# Varieties related to algebraic Group Actions

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### Linear Algebra facts

① Rank Theorem:

$f: K^n \to K^m$ linear map ($K$ a field)

after row and column operations, $f$ can be reduced to the form

$$r\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$ where $r = \operatorname{rank} f$

② $K = \mathbb{C}$ (or any algebraicly closed field)

$f$ a quadratic form

$$f = \sum_{1 \le i \le j \le n} a_{ij} x_i x_j$$

Then the quadratic form can be reduced to a sum of squares (after a linear change of basis in $x_1, \dots, x_n$)

$$f \sim x_1^2 + \dots + x_r^2$$

③ Jordan canonical form:

$f: K^n \rightleftarrows K^n$ endomorphism of $K^n$

After a change of basis in $K^n$, $f$ can be reduced to the form

$$\begin{pmatrix} J_{m_1}^{\lambda_1} & 0 & 0 & 0 & 0 \\ 0 & J_{m_2}^{\lambda_2} & 0 & 0 & 0 \\ 0 & 0 & \ddots & & \\ 0 & 0 & & \ddots & \\ & & & & \end{pmatrix}$$

where $J_m^\lambda$ is an $m \times m$ Jordan block corresponding to the eigenvalue $\lambda$ ie $m\begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ 0 & & & \lambda \end{pmatrix}$

~~Pxxxx~~

These same facts from the point of view of orbits of group actions:

(1') $Hom_k(k^n, k^m) \cong Mat_{m\times n}(k)$

$GL_m(k) = \{m \times m \text{ non-singular matrices}\}$

$GL_n(k) \circlearrowright Mat_{m\times n}(k) \circlearrowleft GL_m(k)$

~~xxxxxxxxxxx Mat~~
(groups act on $Mat_{m\times n}(k)$ by left/right multiplication).

So $GL_n(k) \times GL_m(k)$ acts on $Mat_{m\times n}(k)$.
The orbits of this action are matrices of a given rank

(2') $S_2(k^n) = \{\sum_{i,j} a_{ij} x_i x_j \mid a_{ij} = a_{ji}\}$

↑
symmetric $n\times n$ matrices

$GL_n(k)$ acts by
$(A, X) \longmapsto AXA^T$

This action has finitely many orbits.
(Symmetric matrices of a given rank for an orbit)

(3') $\mathrm{End}(k^n) = \mathrm{Hom}_k(k^n, k^n) = \mathrm{Mat}_{n\times n}(k)$

$GL_n$ acts on $\mathrm{Mat}_{n\times n}(k)$ by conjugation
$(A, X) \mapsto AXA^{-1}$

invariants: $P(X)$ characteristic polynomial of
$X$ $(= \det(X \cdot + I_n))$

this is constant on orbits $\Rightarrow$ don't have finitely many orbits

Thm all polynomials in entries $x_{ij}$ of an $n\times n$ matrix that are $GL_n$-invariant are polynomials in $p_1, \dots p_n$, where $P(X) = \sum_{i=0}^{n} t^i p_{n-i}$

Examples $p_n = \det X$
$p_1 = \mathrm{tr}\, X = x_{11} + x_{22} + \dots + x_{nn}$
$p_i$ = in terms of minors of $X$

$N = \{X \mid p_1(X) = \dots = p_n(X) = 0\}$
$= \{\text{nilpotent matrices}\}$

Typical problems (Representations of Classical Groups)

- $k$ algebraicly closed

Classical groups
$GL_n$, $SL_n$

Given $K^n, \langle\ ,\ \rangle$ a non-degenerate scalar product (symmetric bilinear form)

$O(K^n, \langle\ ,\ \rangle) = \{ g : K^n \to K^n \mid \forall v, w \in K^n, \langle v, w \rangle = \langle gv, gw \rangle \}$

$\cup$

$SO(K^n, \langle\ ,\ \rangle) = \{ X \in O(K^n, \langle\ ,\ \rangle) \mid \det X = 1 \}$

$Sp_{2n}(K)$: given $K^{2n}, \langle\ ,\ \rangle$ - non degenerate anti-symmetric bilinear form

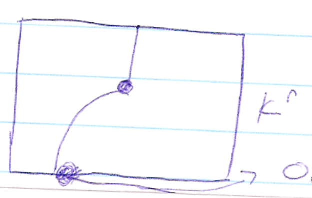
$Sp_{2n}(K) = \{ g : K^n \to K^n \mid \forall v, w \ \langle v, w \rangle = \langle gv, gw \rangle \}$

Simplest representations: those that have finitely many orbits

representation examples

$G \hookrightarrow GL_n$ is a representation of $G$ on $k^n$

orbit closures: ~~algebraic sets~~ (closure in the sense of Algebraic Geometry) algebraic sets, given by polynomial equations.




$\to O_1 \subset \bar{O}_2$ one orbit is contained in closure of another.

Interesting problems

① partial order given by
$$O_1 \leq O_2 \iff O_1 \subset \overline{O_2}$$

② defining equations of orbit closures

$K^n > \overline{O}$ $A = K[x_1, \ldots, x_n]$ polyhomial functions on $K^n$

$I_{\overline{O}} = \{f \in A \mid f|_{\overline{O}} = 0\}$

$A/I_{\overline{O}} = K[\overline{O}]$ coordinate ring of $\overline{O}$

③ Algebraic properties of coordinate rings (information about singularities of orbit closures)

Now, back to original examples:

(1'') $Hom(K^n, K^m) = Mat_{m \times n}(K)$

$A = K[X_{ij}]_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \\ x_{m1} & \cdots & x_{mn} \end{pmatrix}$$

$\mathcal{O}_r$ = {matrices of rank r}
$\mathcal{O}_{r-1} \subset \overline{\mathcal{O}}_r$

$I_{r+1}(X)$ = {$(r+1)\times(r+1)$ minors of the matrix $X$}
· defining operations

· Clear that $I_{r+1}(X)$ vanishes on $\overline{\mathcal{O}}_r$, but not obvious that there are no other polynomial other than those ~~that~~ from that ideal that vanish on $\overline{\mathcal{O}}_r$

So partial order is just
$\mathcal{O}_0 \le \mathcal{O}_1 \le \mathcal{O}_2 \dots \le \mathcal{O}_{\min(m,n)}$

<u>Algebraic properties</u>

(a) normality
$K[\overline{\mathcal{O}}]$ is integrally closed in its field of fractions

<u>Example (of not normal)</u> $K^2 \supset (x^2 - y^2)$

$K[T^3, T^2] \subset K[T]$, $T = x/y$

This property is important because if you know normality (along with something minimal), then
$K[\mathcal{O}] \cong K[\overline{\mathcal{O}}]$

b) Cohen-Macauly property:


variety

Cut it by a random equation,

repeat $d$ times

$K[x]^{(x=\bar{a})} / (f_1, \dots f_d)$ where $f_1, \dots, f_d$ are random equations

has finite~~ly~~ dimension over $K$

<u>representations with finitely many orbits</u> (Kac, 1982)

<u>Ex</u>

~~$E \otimes F \otimes H$~~ $\quad \mathrm{Hom}(K^n, K^m)$

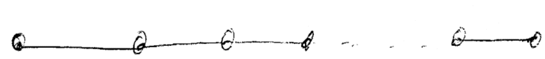
$$\cong$$

$$K^{n*} \otimes K^n$$

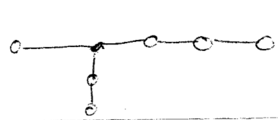
$E \otimes F \otimes H \qquad E \otimes F$

$GL_n(E) \otimes GL_n(F) \otimes GL_n(H)$

<u>Diagrams</u>:



$A_n$



$C^2 \otimes C^3 \otimes C^4$

$E \otimes F \otimes G$ has finitely many orbits iff its T diagram is a Dynkin diagram

$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^n$

$\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$      <u>exception</u> $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^6$

$\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^4$

$\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^5$

$Hom(K^n, K^n) \supset N = \{\text{nilpotent matrices}\}$

Adjoint representation

$G$ -simple graph $\mathfrak{g} = Lie(G)$

Determining algebraic properties for orbit closures in adjoint representations is unsolved and important.