Quantum Information Theory Introduction

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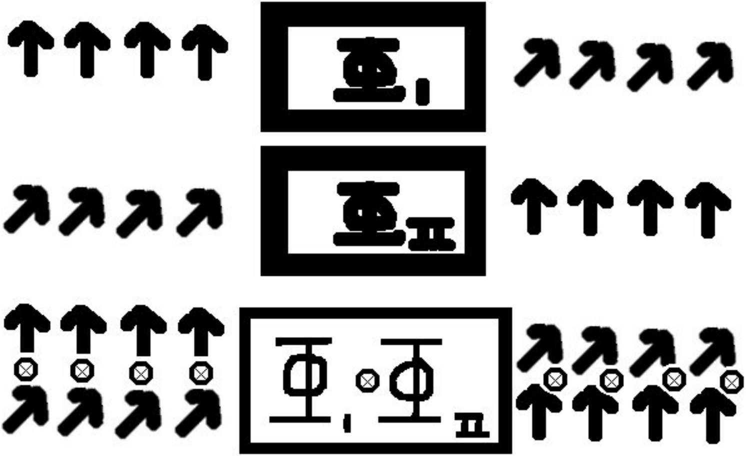
Abstract

I will define a pure matrix. I will show that a density matrix may be decomposed into a superposition of pure matrices called an ensemble. I will discuss the kronecker product of two density matrices. I will discuss the spectrum of the resulting density matrix. I will discuss the tensor product of two channels. I will define the Von Neumann Entropy of a density matrix. I will define the Holevo Mutual information of a channel given an ensemble. I will define the Holevo capacity of a channel. I will discuss the Holevo additivity conjecture. I will discuss numerical attempts. I will discuss closed formulas.

Recall

- ...that a density matrix is a hermitian positive semidefinie nxn matrix of trace 1.
- ...that 2x2 density matrices can be written in terms of the Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ and I.
- ...that for any dimension n there is a monoid of quantum channels that linearly maps nxn density matrices to other density matrices.
- ...that a kraus operator is one way of representing quantum channels.
- ...that the King-Ruskai-Szarek-Werner represents channels geometrically using the Bloch Sphere.

Experiment



Pure Matrix

A density matrix ρ is called a pure matrix if it factors as $\rho = \vec{\rho}\vec{\rho}^*$. Equivalently, pure matrices are defined by having a single nonzero eigenvalue which is necessarily 1.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} 1/2 & -\imath/2 \\ \imath/2 & 1/2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{\imath}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-\imath}{\sqrt{2}} \end{pmatrix}$$

Ensemble Superposition of Pure Matrices

A density matrix may be decomposed into a superposition of pure matrices called an ensemble. A set of eigenvalue-(normalized)eigenvector pairs $\{(\rho_i, \vec{\rho_i})\}$ is an ensemble for ρ if

$$\rho = \sum_i \rho_i \vec{\rho_i}\vec{\rho_i}^* = \rho_1 \vec{\rho_1}\vec{\rho_1}^* + \rho_2 \vec{\rho_2}\vec{\rho_2}^* + \ldots + \rho_n \vec{\rho_n}\vec{\rho_n}^*$$

Alternatively, an ensemble can be thought of as a list of nonnegative real numbers and pure matrices such that the real numbers add up to one.

Kronecker Product of Two Density Matrices

Since the tensor product of two matrices $A \otimes B$ is itself a linear transformation, it can be written as a matrix using the Kronecker Product. The Kronecker product of an nxn matrix A and mxm matrix B is an nmxnm block matrix of Bs scalled by the entries of A

...is Hermitian. ...has trace 1. ...is positive semidefinite.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Spectrum of Tensor Product

If a matrix A has spectrum $\{a_1, ..., a_n\}$ and the matrix B has spectrum $\{b_1, ..., b_m\}$ then the spectrum of the matrix $A \otimes B$ will consist of the nm values $\{a_i b_j\}$ for $1 \leq i \leq n, 1 \leq j \leq m$.

The matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ has eigenvalues $\{0, 1\}$ and the matrix $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ has eigenvalues $\{0, 1\}$ so the matrix

$$\begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ has eigenvalues $\{1, 0, 0, 0\}$

If a matrix A has eigenvectors $\{\vec{a_1}, ..., \vec{a_n}\}$ and the matrix B has eigenvectors $\{\vec{b_1}, ..., \vec{b_m}\}$ then the eigenvectors of the matrix $A \otimes B$ will consist of the nm tensor products $\{\vec{a_i} \otimes \vec{b_j}\}$ for $1 \leq i \leq n, 1 \leq j \leq m$.

Tensor Products of two Channels

Given a basis A of *nxn* matrices and a basis B of *mxm* matrices, any *nmxnm* matrix may be written as a sum of tensor products of the *nxn* and *mxm* basis matrices from A and B. A tensor product of channels lets the channels act on each term in the tensor product individually and then takes their tensor product.

$$M_1 \in \mathbf{B}(\mathbf{C}^n) M_2 \in \mathbf{B}(\mathbf{C}^m) M = M_1 \otimes M_2 \in \mathbf{B}(\mathbf{C}^{nm})$$

$$\Phi_1 : \mathbf{B}(\mathbf{C}^n) \to \mathbf{B}(\mathbf{C}^n)$$

$$\Phi_2 : \mathbf{B}(\mathbf{C}^m) \to \mathbf{B}(\mathbf{C}^m)$$

$$\Phi = \Phi_1 \otimes \Phi_2 : \mathbf{B}(\mathbf{C}^{nm}) \to \mathbf{B}(\mathbf{C}^{nm})$$

$$\Phi(M) = [\Phi_1 \otimes \Phi_2](M_1 \otimes M_2) = \Phi_1(M_1) \otimes \Phi_2(M_2)$$

Example of Tensor Product of Two Channels

Let $M_1 = (I + \sigma_z)/2$, $M_2 = (I + \sigma_y)/2$ and $M = M_1 \otimes M_2$.
$\Phi_1 =$ Identity Operator $\Phi_2 =$ Channel given by Kraus operators $\{I/2, \frac{\sigma_x}{2}, \frac{\sigma_y}{2}, \frac{\sigma_z}{2}\}$ and $\Phi = \Phi_1 \otimes \Phi_2$

$\Phi(M) = [\Phi_1 \otimes \Phi_2](M_1 \otimes M_2)$
$= \Phi_1(M_1) \otimes \Phi_2(M_2)$
$= (I + \sigma_z)/2 \otimes (I - \sigma_y/4)/2$

$$\Phi \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \Phi_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \Phi_2 \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1/2 & 1/8 \\ 1/8 & 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/8 & 0 & 0 \\ 1/8 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Von Neumann Entropy of a Density Matrix

The Von Neumann Entropy $S(\rho)$ of a density matrix ρ is $-Tr[\rho \log_2 \rho]$. This measures the ammount of information that can be transmitted by a single quantum bit. Alternatively this can be written as $-\sum_i \rho_i \log \rho_i$ where the sum is taken over all nonzero eigenvalues.

$S((I+\sigma_z)/2) = -Tr[(I+\sigma_z)/2 \log_2((I+\sigma_z)/2)]$

$$S\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = -Tr[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}] = 0$$

$$S\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} = -Tr[\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} \log_2 1/2 & 0 \\ 0 & \log_2 1/2 \end{pmatrix}]$$

$= -\log_2 1/2 = 1$

Holevo Mutual Information of a Channel given an Ensemble

How much information do we lose by using an entangled state as opposed to pure states? If Φ is a channel and $P = \{(\rho_i, \vec{\rho_i})\}$ is an ensemble for ρ then the Holevo mutual information of the channel given the ensemble is given by;

$$I(\Phi, P) = S(\Phi(\rho)) - \sum_i \rho_i S(\Phi(\vec{\rho_i}\vec{\rho_i}^*))$$

$$S(\Phi(\rho)) - \rho_1 S(\Phi(\vec{\rho_1}\vec{\rho_1}^*)) - \rho_2 S(\Phi(\vec{\rho_2}\vec{\rho_2}^*)) - ... - \rho_n S(\Phi(\vec{\rho_n}\vec{\rho_n}^*))$$

The Holevo Capacity of a Channel

Let $\mathbf{P}$ be the set of ensembles for every $\rho \in \mathbf{B}(\mathbf{C}^n)$.
The Holevo capacity χ of an n dimensional Quantum Channel Φ is

$$\chi(\Phi) = \sup_{P \in \mathbf{P}} I(\Phi, P)$$

The Holevo capacity of the identity channel I_n is
$\chi(I_n) = \sup I(\Phi, P) = \sup S(I_n(\rho)) - \sum_i \rho_i S(I_n(\vec{\rho}_i \vec{\rho}_i^{\,*}))$
$= \sup S(\rho) - \sum_i \rho_i S(\vec{\rho}_i \vec{\rho}_i^{\,*})$
$= \sup S(\rho) - \sum_i \rho_i 0$
$= \sup S(\rho)$
$= S(\text{entirely mixed state}) = n - 1$

Holevo Additivity Conjecture

One might expect that if Φ_1 and Φ_2 are density matrices

$$\chi(\Phi_1 \otimes \Phi_2) = \chi(\Phi_1) + \chi(\Phi_2)$$

This conjecture has been shown to be false for $n >> 1$ by M.B. Hastings. It is open for n = 2.

Numerical Attempts

A number of attempts have been made to find channels disproving the additivity hypothesis. Susumu Osawa, Hiroshi Nagaoka generalize Arimoto-Bluhat algorithms to the quantum realm. Randomly sampling states and channels from the Haar measure has also been explored widely.

Closed Formulas

John Cortese gave the following formula for the relative information of a density matrix R given another density matrix Q in terms of the radii q, r of the Bloch vectors of the matrices and the angle θ between the two vectors.

$$D(R||Q) =$$

$$\frac{1}{2}\log_2(1-r^2) + \frac{r}{2}\log_2\left(\frac{1+r}{1-r}\right) - \frac{1}{2}\log_2(1-q^2) - \frac{r\cos(\theta)}{2}\log_2\left(\frac{1+q}{1-q}\right)$$

Summary

I defined a pure matrix and showed that a density matrix may be decomposed into a superposition of pure matrices called an ensemble. The kronecker product of two density matrices has a spectrum given by the spectrums of the original matrices. Two channels can be combined to create a tensor product of two channels. The Von Neumann entropy of a density matrix was used to define the Holevo Mutual information of a channel given an ensemble. The Holevo capacity of a channel was used to state the Holevo additivity conjecture. Numerical attempts have been made to find a counter-example. Closed formulas have been written for the Holevo Capacity.

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