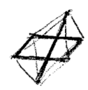
This is an example of a

<u>Pyramid</u>: The convex hull of the union of a $(d-1)$-polytope (called the basis), and a point not in the affine hull of the basis (called the apex)



<u>note</u>: Every face of the pyramid is either a face of the basis or is itself a pyramid with the same apex

Dipyramids convex hull of a $(d-1)$-polytope, K^{d-1} and an interval I, not parallel to the affine hull of K^{d-1}, which intersects K^{d-1} in the relative interior of K^{d-1} and the relative interior of I.

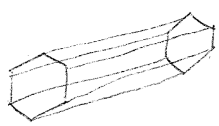


Prisms

K^{d-1} a $(d-1)$-polytope

$I=[0,x]$ a segment not parallel to aff K^{d-1}

prism is the convex hull of K^{d-1} and its translate $K^{d-1}+x$



Example of a prism

The d-cube

$$\{(x_1,\dots,x_d)\in\mathbb{R}^d \,/\, |x_i|\le 1,\ 1\le i\le d\}$$

Neighborly polytopes

Def a polytope is called k-neighborly provided that every k-membered subset, V, of the set vert P determines a proper face $F=\text{conv}\,V$ of P s.t. $V=\text{vert}\,F$

Examples • for $d\geq 1$, every d-polytope is 1-neighborly (That is, every vertex is a proper face)

• Every d-simplex is k-neighborly for each $1\le k\le d$

- 2-neighborly ⇒ The 1-skeleton of the polytope is a complete graph

- Cyclic polytopes (defined below) are k-neighborly for $2k \leq d$. In particular for $k = \lfloor \frac{1}{2} d \rfloor$

<u>Def</u> Take the moment curve M_d in $\mathbb{R}^d$ defined by $x(t) = (t, t^2, t^3, \ldots, t^d)$.
A <u>cyclic polytope</u>, $C(v,d)$, is the convex hull of $v \geq d+1$ points $x(t_i)$ on M_d with $t_1 < t_2 < \ldots t_v$.

<u>Claim</u> if $2k \leq d$, then $C(v,d)$ is k-neighborly

<u>pf</u> Let V be the set of vertices of $C(v,d)$.
Given a set $V_k \subset V$ with k elements, we want a supporting hyperplane, H, s.t. $H \cap C(v,d) = \operatorname{conv} V_k$

for $V_k = \{x(t_i^*) \mid i = 1, \ldots, k\}$
Consider the polynomial
$$p(t) = \prod_{i=1} (t - t_i^*)^2$$

When we expand this polynomial, we get $p(t) = \beta_0 + \beta_1 t + \ldots + \beta_{2k} t^{2k}$ for some coefficients β_i which depend only on the t_i^*s.

Since $2k \le d$, let b be the vector
$b = (\beta_1, \beta_2, \ldots, \beta_{2k}, 0, \ldots, 0)$

let $H = \{x \in \mathbb{R}^d / \langle x, b \rangle = -\beta_0\}$ be a hyperplane

for any $x(t) \in M_d$
$$\langle x(t), b \rangle = \beta_1 t + \beta_2 t^2 + \beta_3 t^3 + \cdots + \beta_{2k} t^{2k} = -\beta_0 + p(t)$$
$$= -\beta_0 + \prod_{i=1}^{k} (t - t_i^*)^2$$

since for each $i = 1, \ldots, k$ $p(t_i^*) = 0$
$x(t_i^*) \in H$ for $1 \le i \le k$

for $x(t) \in M_d \setminus V_k$, (and thus in $V \setminus V_k$)
$$\langle x(t), b \rangle = -\beta_0 + \prod_{i=1}^{k} (t - t_i^*)^2 > -\beta_0$$

Thus H is a supporting hyperplane since all of the vertices of $C(v,d)$ lie on M_d
also, $H \cap V = V_k$

$\operatorname{conv} V_k \subset H \cap C(v,d)$. But $H \cap C(v,d)$ is a face, and is therefore the convex hull of some vertices of $C(v,d)$, and is thus the convex hull of a subset of V_k so

$H \cap C(v,d) \subseteq \operatorname{conv} V_k$.

Thus they are equal.

Thus $\forall d$, $\forall v \ge d+1$, $\exists$ a $\lfloor \frac{1}{2} d \rfloor$-neighborly d-polytope with v- vertices.

we will show This is the best possible (except the simplex)

to do this we need:

Lemma If P is k-neighborly and $1 \le k^* \le k$, then P is k^*-neighborly

pf

- Any k vertices are affinely independant (if not, let v_k be in the affine hull of $v_1, \dots v_{k-1}$. Then v_k is in the face determined by $v_1, \dots, v_{k-1}, w$, but is not a vertex of that face $\Rightarrow$ contradiction)

Thus, every k points not only determine a face, they determine a $(k-1)$-face. Since any $k-1$ face with only k vertices is a simplex, every k points determine a $(k-1)$-simplex.

Given k^* points, they are a subset of k points so they fall in a simplex. Since the simplex is k^*-neighborly, they determine a face of the simplex, therefore a face of P. So P is k^*-neighborly.

Thm (Radon) If A is a $(d+2)$-pointed subset of $\mathbb{R}^d$, then $\exists A', A''$ disjoint

subsets of A s.t. $\operatorname{conv}(A') \cap \operatorname{conv}(A'') \neq \emptyset$
(pf omitted)

Now we are ready to prove:

<u>Prop</u> Say P is a k-neighborly d-polytope and $k > \lfloor \frac{1}{2} d \rfloor$, then $f_0(P) = d+1$ (that is, P is a d-simplex)

← the number of vertices of P

<u>pf</u>

Say $f_0(P) \neq d+1$, then $f_0(P) > d+1$

Let $V \subset \operatorname{vert}(P)$ contain $d+2$ vertices

By Radon, $\exists$ W and Z s.t. $W \cup Z = V$, $W \cap Z = \emptyset$ and $\operatorname{conv} W \cap \operatorname{conv} Z \neq \emptyset$

Wlog, $\operatorname{card} W \leq \lfloor \frac{(d+2)}{2} \rfloor = \lfloor \frac{d}{2} \rfloor + 1 \leq k$

(since $\lfloor \frac{d}{2} \rfloor < 2$ and both are integers)

Since $\operatorname{conv} W \cap \operatorname{conv} Z \neq \emptyset$, every supporting hyperplane H of P which contains W has a non-empty intersection with $\operatorname{conv} Z$.

Therefore, $H \cap Z \neq \emptyset$

Since P is $\operatorname{card} W$ $(\leq k)$-neighborly, W determines a proper face F s.t. $F = \operatorname{conv} W$

Say the supporting hyperplane determining this face is H. Then $H \cap Z \neq \emptyset \subset F$

But W and Z are disjoint so $\exists$ a vertex of P which lies in F but is not a vertex of F, which is a contradiction. $\boxtimes$

Note: $\lfloor \frac{1}{2} d \rfloor$-neighborly polytopes are simply called neighborly d-polytopes.

Note: one of the reasons these are of interest is because they are not simply a generalization of things that happen in 2 or 3 dimensions. Since $\lfloor \frac{1}{2} 2 \rfloor = \lfloor \frac{1}{2} 3 \rfloor = 1$ and every polytope is 1-neighborly, nothing interesting happens in either of those dimensions.

For neighborly polytopes

- # of faces in each dimension is determined by f_0
- $\exists$ a neighborly 4-polytope w/ 8 vertices which is not combinatorially equivalent to $C(8,4)$

Denote the 1-skeleton of a polytope by $G(P)$, the graph of P.

<u>Def</u> A graph G is called $(\mathcal{P}^d)$-realizible if $G = G(P)$ for some d-polytope, P.

<u>Def</u> A graph G is k-connected provided that for every pair of nodes of G, there exist k pairwise disjoint paths in G having those nodes as endpoints

<u>Steinitz's Theorem</u> A graph G is $(\mathcal{P}^3)$-realizible iff G is planar and 3-connected.

(pf omitted)