Semi-Invariants of Tubular Algebras

Tapas Seminar 2008

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Basic Definitions

- Throughout this presentation $Q = (Q_0, Q_1)$ will be a quiver without oriented cycles, where Q_0 is the finite set of vertices and Q_1 is the finite set of arrows. If $a \in Q_1$ is an arrow then ta and ha denote its tail and head respectively.
- A path is a sequence of arrows $p = a_1 a_2 \dots a_s$ with $ta_i = ha_{i+1}$ for all i. We define $tp = ta_s$ and $hp = ha_1$. For each vertex $x \in Q_0$ we also define the trivial path e_x of length 0, satisfying $te_x = he_x = x$. An oriented cycle is a nontrivial path satisfying $hp = tp$.

- Let K be an algebraically closed field. The path algebra KQ is the K-vector space spanned by all paths (including the paths e_x). If p and q are paths, then their product $p \cdot q$ is the concatentation of the paths if $tp = hq$, and is defined 0 otherwise.
- The category $Rep_K(Q)$ of representations of the quiver Q is the category of finite dimensional KQ-modules. If V is a representation of Q (i.e., a finite dimensional KQ-module) then we define $V(x) = e_x V$ for all $x \in Q_0$ and $V(p) : V(tp) \to V(hp)$ is the restriction of multiplication with p to $V(tp) = e_{tp}V$ for every path p.

- The path algebra is graded

$$KQ = \bigoplus_{x,y \in Q_0} e_x KQ e_y.$$

- Let $r \in KQ$ be a relation, i.e.,

$$r = \sum_{i=1}^{s} c_i p_i$$

with p_i a path and $c_i \in K$ for all i.

- We say that the relation r is *admissible* if r is homogeneous with respect to the grading, i.e., there exist $tr, hr \in Q_0$ such that $tp_i = tr$ and $hp_i = hr$ for all i. Let us assume that I is an admissible ideal, i.e., a two sided ideal generated by admissible relations.

- We will call Q/I a quiver with relations. The category $Rep_K(Q/I)$ of representations of Q/I is the category of finite dimensional KQ/I-modules. We may assume that I is generated by admissible relations of length ≥ 2, because otherwise the algebra KQ/I is a factor of a path algebra of a smaller quiver.
- A dimension vector for Q is an element $\alpha \in \mathbf{N}^{Q_0}$, where $\mathbf{N} = \{0, 1, 2, \ldots\}$ is the set of nonnegative integers. We say that a representation V is α-dimensional if $\dim\ V(x) = \alpha(x)$ for all $x \in Q_0$.

- For a dimension vector α we define the representation space by

$$Rep_K(Q,\alpha) = \bigoplus_{a\in Q_1} Hom(K^{\alpha(ta)}, K^{\alpha(ha)}).$$

- Note that every element

$$V = \{V(a) \mid a \in Q_1\} \in Rep_K(Q,\alpha)$$

can be viewed as a representation of Q.

- The groups $GL(Q,\alpha) := \prod_{x\in Q_0} GL(\alpha(x))$ and $SL(Q,\alpha) := \prod_{x\in Q_0} SL(\alpha(x))$ act on $Rep_K(Q,\alpha)$ in a natural way.

- Two representations $V, W \in Rep_K(Q, \alpha)$ are isomorphic if they lie in the same $GL(Q, \alpha)$-orbit.
- We also define

  $$Rep_K(Q/I, \alpha) \subseteq Rep_K(Q, \alpha)$$

  as the Zariski-closed subset defined by $Rep_K(Q/I, \alpha) = \{V \in Rep_K(Q, \alpha) \mid V(r) = 0 \ \forall r \in I$ homogeneous$\}$.

- The space $Rep_K(Q/I,\alpha)$ does not have to be irreducible. We denote its irreducible components by $Rep_K(Q/I,\alpha)_i$ ($i = 1, 2, \ldots, N(Q/I;\alpha)$). We are interested in the rings of semi-invariants

$$SI(Rep(Q/I,\alpha)_i) := K[Rep(Q/I,\alpha)_i]^{SL(Q,\alpha)}.$$

- We recall that the Euler form for Q is a bilinear form on the space $\Gamma := \mathbf{Z}^{Q_0}$ defined by

$$\langle \alpha, \beta \rangle = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha).$$

- Every representation $V \in Rep_K(Q)$ has a canonical resolution

$$0 \to \bigoplus_{a \in Q_1} V(ta) \otimes P_{ha} \to \bigoplus_{x \in Q_0} V(x) \otimes P_x \to V \to 0 \qquad (1)$$

where $P_x = KQe_x$ is the indecomposable projective module associated to the vertex $x \in Q_0$.

- More precisely, denoting by $[x, y] := e_y K Q e_x$ the K-span of all paths from x to y, we have $P_x(y) = [x, y]$ with the linear map $P_x(a)$ acting by the left composition with a. We can also characterize P_x by the property $Hom_R(P_x, W) = W(x)$ for all $W \in Rep_K(Q)$.
- Now (1) implies that $Rep_K(Q)$ is hereditary, and for $W \in Rep_K(Q, \beta)$ we have that the Euler characteristic is equal to

$$\chi(V, W) := \dim_K \ Hom_R(V, W) - \dim_K \ Ext_R(V, W) =$$

$$= \langle \alpha, \beta \rangle.$$

- For a quiver Q with relations I we notice that the indecomposable projective modules again correspond to vertices from Q_0 and the module corresponding to $x \in Q_0$ is just $P'_x := P_x/IP_x$. They are characterized by the property that for each $W' \in Rep_K(Q/I)$ we have $Hom_{R/I}(P'_x, W') = W'(x)$.

- Let $V' \in Rep_K(Q/I, \alpha)$. We construct the module $\bar{P}_0 = \bigoplus_{x \in Q_0} V'(x) \otimes P'_x$. Then for each $V' \in Rep_K(Q/I, \alpha)$ the kernel

$$0 \to V'_{(1)} \to \bar{P}_0 \to V' \to 0$$

has the same dimension vector. We define $\bar{P}_1$ by using the construction of $\bar{P}_0$ for $V'_{(1)}$. Continuing like that we construct the family of projective resolutions of modules from $Rep_K(Q/I, \alpha)$ with fixed terms.

- This construction allows to define the Euler form for the quiver Q with relations I. For two dimension vectors α and β we set

$$\langle\langle\alpha,\beta\rangle\rangle = \sum_{s\geq 0}(-1)^s \ \dim_K \ Ext^s_{R/I}(V',W')$$

where V',W' are the modules from α,β respectively.

Assume Q/I is a quiver with relations, α a dimension vector. Every representation V of dimension vector α of projective dimension 1 with a (minimal) projective resolution

$$0 \to P_1 \to P_0 \to V \to 0$$

defines a determinantal semi-invariant c^V

$$W \mapsto det(Hom_{Q/I}(P_0, W) \to Hom_{Q/I}(P_!, W)$$

on all components of $Rep(Q/I, \beta)$ such that $\langle\langle \alpha, \beta \rangle\rangle = 0$. Such semi-invariant might of course be identically zero on some components.

- A component C of $Rep(Q/I,\beta)$ is faithful if the ideal $J=\{x\in KQ\mid x_{|}C=0\}$ is equal to I, i.e. there are no extra relations satisfied on C.
- **Theorem** (Derksen and Weyman) Let Q/I be a quiver with relations. Let C be a faithful component of $Rep(Q/I,\beta)$. Then the ring of semi-invariants $SI(Q/I,C):=SI(GL(Q,\beta),K[C])$ is spanned by the determinantal semi-invariants c^V.

Tubes and the rings of semi-invariants

- Let $\Lambda = KQ/I$ and denote $Rep(\Lambda, \beta)$ the algebraic variety of the Λ-modules of dimension vector β.
- The family of tubes over Λ of ranks $(m_1, \ldots, m_s)$ is the set of data given below. First, we have a family of indecomposable modules $\{V_t\}$ ($t \in \mathbf{P}^1$) of dimension $\underline{h}$, and the set of modules $E_i^{(j)}$ of dimension vectors $e_i^{(j)}$ ($1 \leq j \leq s$, $0 \leq i \leq m_j$), all of projective dimension 1. The smallest category $Reg(\Lambda)$ containing modules V_t and $E_i^{(j)}$, closed under the extensions and direct summands, is called the category of regular Λ-modules.

- A vector

$$\beta = p\underline{h} + \sum_{j=1}^{s} \sum_{i=0}^{m_j-1} p_i^{(j)} e_i^{(j)}$$

  (where we assume that for each $j = 1, \ldots, s$ we have $min\{p_i^{(j)}; 1 \leq i \leq m_j - 1\} = 0$) is called a regular dimension vector.

- We require that for every regular dimension vector β there exists an irreducible component $Reg(\Lambda, \beta)$ of $Rep(\Lambda, \beta)$

$Reg(\Lambda, \beta)$ is subject to the following conditions:

1. The dimension vectors $e_i^{(j)}$ satisfy the relations

$$\underline{h} = \sum_{i=0}^{m_j - 1} e_i^{(j)}.$$

   for $j = 1, \ldots, s$, and the dimension of the space spanned by $e_i^{(j)}$ equals $\sum_{j=1}^{s} m_j - s + 1$,

2. The decomposition of a general vector in $Reg(\Lambda, \beta)$ is given by the same formula as for the extended Dynkin quivers

3. Dimension vectors $e_i^{(j)}$, and the vectors $e_{[i,k]}^{(j)} := e_i^{(j)} + e_{i+1}^{(j)} + \ldots + e_k^{(j)}$ are Schur roots and the generic modules $E_{[i,k]}^{(j)}$ have projective dimension 1 and injective dimension 1 over Λ
4. The values of the Euler form $\langle\langle e_i^{(j)}, e_k^{(l)} \rangle\rangle = 0$ if $j \neq l$, and $\langle\langle e_i^{(j)}, e_{i+1}^{(j)} \rangle\rangle = -1$, $\langle\langle e_i^{(j)}, e_i^{(j)} \rangle\rangle = 1$

5. The general module in dimension vector $Reg(\Lambda, \underline{h})$ is a 1-paramenter family of modules V_t ($t \in K \cup \infty$), which are also of projective dimension 1 and of injective dimension 1.
6. Every indecomposable module X of projective dimension ≤ 1 orthogonal to $Reg(\underline{h})$ (in the sense that $Hom_\Lambda(X, V_t) = Ext^1_\Lambda(X, V_t) = 0$ for general t) is in the category $Reg(\Lambda)$.
7. The condition 6. implies by results of [DW6] the existence of the semi-invariant c^{V_u} in the coordinate ring $SI(\Lambda, \beta)$.

Theorem. Let $\{V_t, E_i^{(j)}\}$ be a family of tubes of ranks $(m_1, \ldots, m_s)$. Let β be a regular dimension vector. Then the ring of regular semi-invariants $SI_{reg}(\Lambda, \beta)$ has the generators and relations decribed as follows.

a) The non-homogeneous generators are $c^{E_{[i,k]}^{(j)}}$ where $[i, k]$ is an "admissible" path on the j-th circle,

b) The homogeneous generators c^{V_t} which span the space of dimension $p + 1$. We fix a basis $\{c_0, \ldots, c_p\}$ of the weight space $SI_{reg}(\Lambda, \beta)_{\langle \underline{h}, - \rangle}$.

c) There are s relations in $SI_{reg}(\Lambda, \beta)$ each expressing the product of non-homogeneous semi-invariants of index zero on the j-th circle as a linear combination of the semi-invariants $c_0, \ldots, c_p$.

Tubular Algebras

- Give a finite dimensional algebra A_0 and and A_0 module R we denote by $A_0[R]$ the one-point extension of A_0 by R namely the algebra

$$\begin{bmatrix} A_0 & R \\ 0 & k \end{bmatrix}$$

$$= \left\{ \begin{bmatrix} a & r \\ 0 & b \end{bmatrix} \mid a \in A_0, r \in R, b \in k \right\}$$

- The quiver of $A_0[R]$ contains the quiver of A_0 as a full subquiver and there is an additional vertex ω called the extension vertex of $A_0[R]$.
- Similarly, the one-point co-extension $[V]A$ of A by V is defined by $[V]A = ((A^{op})[DV])^{op}$

- Let A_0 be an algebra $E_1, \cdots, E_t$ be A_0 modules and $K_1, \ldots K_t$ branches. Let $A = A_0[E_i, K_i]_{i=1}^{t}$ be inductivetly defined.
- The algebra A is called a tubular extension of A_0 using modules from the tubes provided that the modules $E_1, \ldots E_t$ are pairwise orthogonal modules from the mouth of the tubes.

- Algebras of the form End(T) where T is a preprojective tilting module of a tame hereditary algebra are called **tame concealed algebras.**
- A tubular extension of a tame concealed algebra of tubular type (2,2,2,2), (3,3,3), (4,4,2) or (6,3,2) is called a **tubular algebra**
- An algebra will be said to be cotubular provided that the opposite algebra A^{op} is tubular.

- Let $A = (A_0, A_\infty)$ be an algebra which is an extension of A_0 and a coextension of A_∞
- Let α_0 be the positive radical generator of A_0 and α_∞ be the positive radical generator for A_∞. Then $\underline{h} = p\alpha_0 + q\alpha_\infty$ with $p, q \in \mathbb{Z}_+$
- For the one parameter family of modules V_t, we have that $t = \frac{q}{p}$.

Shrinking Functors

- For a tubular algebra $A = (A_0, A_\infty)$, we can define a left shrinking functor $\Sigma_L = Hom(T, -)$ with $T = T_0 + T_P$ where T_0 is in the preprojective component of A and T_P is a projective module not in the preprojective component.
- Similarly we can define a right shrinking functor $\Sigma_R = Hom(-, S)$ where $S = S_0 + S_Q$ with S_0 in the preinjective component and S_Q a injective module not in the preinjective component.
- We can associate a linear transformation with Σ_L and Σ_R namely σ_l and σ_r

- There exists left and right shrinking functors such that

$$(p\alpha_0 + q\alpha_\infty)\sigma_l = (p+q)\alpha_0 + q\alpha_\infty$$

  and

$$(p\alpha_0 + q\alpha_\infty)\sigma_r = p\alpha_0 + (p+q)\alpha_\infty$$

- So basically we can use shrinking modules to shift from $t = 1$ to $t = \frac{q}{p}$ for any $\underline{h} = p\alpha_0 + q\alpha_\infty$.