

Simplicial Objects in Algebraic Topology

Goal: Study the relation between Topological spaces and simplicial sets, using Quillen model categories (more on those later).

Let Top be the category of topological spaces that are Hausdorff and compactly generated. We would like to work with the homotopy category instead. That is, given two morphisms

$X \begin{smallmatrix} \xrightarrow{f} \\ \xrightarrow{g} \end{smallmatrix} Y$ we say that f and g are *homotopic* (and write $f \sim_H g$) if there is a continuous function

$$H : X \times [0, 1] \rightarrow Y$$

such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$.

Homotopy (\sim_H) has a couple of nice properties. It is an equivalence relation on the set of morphisms of Top , and it is compatible with composition. By compatible with composition, we mean that if $f \sim_H f'$ and $g \sim_H g'$, then $f \circ g \sim_H f' \circ g'$. These properties allow us to form a new category, HTop , with the same objects as Top and with $\text{Hom}_{\text{HTop}}(X, Y) = \text{Hom}_{\text{Top}}(X, Y) / \sim_H$.

Let Δ be the simplicial category, whose objects are the totally ordered finite sets $\underline{n} = \{0 < 1 < \dots < n\}$, and whose morphisms are monotonic non-decreasing maps. Then the category of simplicial sets, $\Delta^{\text{op}} \text{Set}$, is the category of functors from Δ^{op} to Set . In particular, the standard n -simplex Δ^n is the functor $\text{Hom}_{\Delta}(-, \underline{n})$.

Now that we have these categories, we can define adjoint functors between them:

$$\Delta^{\text{op}} \text{Set} \begin{smallmatrix} \xrightarrow{|\cdot|} \\ \xleftarrow{\text{Sing}} \end{smallmatrix} \text{Top} ,$$

defined as follows. We define $|\Delta^n| = \{(t_0, \dots, t_n) \mid t_i \geq 0, \sum t_i = 1\}$, a subspace of \mathbb{R}^{n+1} . Then given any $X \in \Delta^{\text{op}} \text{Set}$, we define $|X|$ as the colimit of $|\Delta^n|$ as $\Delta^n \rightarrow X$. For the other functor, we define $\text{Sing}(X)$ as $\text{Hom}_{\text{Top}}(|X|, X)$.

We already know how to reduce Top to HTop by identifying homotopic maps. Is there a reasonable way to do this for $\Delta^{\text{op}} \text{Set}$? Our first inclination might be to notice that $|\Delta^1| = I$, so that in analogy to the previous case, we will say that if we have two morphisms in $\Delta^{\text{op}} \text{Set}$,

$X \begin{smallmatrix} \xrightarrow{f} \\ \xrightarrow{g} \end{smallmatrix} Y$ that f and g are homotopic (\sim_H) if there is a map

$$H : X \times \Delta^1 \rightarrow Y$$

such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$.

Unfortunately, if we define homotopy in this way, then \sim_H is neither an equivalence relation, nor is it compatible with composition! So we need a smarter idea.

Quillen's idea is to enrich Top with Serre fibrations, weak equivalences, and cofibrations. We can then find analogues in $\Delta^{\text{op}} \text{Set}$ and these will help us do homotopy theory. In all the following definitions, let f be a map from X to Y (in Top).

Definition: f is a *Serre fibration* if there exists an H that makes the diagram below commute:

$$\begin{array}{ccc} D^n & \longrightarrow & X \\ i_0 \downarrow & \nearrow H & \downarrow f \\ D^n \times I & \longrightarrow & Y \end{array}$$

Definition: f is a *weak equivalence* if $\pi_r(X, x_0) \cong \pi_r(Y, f(x_0))$ for all $x_0 \in X$

Definition: f is a *cofibration* if whenever $p : A \rightarrow B$ is a Serre fibration and a weak equivalence, there is a map $q : Y \rightarrow A$ making the diagram below commute:

$$\begin{array}{ccc} X & \longrightarrow & A \\ f \downarrow & \nearrow q & \downarrow p \\ Y & \longrightarrow & B \end{array}$$

Theorem [Whitehead]: $X \sim_H Y \iff X$ and Y are weakly equivalent.

Now let's see how we can carry the above definitions over to $\Delta^{\text{op}} \text{Set}$. We define Λ_k^n to be the subcomplex of Δ_n consisting of the boundary of Δ_n minus the k^{th} face. Then the *Kan fibrations* in $\Delta^{\text{op}} \text{Set}$ are the maps $f : X \rightarrow Y$ such that for each $k \leq n$, there is a map H that makes the diagram below commute:

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ i \downarrow & \nearrow H & \downarrow f \\ \Delta^n & \longrightarrow & Y \end{array}$$

We define the weak equivalences in $\Delta^{\text{op}} \text{Set}$ as the maps $f : X \rightarrow Y$ such that the induced map $|f| : |X| \rightarrow |Y|$ is a weak equivalence in Top. Finally, we define the cofibrations in $\Delta^{\text{op}} \text{Set}$ to be the injective maps.

With the above enrichments (i.e., Serre or Kan fibrations, weak equivalences, and cofibrations), both Top and $\Delta^{\text{op}} \text{Set}$ form a Quillen model category. That is, they satisfy the following:

1. They are closed under limits and colimits.
2. If two out of three of the functions f , g , and $g \circ f$ are weak equivalences, then so is the third.
3. Weak equivalences, fibrations, and cofibrations are closed under retracts.
4. Every map can be factored into the composition of a cofibration and a fibration that is also a weak equivalence. Every map can also be factored into the composition of a cofibration that is a weak equivalence, and a fibration.
5. If i is a cofibration and φ is a fibration, then whenever one of them is a weak equivalence, there is a map H that makes the diagram below commute:

$$\begin{array}{ccc}
 A & \longrightarrow & X \\
 i \downarrow & \nearrow H & \downarrow \varphi \\
 B & \longrightarrow & Y
 \end{array}$$

Now, let M be a Quillen model category. We define a new category, $\text{Ho}(M)$, with the same object class as M , and with all the weak equivalences made invertible. In this new category, we can work with homotopy as follows. Consider two maps $X \begin{smallmatrix} \xrightarrow{f} \\ \xrightarrow{g} \end{smallmatrix} Y$. We say that f is *left homotopic to g* ($f \sim^l g$) if there is a map H and some cylinder $\text{Cyl } X$ that make the diagram below commute:

$$\begin{array}{ccc}
 X \amalg X & \xrightarrow{(f,g)} & Y \\
 \downarrow \nabla & \searrow & \nearrow H \\
 & & \text{Cyl } X \\
 & \swarrow \sim & \downarrow \\
 X & &
 \end{array}$$

We say that f is *right homotopic to g* ($f \sim^r g$) if there is a map K and some Path Y that makes the diagram below commute:

$$\begin{array}{ccc}
 X & & \\
 (f,g) \downarrow & \searrow K & \\
 Y \times Y & \xleftarrow{F} & \text{Path } Y \\
 \Delta \uparrow & \nearrow \sim & \\
 Y & &
 \end{array}$$

Then we say that f is *homotopic to g* ($f \sim_H g$) if $f \sim^\ell g$ and $f \sim^r g$. This forms an equivalence relation and it is compatible with composition.

The main fact that helps us is that if X and Y are fibrant and cofibrant, then left homotopy and right homotopy coincide, so to determine whether two maps are homotopic, it suffices to determine whether they are left homotopic or right homotopic.

Now that we know how to do homotopy theory with Quillen model categories, we can return to the original goal of studying the relationship between $\Delta^{\text{op}} \text{Set}$ and Top by looking at their homotopy categories. In particular, the functors $|\cdot|$ and Sing defined on the first page induce functors between the homotopy categories:

$$\text{Ho}(\Delta^{\text{op}} \text{Set}) \begin{array}{c} \xrightarrow{|\cdot|} \\ \xleftarrow{\text{Sing}} \end{array} \text{Ho}(\text{Top}) ,$$

Theorem [Quillen]: The above functors are category equivalences.

Now we'll look at a similar problem, and again form a Quillen model category to form a coherent homotopy theory. Consider the pair of adjoint functors

$$\text{Set} \begin{array}{c} \xrightarrow{\mathbb{Z}} \\ \xleftarrow{\text{Forget}} \end{array} \text{Ab} ,$$

which induces a pair of functors

$$\Delta^{\text{op}} \text{Set} \begin{array}{c} \xrightarrow{\mathbb{Z}} \\ \xleftarrow{\text{Forget}} \end{array} \Delta^{\text{op}} \text{Ab} ,$$

Can we enrich $\Delta^{\text{op}} \text{Ab}$ with fibrations, cofibrations, and weak equivalences to form a Quillen model category? It turns out to be pretty easy. We'll say that f is a fibration if $\text{Forget}(f)$ is a Kan fibration; f is a weak equivalence if $\text{Forget}(f)$ is a weak equivalence; and $i : A \rightarrow B$ is a cofibration if there is a map H that makes the below diagram commute whenever φ is a fibration and a weak equivalence:

$$\begin{array}{ccc} A & \longrightarrow & X \\ i \downarrow & \nearrow H & \downarrow \varphi \\ B & \longrightarrow & Y \end{array}$$

These enrichments do make $\Delta^{\text{op}} \text{Ab}$ into a Quillen model category, so $\text{Ho}(\Delta^{\text{op}} \text{Ab})$ exists.

Finally, it can be shown that $\Delta^{\text{op}} \text{Ab}$ is equivalent to the category $\text{Ch}^- \text{Ab}$ by the functor N sending A_n to $\bigcap_{i=0}^{n-1} \ker d_i$.