Quantum Information Theory Introduction

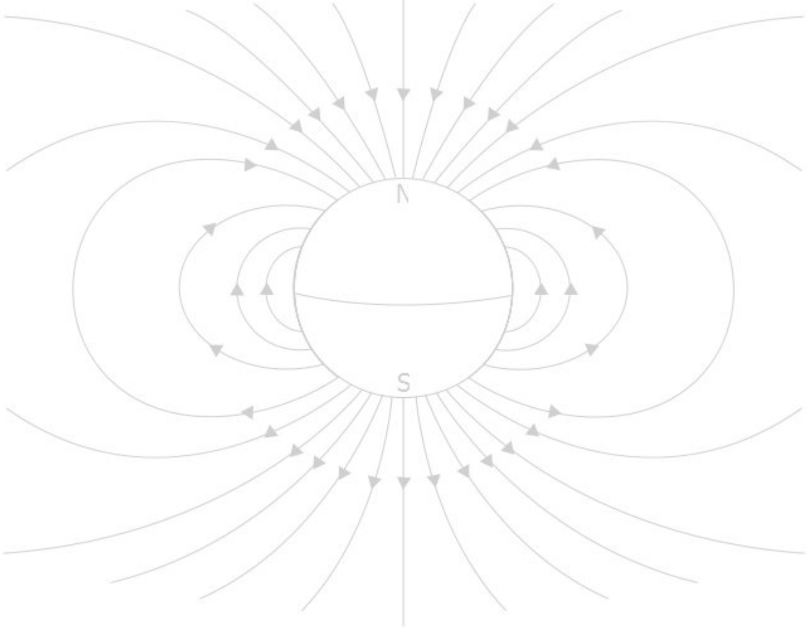
Robert Paul Chase

June 1, 2010

Abstrct

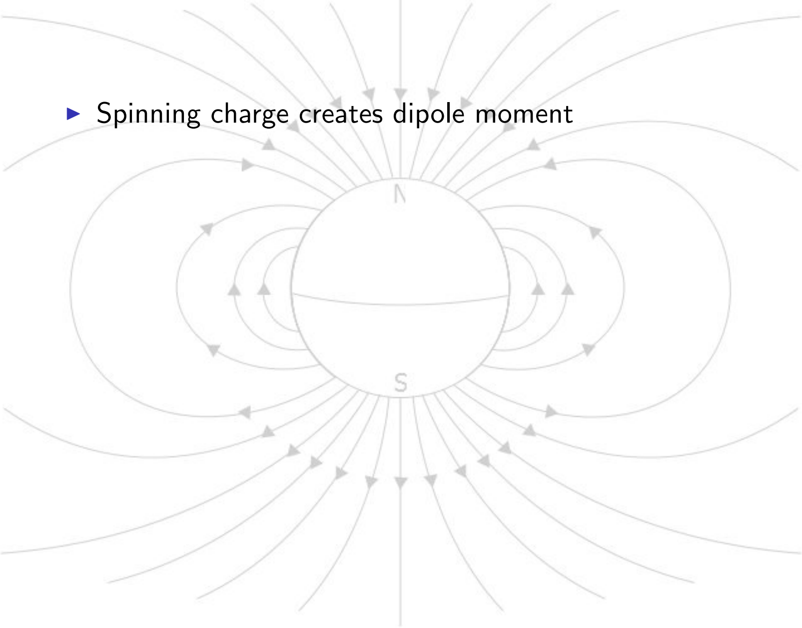
I will use the Stern Gerlach experiment as a motivating example. I will define a quantum state. I will define a Hermitian matrix. I will define a positive semidefinite matrix. I will define a density matrix. I will explain that the eigenvectors of a density matrix correspond to outcomes of experiments. I will explain that eigenvalues correspond to the probabilities of those outcomes. I will define a completely positive map. I will define a quantum channel. I will show that a sum of conjugations by a certain set of matrices gives a channel and conversely define the set of Kraus operators of a channel. I will define the Pauli matrices. I will define the Bloch Sphere. I define the King-Ruskai-Szarek-Werner matrix and show that it also represents a channel.

Stern Gerlach Experiment



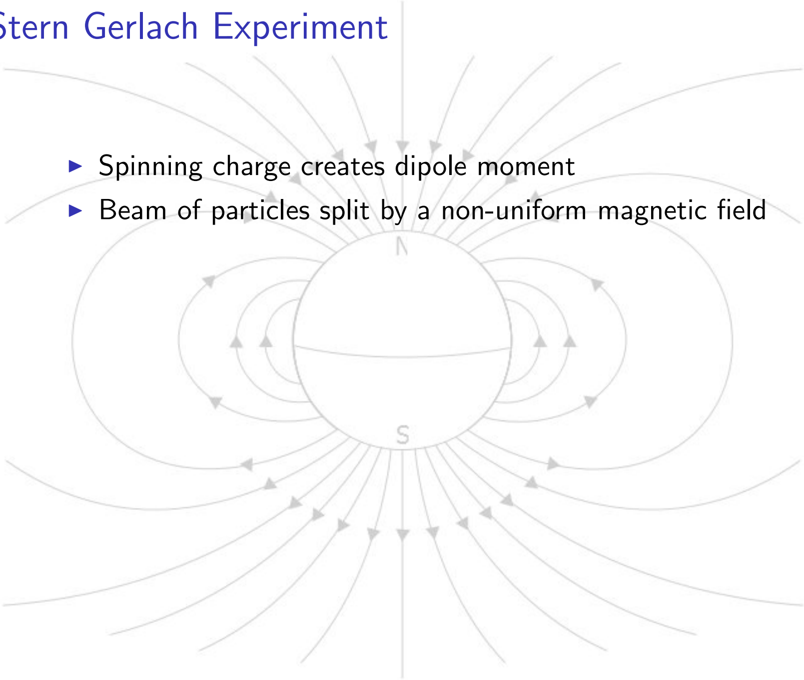
Stern Gerlach Experiment

- Spinning charge creates dipole moment

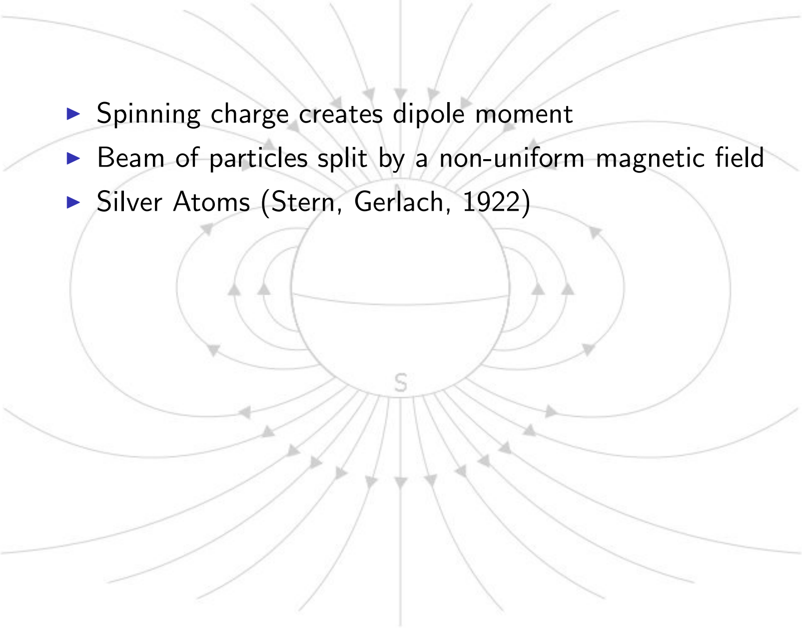


Stern Gerlach Experiment

- Spinning charge creates dipole moment
- Beam of particles split by a non-uniform magnetic field



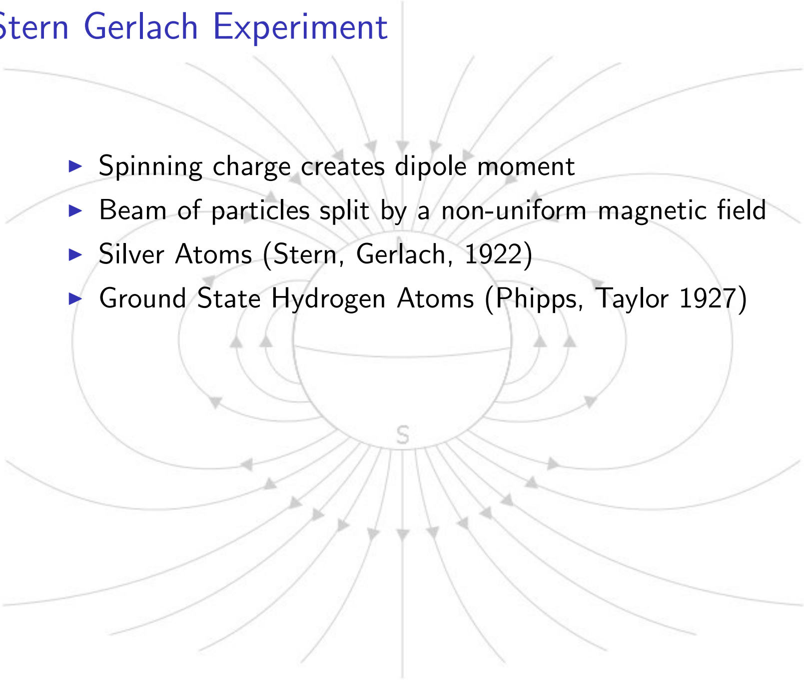
Stern Gerlach Experiment



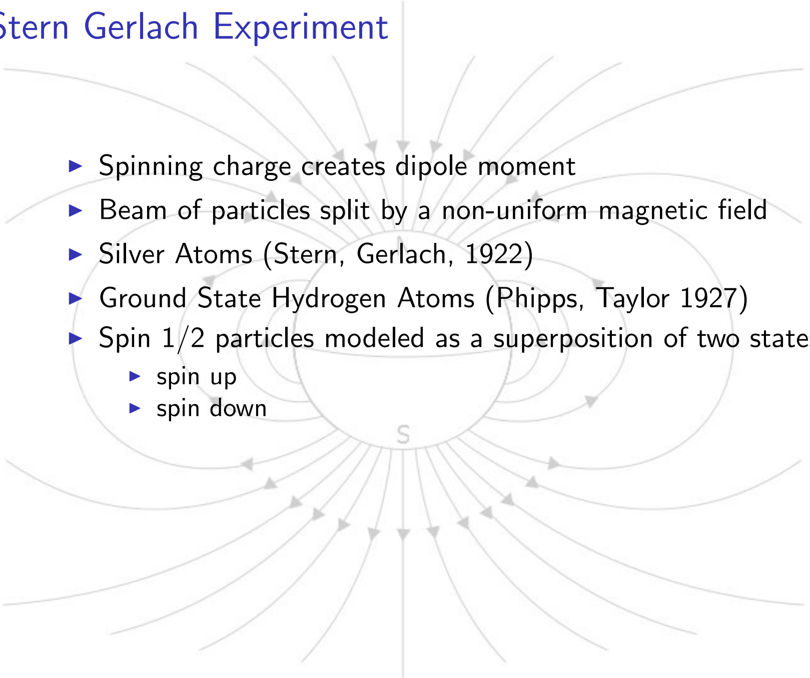
- Spinning charge creates dipole moment
- Beam of particles split by a non-uniform magnetic field
- Silver Atoms (Stern, Gerlach, 1922)

Stern Gerlach Experiment

- Spinning charge creates dipole moment
- Beam of particles split by a non-uniform magnetic field
- Silver Atoms (Stern, Gerlach, 1922)
- Ground State Hydrogen Atoms (Phipps, Taylor 1927)



Stern Gerlach Experiment

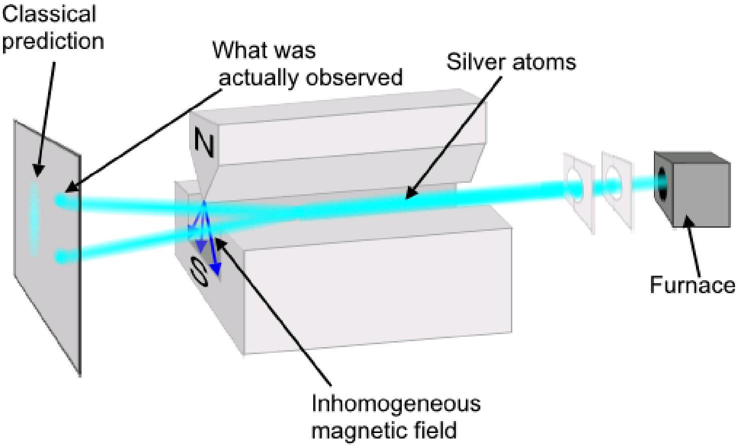


- Spinning charge creates dipole moment
- Beam of particles split by a non-uniform magnetic field
- Silver Atoms (Stern, Gerlach, 1922)
- Ground State Hydrogen Atoms (Phipps, Taylor 1927)
- Spin 1/2 particles modeled as a superposition of two states
  - spin up
  - spin down

Stern Gerlach Experiment

- Spinning charge creates dipole moment
- Beam of particles split by a non-uniform magnetic field
- Silver Atoms (Stern, Gerlach, 1922)
- Ground State Hydrogen Atoms (Phipps, Taylor 1927)
- Spin 1/2 particles modeled as a superposition of two states
  - spin up
  - spin down
- Spin 1 particles modeled as superposition of three states
  - spin up
  - spin middle
  - spin down

Stern Gerlach Aparatus



A beam of particles is split by a nonuniform magnetic field.

Quantum States

Definition

An n-dimensional quantum state is an element of $\mathbf{C}^n$ of unit length.

- Length is defined by an inner product $< u, u >$.
- Unitary transformations are such that $U^*U = I$.
- States are mapped to other states by unitary transformations. If $< u, u >= 1$ then $< uU^*, Uu >= uU^*Uu =< u, u >= 1$ too.

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} -\imath \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ \imath \end{pmatrix} / \sqrt{2}$$

Hermitian Matrices

Definition

A complex nxn matrix H is Hermitian iff the conjugate of the transpose of H denoted by H^* is H itself.

- Hermitian matrices have real eigenvalues.
- Hermitian matrices have orthogonal eigenvectors that span $\mathbf{C}^n$.
- Homogeneity: A Hermitian matrix times a real number is another Hermitian matrix.
- Superposition: The sum of two Hermitian matrices is Hermitian.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Positive Semidefinite Matrices

Definition

An nxn Hermitian matrix H is positive semi-definite if $x^*Hx \geq 0$ for all vectors $x \in \mathbf{C}$. This is equivalent to H having nonnegative eigenvalues.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\imath \\ \imath & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Density Matrices

Definition

An nxn complex matrix ρ is an n-dimensional density matrix if

- ρ is Hermitian
- ρ is positive-semidefinite
- ρ has trace 1

Equivalently, a Hermitian matrix ρ is a density matrix if its eigenvalues are all non-negative and sum to one. The set of all n-dimensional density matrices is denoted by $\mathbf{B}(\mathbf{C}^n)$ in honor of the Swiss Nobel Laureate Felix Bloch.

$$\lambda_i(\rho) \geq 0 \sum_i \lambda_i(\rho) = 1$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1/2 & -\imath/2 \\ \imath/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Density Matrices can represent indeterminant systems.

From Quantum States to Density Matrices

Given a known quantum state s, a density matrix representing s written ρ may be obtained from the formula $\rho = ss^*$ because the matrix ss^* is Hermitian, positive semidefinite with eigenvalues $\{1, 0\}$ and has trace 1. Only 'pure' density matrices may be obtained as an outer product of a single state in this fashion. This construction destroys any notion of phase contained in the state; if $\tilde{s} = Exp[\imath t]s$ then $\tilde{s}\tilde{s}^* = ss^*$. On the other hand, density matrices can represent systems where the state of a particle is unknown.

Interpretation of Eigenvector

Every outcome of an experiment can be represented by a state. The normalized eigenvectors of a density matrix correspond to states. For example, the density matrix

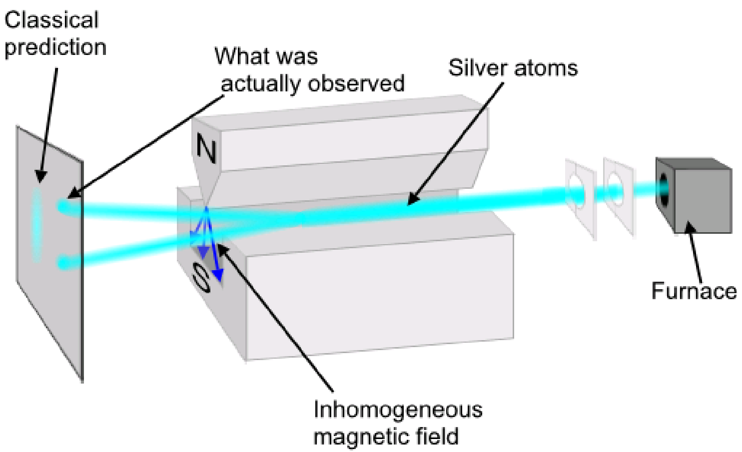
$$\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

is referred to as the completely mixed state. It has normalized eigenvectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

which refer to the 'spin-up state' and the 'spin-down state' respectively.

Stern Gerlach Aparatus Revisited



Before the beam is split, a particle can only be represented by a density matrix. After the beam has been measured, the particle is represented by a quantum state or a pure matrix.

Interpretation of Eigenvalues

To each eigenvector there is an associated eigenvalue and these eigenvalues correspond to the probabilities of obtaining the states represented by the eigenvectors. If a particular beam of particles subjected to a nonuniform magnetic field is represented by the density matrix

$$\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

with eigenvectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and eigenvalues $\{1/2, 1/2\}$, then a particular particle in the beam will go up with probability $1/2$ and down with probability $1/2$.

Completely Positive Maps

Definition

A map $\Phi : \mathbf{C}^{nxn} \rightarrow \mathbf{C}^{mxm}$ is called positive if the positive semidefiniteness of ρ implies the positive semidefiniteness of $\Phi(\rho)$.
Positive maps preserve the positive semidefiniteness property.
A map $\Phi : \mathbf{C}^{nxn} \rightarrow \mathbf{C}^{mxm}$ is k-positive if

$$I_k \otimes \Phi : \mathbf{C}^{kxk} \otimes \mathbf{C}^{nxn} \rightarrow \mathbf{C}^{kxk} \otimes \mathbf{C}^{mxm}$$

is positive.
A map Φ is called completely positive if it is k-positive for all k.

Example of Positive, 1-Positive Map that is not 2-Positive

The map $f(x) = x^{tr}$ is positive because the eigenvalues of x and x^{tr} coincide.
$I_1 \otimes f = f$ is therefore positive so f is 1-positive.
The map $f(x) = x^{tr}$ is not 2-positive because a counterexample

$$(I_2 \otimes f) \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

can be found such that the image of a matix with nonnegative eigenvalues has negative eigenvalues.

The Transpose is not 2-Positive

The original matrix has eigenvectors with nonnegative eigenvalues

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 0-\lambda & 1 & 0 \\ 0 & 1 & 0-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2(\lambda^2-1)$$

Quantum Channel

Definition

A map $\Phi : \mathbf{B}(\mathbf{C}^n) \to \mathbf{C}^{mxm}$ is a quantum channel if it is

- Linear
- Completely positive
- Trace preserving

For instance, the operator $\Phi(\rho) = (\rho - I/2)/2 + I/2 = \rho/2 + I/4$ is a channel.

Quantum channels map density matrices to density matrices.

Kraus Operators

Theorem

We assume that the set of nxm matrices $\{B_i\}$ is such that $\sum_i B_i B_i^* \leq I$. Then we call the B_i Kraus Operators and the map $\Phi(\rho) = \sum_i B_i^* \rho B_i : B(\mathbf{C}^n) \to \mathbf{C}^{mxm}$ is a quantum channel.

Theorem

Let $\Phi : B(\mathbf{C}^n) \to B(\mathbf{C}^m)$ be a quantum channel. There exists a set of matrices $\{B_i\}$ such that $\Phi(\rho) = \sum_i B_i^* \rho B_i$ that need not be unique. For any given Channel there is a set of Kraus Operators with minimal cardinality and this cardinality is called the Rank of the Channel.

We will first develop some notation and then show some examples of Kraus Operators.

Pauli Matrices

The following three matrices are called the Pauli matrices.

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sigma_y = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix} \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

They are traceless, have determinant 1 and satisfy the following relations;

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = -\imath\sigma_x\sigma_y\sigma_z = I_2$$

The set of real linear combinations of these matrices is equal to the set of traceless Hermitian 2x2 matrices. Adding real scalar multiples of the identity matrix to such spans yields the set of all Hermitian Matrices.

Bloch Sphere

Any density matrix ρ may be written as a real linear combination of Pauli matrices and the identity matrix. In particular, any density matrix may be written as a real linear combination of Pauli matrices and 1/2 times the identity matrix.

$$\rho = (I + w_x\sigma_x + w_y\sigma_y + w_z\sigma_z)/2$$

We write the vector of w_is as $\vec{w}$, call this the Pauli vector and denote its length by r. The extended Pauli vector includes a weight of 1/2 corresponding to the weight of the identity. With a slight abuse of notation we may write any density matrix in terms of weights and Pauli matrices

$$\rho = \vec{w} \bullet \vec{\sigma}/2$$

It turns out the the eigenvalues of ρ are $\frac{1+r}{2}$ and $\frac{1-r}{2}$. These are only nonnegative if $0 \leq r \leq 1$ so the set of valid density matrices corresponds to the set of Pauli vectors of unit length or less. We call this set the Bloch Sphere and denote it by $B(\mathbf{C}^2)$.

Kraus Operator Examples

Example: The set $\{I\}$ is a set of Kraus Operators and so is $\{\frac{I}{\sqrt{2}}, \frac{I}{\sqrt{2}}\}$.
The rank of the identity channel is 1.
Example: The set $\{I/2, \frac{\sigma_x}{2}, \frac{\sigma_y}{2}, \frac{\sigma_z}{2}\}$ is a set of Kraus Operators.
Let us consider the action of these operators on the density matrix $\rho = (I + \sigma_x)/2$

$$\sum_i B_i B_i^* = \frac{I}{2}\frac{I}{2} + \frac{\sigma_x}{2}\frac{\sigma_x}{2} + \frac{\sigma_y}{2}\frac{\sigma_y}{2} + \frac{\sigma_z}{2}\frac{\sigma_z}{2} = I \leq I$$

$$\sum_i B_i \rho B_i^* = \sum_i B_i I/2 B_i^* + \sum_i B_i \sigma_x/2 B_i^*$$
$$= I/2 + \frac{\sigma_x}{2}\frac{\sigma_x}{2}\frac{\sigma_x}{2} + \frac{\sigma_y}{2}\frac{\sigma_x}{2}\frac{\sigma_y}{2} + \frac{\sigma_z}{2}\frac{\sigma_x}{2}\frac{\sigma_z}{2}$$
$$= I/2 + \frac{\sigma_x}{8} + \imath\frac{\sigma_y}{2}\frac{\sigma_z}{4} + \imath\frac{\sigma_y}{4}\frac{\sigma_z}{2}$$
$$= I/2 + \frac{\sigma_x}{8} - \frac{\sigma_x}{8} - \frac{\sigma_y}{8}$$
$$= I/2 - \sigma_x/8$$

King-Ruskai-Szarek-Werner Matrix

Since a quantum channel linearly maps density matrices to density matrices, a quantum channel linearly maps Pauli vectors to Pauli vectors.

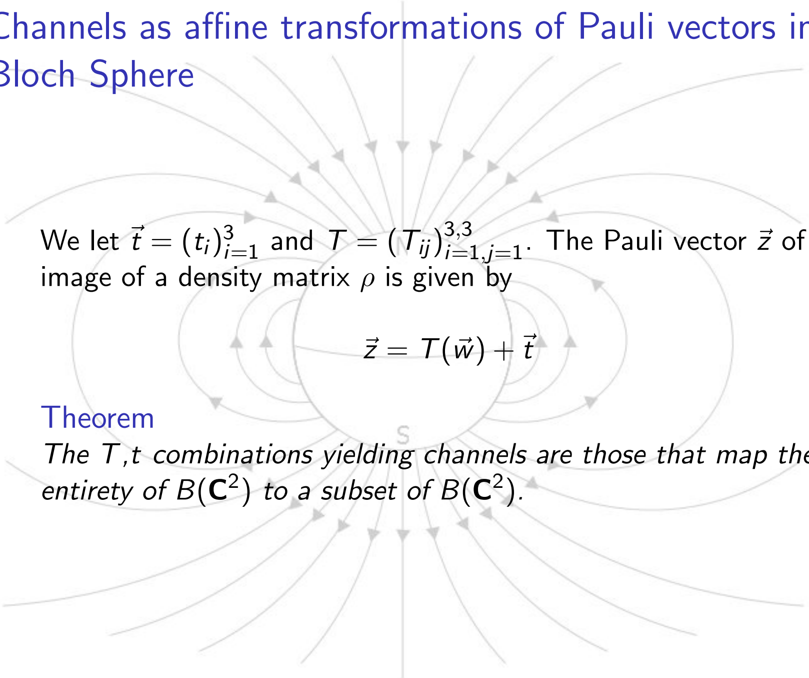
Any quantum channel preserves trace by definition, and the only contribution to the trace of a 2x2 density matrix comes from the identity, so a quantum channel fixes the first weight of the extended Pauli vector.

Therefore any channel $\Phi(\rho)$ may be written in the Pauli basis as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ t_1 & T_{11} & T_{12} & T_{13} \\ t_2 & T_{21} & T_{22} & T_{23} \\ t_3 & T_{31} & T_{32} & T_{33} \end{pmatrix} \begin{pmatrix} 1 \\ w_x \\ w_y \\ w_z \end{pmatrix}$$

This corresponds to an affine transformation of the Pauli vector or a linear transformation followed by a translation.

Channels as affine transformations of Pauli vectors in the Bloch Sphere



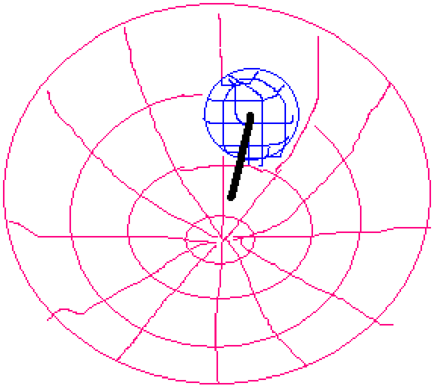
We let $\vec{t} = (t_i)_{i=1}^{3}$ and $T = (T_{ij})_{i=1,j=1}^{3,3}$. The Pauli vector $\vec{z}$ of the image of a density matrix ρ is given by

$$\vec{z} = T(\vec{w}) + \vec{t}$$

Theorem

The T,t combinations yielding channels are those that map the entirety of $B(\mathbf{C}^2)$ to a subset of $B(\mathbf{C}^2)$.

Graphical Interpretation of King-Ruskai-Szarek-Werner



In general, a channel may shrink and rotate the Bloch Sphere before translating it from the center.

Summary

Particles in the Stern Gerlach experiment can be modeled as normalized complex vectors. The definition of hermitian matrices and completely positive matrices appear in the definition of density matrices. The eigenvectors of a density matrix correspond to outcomes of experiments. The eigenvector associated eigenvalues correspond to the probabilities of experimental outcomes. Quantum channels map density matrices to density matrices and can be represented as a sum of conjugations by Kraus operators. Pauli matrices are a basis of the traceless 2x2 Hermitian matrices. The Bloch Sphere is the set of Pauli matrix weight vectors which yield density matrices. The King-Ruskai-Szarek-Werner matrix represents a channel as an affine transformation of the Pauli vector.

References I

- http://www.springerlink.com/content/q47427674u801u22/
- http://prola.aps.org/abstract/PR/v29/i2/p309_1
- The absolute value is not completely positive. http://www.math.wm.edu/~ckli/ima/note-5.pdf
- Positive maps that are not completely positive are the difference of two completely positive maps. http://arxiv.org/abs/quant-ph/0001053
- transposes are not completely positive. http://panic.berkeley.edu/~shoyer/pubs/math97.pdf
- John Cortese http://arxiv.org/PS_cache/quant-ph/pdf/0207/0207128v1.pdf
- Nielsen and Chuang reference, Theorems 8.1 and 8.3.

References II

Good photo http://infoproc.blogspot.com/2005/08/is-hilbert-space-discrete.html