Auslander-Reiten theory; Representation and other dimensions

Gordana Todorov, Tapas seminar, Northeastern University, May 20, 2008

Algebra and modules

- Λ - artin algebra, i.e. algebra which is finitely generated module over its center, which is a commutative artin ring.
- Examples: finite dimensional algebras over fields, all commutative artin rings, path algebras of finite quivers with no oriented cycles, quotients of such path algebras, group algebras for finite groups, exterior algebras
- modΛ - category of finitely generated Λ-modules.

Finitely presented functors

- $fp((\text{mod}\,\Lambda)^{op}, Ab)$ - the category of finitely presented contravariant functors $\mathcal{F} : \text{mod}\,\Lambda \to Ab$, where Ab is the category of abelian groups, i.e.:

$$Hom_\Lambda(\ ,B) \to Hom_\Lambda(\ ,C) \to \mathcal{F} \to 0$$

- $(\ ,C) = Hom_\Lambda(\ ,C) : \text{mod}\,\Lambda)^{op} \to Ab$ is a finitely generated projective functor, and all finitely generated projective functors are isomorphic to such functors.
- Since $(\text{mod}\,\Lambda)^{op}$ abelian category, it has kernels, so finitely presented functors have presentation:

$$0 \to (\ ,A) \to (\ ,B) \to (\ ,C) \to \mathcal{F} \to 0,$$

  where $0 \to A \to B \to C$ is a (left) exact sequence in $\text{mod}\,\Lambda$.
- Finitely presented injective functors are given by injective presentations of Λ-modules $0 \to A \to I_0(A) \to I_1(A)$ as:

$$0 \to (\ ,A) \to (\ ,I_0(A)) \to (\ ,I_1(A)) \to E(\ ,A) \to 0.$$

- Functors which vanish on Λ (projective modules) have projective presentations which are given by short exact sequences $0 \to A \to B \to C \to 0$.
- $\text{Ext}^1_\Lambda(\ ,A)$ is a finitely presented functor which vanishes on projectives and has a presentation given by $0 \to A \to I_0(A) \to \Sigma(A) \to 0$ as:

$$0 \to (\ ,A) \to (\ ,I_0(A)) \to (\ ,\Sigma(A)) \to \text{Ext}^1_\Lambda(\ ,A) \to 0$$

- $\text{Ext}^1_\Lambda(\ ,A)$ is a subfunctor of $E(\ ,A)$, actually it is the maximal subfunctor which vanishes on projective modules.

- For each indecomposable Λ-module A, the functors $\mathrm{Ext}^1_\Lambda(\ ,A)$ and $E(\ ,A)$ have unique simple finitely presented subfunctor $\mathcal{S}$. The unique (up to isomorphism) minimal presentation of this functor is given by a short exact sequence (unique up to isomorphism):

$$0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0. \qquad (*)$$

- Some maps between these functors:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & (\ ,A) & \longrightarrow & (\ ,B) & \longrightarrow & (\ ,C) & \longrightarrow & \mathcal{S} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (\ ,A) & \longrightarrow & (\ ,I_0(A)) & \longrightarrow & (\ ,\Sigma(A)) & \longrightarrow & \mathrm{Ext}^1_\Lambda(\ ,A) & \longrightarrow & 0 \\
 & & \| & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (\ ,A) & \longrightarrow & (\ ,I_0(A)) & \longrightarrow & (\ ,I_1(A)) & \longrightarrow & E(\ ,A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & (\ ,I_0(A)) & \longrightarrow & (\ ,I_0(A))\coprod(\ ,\Sigma(A)) & \longrightarrow & (\ ,I_1(A)) & \longrightarrow & \overline{E}(\ ,A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \| & & \\
0 & \longrightarrow & 0 & \longrightarrow & (\ ,\Sigma(A)) & \longrightarrow & (\ ,I_1(A)) & \longrightarrow & \overline{E}(\ ,A) & \longrightarrow & 0.
\end{array}$$

Almost split sequences/ AR sequences

- *Almost split sequence* is a sequence which satisfies the following conditions:
  - The sequence does not split.
  - Both end terms A and C, are indecomposable.
  - For each indecomposable module X, every non-isomorphism $X \xrightarrow{a} C$ factors through f.
- Sequences (*) are almost split sequences; they are also called *Auslander-Reiten sequence*, or *AR sequence*.
- EXISTENCE and UNIQUENESS THEOREM (Auslander-Reiten)
  a. For each non-injective indecomposable Λ-module A, there exists an almost split sequence $0 \to A \to B \to C \to 0$.
  b. This sequence is unique up to isomorphism of sequences.
- The module C is uniquely determined by A and can be described as $C = TrDA$, where TrD is a functor on appropriate quotient categories:

$$\overline{\mathrm{mod}}\Lambda \xrightarrow{D} \underline{\mathrm{mod}}\Lambda^{op} \xrightarrow{Tr} \underline{\mathrm{mod}}\Lambda.$$

The functors DTr and TrD are often denoted by τ and τ^{-1} and are called Auslander-Reiten translations, or AR-translations.

Irreducible morphisms

- Consider almost split sequences (decompose into indecomposable summands): $0 \to A \xrightarrow{(g_i)} \coprod_i B_i \xrightarrow{\Sigma f_i} C \to 0.$
- The maps $\{f_i\}$ and $\{g_i\}$ cannot be factored (except with isomorphisms).
- A map $X \xrightarrow{f} Y$ is called *irreducible* if in any factorization $f = b \cdot a$ either a is a split monomorphism or b is a split epimorphism.
- Theorem (Auslander-Reiten) If Λ is an artin algebra of finite representation type, then every map in modΛ is a sum of compositions of irreducible maps. In the general case, there is also a summand which is in the infinite radical.

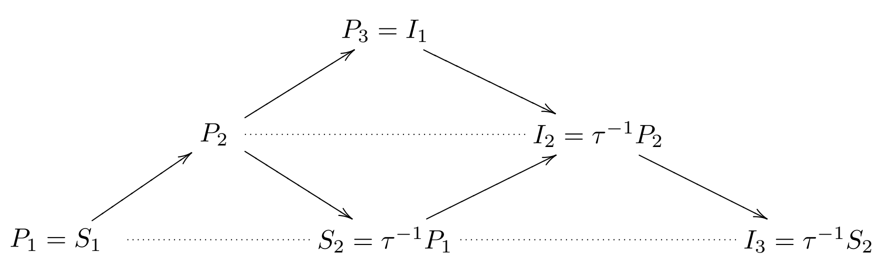
Auslander-Reiten quiver/ AR quiver

- Vertices - for each indecomposable module M (actually isomorphism class) associate vertex $[M]$.
- Arrows - $[M] \to [N]$ if there is an irreducible map $M \to N$.
- Example: Consider left modules over the algebra:

$$\Lambda = \begin{bmatrix} \mathbb{k} & \mathbb{k} & \mathbb{k} \\ 0 & \mathbb{k} & \mathbb{k} \\ 0 & 0 & \mathbb{k} \end{bmatrix}$$

$$P_1 \quad P_2 \quad P_3$$

There are three non isomorphic indecomposable projective Λ-modules $\{P_1, P_2, P_3\}$, corresponding simple Λ-modules $\{S_1 = P_1, S_2, S_3\}$, and corresponding indecomposable injective Λ-modules $\{I_1 = P_3, I_2, I_3 = S_3\}$, total of 6 modules. AR quiver for this algebra:



[This concludes the topics actually covered during the talk. The following pages contain related topics that the speaker considers important in order to provide a more complete overview of the subject.]

Quiver representations

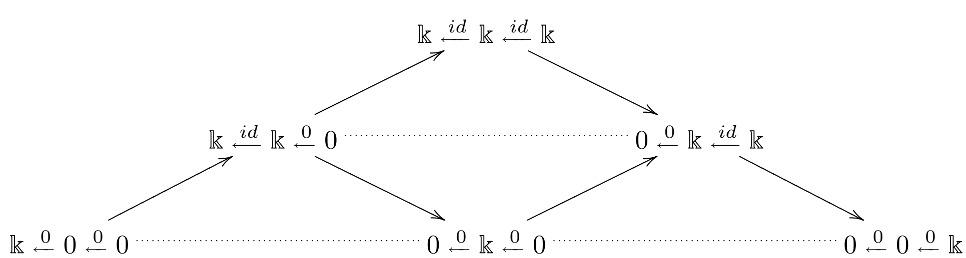
- The above Λ-modules can be viewed as representations of the quiver

$$Q = \cdot_1 \leftarrow \cdot_2 \leftarrow \cdot_3.$$

- *Quiver* $Q = (Q_0, Q_1)$ is a collection of *vertices* Q_0 and *arrows* Q_1.
- A $\Bbbk$-*representation* of Q is $(\{V_i\}_{i \in Q_0}, \{\varphi_a : V_i \to V_j\}_{(i \xrightarrow{a} j) \in Q_1})$, where V_i are $\Bbbk$-vector spaces and φ_a are $\Bbbk$-linear maps.
- A *morphism* between representations

  $$(\{V_i\}_{i \in Q_0}, \{\varphi_a : V_i \to V_j\}_{(i \xrightarrow{a} j) \in Q_1}) \xrightarrow{f} (\{W_i\}_{i \in Q_0}, \{\psi_a : W_i \to W_j\}_{(i \xrightarrow{a} j) \in Q_1})$$

  is $f = \{f_i\}_{i \in Q_0}$ such that $f_j \cdot \varphi_a = \psi_a \cdot f_i$ for each $(i \xrightarrow{a} j) \in Q_1$.
- *Path algebra* is $\Bbbk$-vector space of all paths. Multiplication is defined by composition of paths. Path algebra is denoted by $\Bbbk Q$.
- $rep_{\Bbbk} Q \simeq \operatorname{mod} \Bbbk Q$, i.e. the category of finitely generated representations of Q over $\Bbbk$ is equivalent to the category of finitely generated modules over the path algebra $\Bbbk Q$.
- The path algebra of the above quiver is isomorphic to Λ in the previous example. The indecomposable representations are:
  $P_1 = S_1 = (\Bbbk \xleftarrow{0} 0 \xleftarrow{0} 0)$, $P_2 = (\Bbbk \xleftarrow{id} \Bbbk \xleftarrow{0} 0)$, $P_3 = I_1 = (\Bbbk \xleftarrow{id} \Bbbk \xleftarrow{id} \Bbbk)$,
  $S_2 = (0 \xleftarrow{0} \Bbbk \xleftarrow{0} 0)$, $I_2 = (0 \xleftarrow{0} \Bbbk \xleftarrow{id} \Bbbk)$, $I_3 = S_3 = (0 \xleftarrow{0} 0 \xleftarrow{0} \Bbbk)$



Arrows indicate the irreducible maps.
Dotted lines indicate almost split sequences.

Endomorphism algebras

- If algebra Λ is of *finite representation type*, i.e. there are only finitely many non-isomorphic indecomposable Λ modules, then the category of finitely presented functors is equivalent to the category of finitely generated modules over the artin algebra $\Gamma = \mathrm{End}_\Lambda(M)^{op}$, where $M = \coprod M_i$, the sum taken over all representatives of indecomposable modules. Γ is called *Auslander algebra.*
- Auslander algebra has global dimension 2, i.e. projective dimension of any Γ-module is less then or equal to 2.
- $\mathrm{gl.dim} fp((\mathrm{mod}\,\Lambda)^{op}, Ab) \leq 2$.
- For any Λ-module M, let $\Gamma = \mathrm{End}_\Lambda(M)^{op}$. Consider the functor:

$$\mathrm{mod}\,\Lambda \xrightarrow{Hom_\Lambda(M,\ )} \mathrm{mod}\,\Gamma$$

  - Indecomposable projective Γ-modules are $Hom_\Lambda(M, M_i)$, where M_i are indecomposable summands of M.
  - Homological properties are easy to keep track of.
  - Reflection functors, APR tilts are all of this type.
  - Tilting theory
  - Reduction to smaller algebras is often done this way, by letting M be a sum of fewer projective modules.
  - Reduction to smaller quivers is often of this form.
  - Ringel's shrinking functors are such.

Representation dimension
$rep.dim\Lambda$

- *Representation dimension of* Λ is defined to be:

$$rep.dim\Lambda = inf\{gl.dim\Gamma | \Gamma = \mathrm{End}_\Lambda(M)^{op}, M \text{generator cogenerator} \in \mathrm{mod}\,\Lambda\}$$

$$= inf\{gl.dim\Gamma | \Gamma = \mathrm{End}_\Lambda(\Lambda \oplus D\Lambda \oplus M)^{op}, M \in \mathrm{mod}\,\Lambda\}$$

- Theorem (Auslander) Artin algebra Λ is of finite representation type if and only if $rep.dim\Lambda = 2$.
- Many classes of algebras were shown to have $rep.dim\Lambda \leq 3$:
  - $rad^3\Lambda$
  -

  - Preprojective algebras (Andy Carroll will talk about these algebras).

- *Finitistic dimension conjecture* was proved for algebras having $rep.dim\Lambda \leq 3$.

- For more then 10 years we didn't know examples of algebras with larger representation dimension.

- Theorem (Rouquier) There are algebras of any representation dimension; $rep.dim(\Lambda V) = n$ if $dim_{\Bbbk} V = n$. (ΛV) is the exterior algebra.

- Theorem (Iyama) $rep.dim\Lambda \leq \infty$.

- Theorem (Dugas) If $\underline{\mathrm{mod}}\Lambda \xrightarrow{equiv} \underline{\mathrm{mod}}\Lambda'$ then $rep.dim\Lambda = rep.dim\Lambda'$.

- Theorem (Oppermann) Gives general method for establishing lower bounds for the representation dimension of a given algebra, or clas of algebras.

- Many open questions:
  - What makes $rep.dim\Lambda \geq 4$?
  - Find other algebras with $rep.dim\Lambda \geq 4$.
  - Find $gl.dim\,\mathrm{End}_\Lambda(\Lambda \oplus D\Lambda)^{op}$
  - Find $inf\{gl.dim\Gamma | \Gamma = \mathrm{End}_\Lambda(\Lambda \oplus M)^{op}\}$, i.e. consider only generators.
  - Find $inf\{gl.dim\Gamma | \Gamma = \mathrm{End}_\Lambda(D\Lambda \oplus M)^{op}\}$, i.e. consider only cogenerators.
  - Relations: $rep.dim$, $gen.rep.dim$, $gl.dim\,\mathrm{End}_\Lambda(\Lambda \oplus D\Lambda)^{op}$.
  - Generalize Dugas theorem to other contravariantly finite subcategories.
  - Filter the category of modules by contravariantly finite subcategories and study representation dimension.

Finitistic dimension conjecture

- Definitions:
  $fin.dim = sup\{pdM \mid pdM < \infty,\ M \in \mathrm{mod}\,\Lambda\}$
  $Fin.dim = sup\{pdM \mid pdM < \infty,\ M \in \mathrm{Mod}\,\Lambda\}$

- Conjectures:
  Little fin.dim.conj: $fin.dim < \infty$
  Big fin.dim.conj: $Fin.dim < \infty$

- History: see Birge Huisgen-Zimmermann
- Examples:
  - If $gl.dim\Lambda = n$ then $fin.dim\Lambda = n$.
  - Let Q be a finite quiver, no oriented cycles. Then the path algebra
  - If Λ is selfinjective then $fin.dim\Lambda = 0$ since the only modules of finite projective dimension are projectives.
- Known results:
  - (Green-Kirkman-Kuzmanovich) Monomial algebras $fin.dim\Lambda < \infty$.
  - (Igusa-Zacharia) Monomial algebras.
  - (Green-HuisgenZimmerman) $rad^3\Lambda = 0$ implies $fin.dim\Lambda < \infty$.
  - (Auslander-Reiten) $\mathcal{P}^{<\infty}(\Lambda) = \{X \mid pd_\Lambda X \leq \infty\}$ is contravariamtly finite implies $fin.dim\Lambda < \infty$.
  - (Igusa-T.) $rep.dim\Lambda \leq 3$ implies $fin.dim\Lambda < \infty$.
- Questions - Conjecture is still open for $rep.dim\Lambda \geq 4$.

Dimensions of subcategories of Triangulated categories

$dim_{\mathcal{T}}\mathcal{C}$

- Let $\mathcal{T}$ be a triangulated category. Let $\mathcal{C}$ and $\mathcal{D}$ be subcategories of $\mathcal{T}$.
  Define:
  $\langle\mathcal{C}\rangle$ = the full subcategory of $\mathcal{T}$ consisting of all direct summands of finite direct sums of shifts of objects in $\mathcal{C}$.
  $\mathcal{C} * \mathcal{D}$ = the full subcategory of $\mathcal{T}$ consisting of all objects M such that there exists a distinguished triangle $C \to M \to D \to C[1]$ with $C \in \mathcal{C}$ and $D \in \mathcal{D}$.
  $\langle\mathcal{C}\rangle_1 := \langle\mathcal{C}\rangle$
  $\langle\mathcal{C}\rangle_n := \mathcal{C} * \langle\mathcal{C}\rangle_{n-1}$
- Dimension of subcategories of triangulated categories $\mathcal{T}$:

$$dim_{\mathcal{T}}\mathcal{C} := inf\{d \in \mathbb{Z} \mid \exists \text{ an object } M \in \mathcal{T} \text{ such that } \mathcal{C} \subseteq \langle M\rangle_{d+1}\}.$$

- Examples of triangulated categories:
  $D^b = D^b(\operatorname{mod}\Lambda)$ = the derived category of bounded complexes
  $\underline{\operatorname{mod}}\Lambda$ = the stable category of finitely generated modules
- Theorem (Rouquier, Oppermann)
  $dim_{D^b}\operatorname{mod}\Lambda \leq dim_{D^b}D^b = dimD^b(\operatorname{mod}\Lambda) \leq rep.dim\Lambda$.
- Remark $der.dim\operatorname{mod}\Lambda = dim_{D^b}\operatorname{mod}\Lambda$ by definition.

Comparison of different dimensions

- The following graph includes several dimensions mentioned previously.
- $ll\Lambda$ denotes Loewy length of λ.