Abstract Polytopes and Symmetry

To motivate the idea of abstract polytopes, we start by examining the regular polyhedra and their properties. There are 5 regular convex polyhedra: the tetrahedron, cube, octahedron, dodecahedron, and icosahedron. These are the so-called Platonic Solids. How should we generalize regular convex polyhedra to higher dimensions?

Let $P \subset E^n$, where E^n is n-dimensional Euclidean space. Recall that P is convex if given any two points x and y in P, the line segment connecting them is also in P. (Algebraically, we say that $\forall x, y \in P$, $\forall\ 0 < \lambda < 1$, $\lambda x + (1 - \lambda)y \in P$.) Given any set S we can take its *convex hull*, which is defined to be the intersection of all convex sets which contain S. Then P is an n-dimensional polytope (also called an n-polytope) if it is the convex hull of finitely many points of E^n and its interior is nonempty.

When $n = 2$, we get the convex polygons, and when $n = 3$, we get the convex polyhedra. In general, the boundary of an n-polytope P decomposes into finitely many $(n-1)$-dimensional polytopes (the *facets* of P). All the boundary features (of any dimension) are the *faces* of P, and we call a k-dimensional face a k-face of P. For example, the cube has six 2-faces, twelve 1-faces (the edges), and eight 0-faces (the vertices).

In addition to having polytopal faces, a polytope has polytopal *vertex figures*: the vertices adjacent to a given vertex form the outline of an $(n - 1)$-polytope.

One natural 4-polytope is the 4-cube: take a 3-cube and shift it along 4-space, forming a "prism". Another nice 4-polytope is the 24-cell; it is composed of 24 octahedra put together in 4-space. If the dimensions are chosen properly, one can obtain a regular version of these - in other words, one where each face has the same dimensions. We'll make the notion of regularity more precise in a moment.

Given an n-polytope P, a *flag* of P is a maximal chain of faces of P: $F_0 \subset F_1 \subset ... \subset F_n = P$, where F_i is an i-face of P. Then we say that P is *regular* if its symmetry group acts transitively on the flags of P; i.e., some symmetry of P carries a chosen flag to any other flag. The allowable symmetries here are the isometries that map P to itself; e.g., reflections and rotations.

When $n = 2$, the regular convex poltyopes are the regular p-gons for $p \geq 3$. We use the Schläfli symbol $\{p\}$ for the regular p-gon. When $n = 3$, the regular convex polytopes are the Platonic solids, and each one has a Schläfli symbol $\{p, q\}$, where the faces of the solid are regular p-gons, and the vertex figures are regular q-gons.

The table below lists the regular convex 4-polytopes:

Name	Schläfli symbol	number of facets	size of symmetry group
simplex	{3, 3, 3}	5	120
cross-polytope	{3, 3, 4}	16	384
hypercube	{4, 3, 3}	8	384
24-cell	{3, 4, 3}	24	1152
600-cell	{3, 3, 5}	600	14400
120-cell	{5, 3, 3}	120	14400

The Schläfli symbol $\{p, q, r\}$ means that the polytope has facets of type $\{p, q\}$ and vertex figures of type $\{q, r\}$. In general, the polytope with Schläfli symbol $\{p_1, p_2, ..., p_{n-1}, p_n\}$ has facets of type $\{p_1, ..., p_{n-1}\}$ and vertex figures of type $\{p_2, ..., p_n\}$.

When $n \geq 5$, it turns out there are only 3 regular n-polytopes for each n:

Name	Schläfli symbol	number of facets	size of symmetry group
n-simplex	{3, ..., 3}	$n + 1$	$(n + 1)!$
n-cross-polytope	{3, ..., 3, 4}	2^n	$2^n n!$
n-cube	{4, 3, ..., 3}	$2n$	$2^n n!$

The symmetry group of a regular polytope is a Coxeter group generated by reflections about the walls of the fundamental region of the polytope. For example, we can divide the 3-cube into 3-simplices by taking a flag of the 3-cube and taking the midpoints of each element as the vertices of the simplex. Then 1 of the 2-faces of the simplex is on the boundary of the cube; the remaining three 2-faces provide the reflections that generate the group.

If we relax the convexity requirement, we get some more regular polytopes: in two dimensions, for instance, we have the pentagram $\{5/2\}$, and in three dimensions, we have the Kepler-Poinsot Polyhedra $\{5/2, 3\}, \{3, 5/2\}, \{5/2, 5\}$, and $\{5, 5/2\}$, all of which have the same symmetry group as the icosahedron.

Now let's drop all geometric requirements and look at polytopes combinatorially:

We define an *abstract polytope P of rank n* to be a ranked poset with several properties (to follow). We call the elements of the poset the *faces* of P, with an element of rank i being an i-face of P. As in the case of geometric polytopes, we call the 0-faces vertices, the 1-faces edges, and the $(n - 1)$-faces facets.

The properties we require other than the partial order are:

- P has unique faces F_{-1} and F_n of rank -1 and n, respectively.
- Each flag of P contains exactly $n + 2$ faces.
- P is connected.

- All intervals of rank 1 are diamonds. That is, given an $(i+1)$-face F_{i+1} and an $(i-1)$-face F_{i-1}, there are exactly two i-faces, F_i and F_i^*, such that $F_{i-1} < F_i, F_i^* < F_{i+1}$.

Let $\Gamma(P)$ be the group of (combinatorial) automorphisms of P. Then we say that P is *regular* if the action of $\Gamma(P)$ on P is flag-transitive. The regular 0-, 1-, and 2-polytopes are essentially the same as before. We do get some new regular 3-polytopes, however. We can view the general regular 3-polytope as a tesselation of a closed surface. For example, we can project the faces of a Platonic solid onto a concentric sphere to obtain a tesselation of the sphere. In general, among surfaces of any given genus g, there are finitely many possible tesselations of that surface; all of them are known for $g \leq 100$.

How can we extend this to 4-polytopes? Consider a polytope of type $\{4, 4, 3\}$. Such a polytope would have facets of type $\{4, 4\}$ (torus maps), and vertex figures of type $\{4, 3\}$ (cubes). So to build this polytope, let us start with a solid torus, tessellated with squares. To fulfill the "diamond condition", every square face must lie on 2 tori. Each vertex of this polytope is contained in 6 tori, arranged cubically. That is, each vertex has 6 neighbors that form the vertices of a cube.

We can also define a regular abstract polytope algebraically. We do this using C-groups, which are quotients of Coxeter groups . A C-group is a group $\Gamma =< \rho_0, ..., \rho_{n-1} >$ satisfying $\rho_i^2 = 1$ for all i, $(\rho_i\rho_j)^2 = 1$ whenever $|i-j| \geq 2$, and there are numbers $p_1, ..., p_{n-1}$ such that $(\rho_i\rho_{i+1})^{p_{i+1}} = 1$.

Given an abstract polytope P, the group of symmetries $\Gamma(P)$ is a C-group. In particular, fix a base flag F of P. Then by the diamond condition, for each i between 0 and $n-1$, there is a flag $F^{(i)}$ that differs from F in the i^{th} rank only; we say that these two flags are i-adjacent. Then if we let ρ_i be the automorphism that sends F to $F^{(i)}$, the elements ρ_i will satisfy the above relations.

Similarly, we can construct a regular polytope P given the C-group Γ. We define the j-faces of P to be the right cosets of $\Gamma_j =< \rho_i \mid i \neq j >$. Then we say that $\Gamma_j\varphi \leq \Gamma_k\psi \iff j \leq k$ and $\Gamma_j\varphi \cap \Gamma_k\psi \neq \emptyset$. It turns out that we actually have a one-to-one correspondence between C-groups and regular polytopes, so it's possible to study the polytopes algebraically instead of combinatorially.

Further questions:

- Amalgamation: Given polytopes P and Q, when is there a regular polytope with P as its facets and Q as its vertex figures? If such a polytope exists, it is universal, and we use the symbol $\{P, Q\}$ for it.
- Extension: Given a polytope $P = \{p_1, ..., p_{n-1}\}$, for what values of p_n does the polytope $\{p_1, ..., p_n\}$ exist? (Partial answer: any even value of p_n works [Daniel Pellicer-Covarrubias 2006]).