3-Arc Transitive Cubic Graphs and Abstract 4-Polytopes

Let G be a finite, connected, simple cubic graph. We call a sequence of vertices $[v] = [v_0, v_1, ..., v_t]$ a t-arc if $v_i v_{i+1}$ is an edge of G and $v_{i-1} \neq v_{i+1}$ $\forall i$. Then there is a natural action of $\mathrm{Aut}(G)$ on the t-arcs of G. We say that G is t-transitive if $\mathrm{Aut}(G)$ acts transitively on the t-arcs of G, but not on the $(t+1)$-arcs.

Let $[v] = [v_0, ..., v_t]$ be a t-arc in a cubic graph. Then v_t has two neighbors other than v_{t-1}; call these neighbors y_1 and y_2. Then there are two natural t-arcs $[v^{(1)}] = [v_1, ..., v_t, y_1]$ and $[v^{(2)}] = [v_1, ..., v_t, y_2]$; we call these the *successors* of $[v]$.

Theorem [Tutte 1947]: There are no finite, connected, simple cubic graphs which are t-transitive for $t > 5$.

Theorem [Tutte 1966]:

1. $\mathrm{Aut}(G)$ acts transtively on t-arcs if and only if there exists a t-arc $[v]$ such that there are automorphisms g_i that carry $[v]$ to $[v^{(i)}]$ for $i = 1, 2$.
2. If $[v]$ is a t-arc in a t-transitive graph G, then $\mathrm{Aut}(G)$ acts regularly on t-arcs.

Theorem: If G is t-transitive, then $|\mathrm{Aut}(G)| = |V(G)| \times 3 \times 2^{t-1}$.

Let's look at the Petersen Graph G (F10A in the Foster census). Consider its canonical double cover $D(G)$ (F20B). That is, $V(D(G)) = V(G) \times \mathbb{Z}_2$, with $\mathbb{Z}_2 = \{1, z\}$, and (v_1, k_1) is adjacent to (v_2, k_2) if and only if v_1 is adjacent to v_2 in G and $k_1 k_2 = z$. We can think of this as taking a "red" copy of $V(G)$ and a "blue" copy of $V(G)$, and connecting each red vertex to the blue copies of its neighbors. When we do this for the Petersen Graph, we can identify the red vertices with midpoints of edges of the projection of the 4-simplex, and we can identify the blue vertices with midpoints of 2-faces.

Let P be the 4-simplex; that is, the convex hull of 5 equally spaced points in $\mathbb{R}^4$. Then P has $5! = 120$ symmetries; each one gives rise to an automorphism of $D(G)$. Since P is self-dual, there are 240 extended symmetries, and these give rise to all the automorphisms of G. So we have that $|\mathrm{Aut}(D(G))| = 240 = 20 \times 3 \times 2^{3-1}$, so $D(G)$ is 3-transitive (and so is the Petersen Graph).

Now let's determine the transitivity of a general class of graphs that arise in studying 4-polytopes.

Definition: Let P be a rank 4 polytope. Then the medial layer graph of P, $M(P)$, is the restriction of the Hasse diagram of P to rank 1 and 2 faces.

Note: $M(P)$ is bipartite and simple, and finite if P is finite. Since P is flag connected, $M(P)$ is connected. We want to study the case where $M(P)$ is cubic, which means that any 2-face contains 3 1-faces, and any 1-face is contained in 3 2-faces. Thus we want to study regular polytopes P of type $\{3, q, 3\}$.

The automorphism group of the polytope P has 4 generators $\rho_0, ..., \rho_3$, with the properties that $(\rho_0\rho_1)^3 = (\rho_1\rho_2)^q = (\rho_2\rho_3)^3 = \rho_i^2 = \epsilon$ and $\langle\rho_i \mid i \in I\rangle \cap \langle\rho_i \mid i \in J\rangle = \langle\rho_i \mid i \in I \cap J\rangle$. We can use these properties to define a polytope P algebraically. Let $\Gamma_i = \langle\rho_j \mid i \neq j\rangle$. Then the cosets $\Gamma_j\varphi$ are the j-faces of P. We define

$$\Gamma_j\varphi \leq \Gamma_k\Psi \iff j \leq k \text{ and } \Gamma_j\varphi \cap \Gamma_k\Psi \neq \emptyset.$$

Defining P in this way, the medial layer graph $M(P)$ is connected, simple, and cubic.

Fix a base flag $F = (F_{-1}, F_0, ..., F_4)$ in P such that ρ_i sends F to its i-adjacent flag. Let $v_1 = F_1$ and $v_2 = F_2$; these are both vertices of $M(P)$. There is a cycle of length $2q$ in $M(P)$ that alternates between 1-faces and 2-faces; let the vertices of this cycle be $v_0, v_1, ..., v_{2q-1}$; this includes v_1 and v_2 as defined above. Since $M(P)$ is cubic, each of these vertices has a neighbor not in the cycle; let w_i be the neighbor of v_i. If $q = 2$, then $M(P) = K_{3,3}$; otherwise all the vertices w_i are distinct.

Let δ be the dual automorphism that reverses F. Then $\delta\rho_i\delta = \rho_{3-i}$. Furthermore, since δ swaps v_1 and v_2 and it fixes the cycle, it must swap v_0 and v_3, v_{2q-1} and v_4, etc. The automorphism ρ_0 fixes v_1, v_2, v_0, and w_1, and swaps v_{2q-1} with w_0, v_3 with w_2, etc. It can be shown that the automorphisms $\delta\rho_2$ and $\delta\rho_2\rho_0$ send the 3-arc $[v_0, v_1, v_2, v_3]$ to its 2 successors, which is enough to show that $M(P)$ is at least 3-transitive. Tutte's theorem tells us that $M(P)$ is at most 5-transitive. To show that $M(P)$ is 3-transitive (and not 4- or 5-transitive), we consider the automorphism $\alpha = (\rho_3\rho_2\rho_0)$. For $M(P)$ to be 4-transitive, there can't be any elements of order 6 that stabilize a vertex; α stabilizes v_1 and has order 6. For $M(P)$ to be 5-transitive, the elements of order 6 in a vertex stabilizer must permute the 6 elements that are a distance 2 away from that vertex in two disjoint 3-cycles. However, α permutes the elements that are a distance 2 away from v_1 in a 6-cycle. So $M(P)$ is 3-transitive.

Conclusion: If P is a regular self-dual polytope of type $\{3, q, 3\}$, then $M(P)$ is connected, cubic, and 3-transitive.

Now, what can we say in the reverse direction? That is, given a graph G, when could it be the medial layer graph of a polytope? We first notice that:

- ρ_0 fixes $N(v_1)$.
- ρ_1 fixes $N(w_2)$.
- ρ_2 fixes $N(w_1)$.

- ρ_3 fixes $N(v_2)$.

Given a 3-transitive, connected, cubic graph G, define ρ_0 to be the unique automorphism that fixes $N(v_1)$, and likewise define ρ_1, ρ_2, and ρ_3. Let δ be the unique automorphism which reverses $[v_0, v_1, v_2, v_3]$. Then these automorphisms satisfy:

1. $\rho_i^2 = 1$.
2. $(\rho_0\rho_1)^3 = (\rho_1\rho_2)^q = (\rho_2\rho_3)^3 = 1$.
3. Γ_0 and Γ_3 are C-groups. (In particular, they satisfy the necessary intersection property).
4. If G is bipartite and 3-transitive, then $\mathrm{Aut}(G) \cong$ extended symmetry group of P if $\Gamma_0 \cap \Gamma_3 =< \rho_1, \rho_2 >$.

So it seems that: if G is a bipartite, 3-transitive graph, one of the following is true:

1. G is $M(P)$ for some P.
2. $\Gamma_0 = \Gamma_3 = \mathrm{Aut}(G)$. (i.e., ρ_3 is redundant)
3. $\Gamma_0 \cap \Gamma_3 =< \rho_1, \rho_2 > \cup < \rho_1, \rho_2 > \alpha$ (only if $Z(G) \neq 1$.)

Future work:

1. From the combinatorial data of a given graph, can we tell which of the above cases applies?
2. Can we modify any case 3 graphs to get a case 1 or 2 graph?