Solutions of mKdV in classes of functions unbounded at infinity

Peter Topalov

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The modified Korteweg - de Vries equation (mKdV) is the non linear partial differential equation:

$$r_t - 6r^2 r_x + r_{xxx} = 0. \tag{1}$$

It has applications to quantum physics and is related to the Korteweg - de Vries equation (KdV):

$$q_t - 6q q_x + q_{xxx} = 0. \tag{2}$$

The equation 1 is referred to as the defocusing case. If the minus sign is changed to a plus, we get the equation:

$$r_t + 6r^2 r_x + r_{xxx} = 0 \tag{3}$$

which is also related to the Korteweg - de Vries equation and is referred to as the focusing case. The defocusing case, which is the object of this talk, is the less complicated of the two.

In solving the defocusing case, we have a function $r_0(x)$ as initial data and we want to find a function $r(t,x)$ such that

$$\begin{cases} r_t - 6r^2 r_x + r_{xxx} = 0, \\ r(0,x) = r_0(x). \end{cases} \tag{4}$$

We require the initial data and the solution to have some particular asymptotics and we express these conditions by requiring that these functions lie in some particular functional space. We consider two possibilities.

1. The space $\mathcal{O}_\beta(I \times \mathbb{R})$. $I = (a,b)$ is an open interval in $\mathbb{R}$ and this is were the variable t belongs, while $\mathbb{R}$ is where the variable x is allowed to vary; $\beta \in \mathbb{R}$ and $\beta < \frac{1}{2}$.

This space contains smooth functions, i.e. it is a subsapce of $\mathcal{C}^\infty(I\times\mathbb{R})$, that satisfy the following condition: $\forall J$ compact subset of I, $\forall i,j \geqslant 0$, $\exists C_{J,i,j} > 0$ such that:

$$|\partial_t^i\partial_x^j r(t,x)| \leqslant C_{J,i,j}|x|^{\beta-j} \qquad \forall t\in J, \forall x \text{ s.t. } |x|\geqslant 1. \tag{5}$$

This condition affects the behaviour of the function at $\pm\infty$; specifically if β is close to $\frac{1}{2}$ and $j=1$, r behaves roughly as $\sqrt{x}$ at infinity, hence it is unbounded.

2. The Schwartz space $\mathcal{S}^\beta(I\times\mathbb{R})$ (I and $\mathbb{R}$ have the same meaning as before). Again $\beta\in\mathbb{R}$ and $\beta<\frac{1}{2}$. This space is actually a subspace of the previous one, i.e. $\mathcal{S}^\beta(I\times\mathbb{R}) \subseteq \mathcal{O}_\beta(I\times\mathbb{R})$. Functions in this space have asymptotics given roughly by the fomula

   $$r(t,x) \sim \sum_{k\geqslant 0} a_k^\pm(t)(\pm x)^{\beta_k} \qquad (\pm x)\to\infty; \tag{6}$$

   this can be thought of as a Taylor expansion at infinity. The exponents β_k are taken to satisfy the following conditions:

   - $\beta_0 = \beta > \beta_1 > \beta_2 > \ldots$,
   - $\lim\limits_{k\to\infty} \beta_k = -\infty$.

   This space contains functions that are smooth in both variables and unbounded at infinity.

The main result here is the following:

Theorem. $\forall r_0 \in \mathcal{O}_\beta(\mathbb{R})$ (resp. $\forall r_0\in\mathcal{S}^\beta(\mathbb{R})$) with $\beta<\frac{1}{2}$, there exists a unique global solution of the defocusing mKdV equation in $\mathcal{O}_\beta(\mathbb{R}\times\mathbb{R})$ (resp. in $\mathcal{S}^\beta(\mathbb{R}\times\mathbb{R})$).

The proof has a more geometric flavour and is based on a result for the KdV equation. We introduce an application of functional spaces B known as Miura transform; this acts as follows:

$$r(x) \longmapsto B(r(x)) = r'(x) + r^2(x). \tag{7}$$

If we want to work with smooth functions, then we can take this application to go from $\mathcal{C}^\infty(\mathbb{R})$ to $\mathcal{C}^\infty(\mathbb{R})$. If $r_0\in\mathcal{O}_\beta(\mathbb{R})$ is the initial data for our mKdV equation, we can apply the Miura transform to it and we get a function $q_0\in\mathcal{O}_\delta(\mathbb{R})$, where

$$\delta = \max\{\beta-1, 2\beta\}. \tag{8}$$

Now we solve the KdV equation with initial data q_0, which is possible by previous results, and get a function $q(t,x)$. At this point, if we could invert the Miura transform, we would be able to pull back the function q to a solution of the original mKdV equation. But the Miura transform is not invertible and $B^{-1}(q)$ is an "interval" of functions rather than a single function.

To proceed, we consider the function

$$\psi_0(x) = e^{\int_0^x r_0(s)ds}, \tag{9}$$

and the Schrödinger operator

$$L_{q(t)} = -\partial_x^2 + q(t). \tag{10}$$

The function ψ_0 is a generalized eigenfunction of L_{q_0} with eigenvalue zero, i.e.

$$L_{q_0}(\psi_0) = 0; \tag{11}$$

it is used as initial data for the first order partial differential equation

$$\begin{cases} \psi_t = Q(q(t)\psi) \\ \psi|_{t=0} = \psi_0 \end{cases} \tag{12}$$

which always admits a solution $\psi(t)$[1]. Intuitionally, we are fixing an initial vector at q_0, where q_0 is thought of as the initial point for the curve $q(x)$ in the KdV phase space; then we apply a parallel transform to move the vector along the curve $q(x)$. Finally we get a solution of the mKdV equation by taking:

$$r(t) = \frac{\psi_x(t)}{\psi(t)}. \tag{13}$$

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[1] The operator Q is defined as follows. First we introduce the operator

$$A_{q(t)} = -4\partial_x^3 + 3q \circ \partial_x + 3\partial_x \circ q;$$

this, together with the Schrödiger operator, forms a Lax pair and defines the equation

$$\dot{L} = [A, L]$$

which is equivalent to the KdV equation. Then we take Q to be the remainder of the division of A by L. Explicitly:

$$Q = 2q(t)\partial_x + q_x.$$

The focusing case has been studied by John Gonzalez, who recently proved the existence of a local solution. It is not clear whether it is unique or not and whether it can be extended to a global solution. Applying the Miura transform in this situation does not produce a solution of the KdV equation. Instead we apply the transform

$$r(x) \longmapsto r'(x) + ir^2(x) \tag{14}$$

and we get a complex valued solution of the KdV equation. But there are no results known for this case.