5/13/10 Gabe Cunningham

Chiral Polytopes – Tapas 2010

Let us start by recapping the definition of an abstract polytope:

Briefly, an *abstract polytope* is a ranked poset $\mathcal{P}$ such that:

(i) $\mathcal{P}$ has a single maximal element (face) and a single minimal element (face),

(ii) All flags of $\mathcal{P}$ have the same length,

(iii) All sections of $\mathcal{P}$ are connected, and

(iv) Given a face F of rank i and a face H of rank $i+2$ such that $F < H$, there are precisely two faces G_1 and G_2 (of rank $i+1$) such that $F < G_j < H$ for $j = 1, 2$.

From now on, I'm going to just say polytope instead of abstract polytope.

Given a polytope $\mathcal{P}$, its automorphism group $\Gamma(\mathcal{P})$ consists of bijections that preserve rank and incidence. We say that an abstract polytope $\mathcal{P}$ is regular if the action of $\Gamma(\mathcal{P})$ on the flags is transitive. We define the string Coxeter group $[p_1, \ldots, p_{n-1}]$ to be the group generated by $\rho_0, \ldots, \rho_{n-1}$ such that $\rho_i^2 = 1$, $(\rho_{i-1}\rho_i)^{p_i} = 1$, and $(\rho_i\rho_j)^2 = 1$ if $|i-j| \geq 2$. We saw last time that the automorphism group of a regular polytope is a quotient of a string Coxeter group that satisfies the "intersection condition":

$$\langle \rho_j \mid j \in J \rangle \cap \langle \rho_j \mid j \in K \rangle = \langle \rho_j \mid j \in J \cap K \rangle,$$

and we call such quotients string C-groups.

It turns out that from a string C-group Γ we can build a regular polytope $\mathcal{P}(\Gamma)$ such that $\Gamma(\mathcal{P}(\Gamma)) = \mathcal{P}$. In particular, letting $\Gamma_j = \langle \rho_i \mid i \neq j \rangle$, we define the j-faces of $\mathcal{P}(\Gamma)$ to be the right cosets $\Gamma_j\varphi$ with $\varphi \in \Gamma$. Then we say that two faces are incident if they intersect as cosets. What's nice about this is that we can study abstract regular polytopes just by working with their automorphism groups – there is a one-to-one correspondence between string C-groups with a distinguished set of generators and regular polytopes.

Now, given a string C-group Γ, we can look at its *rotation subgroup* Γ^+, generated by $\sigma_1, \ldots, \sigma_n$, where $\sigma_i = \rho_{i-1}\rho_i$. If Γ is a quotient of $[p_1, \ldots, p_{n-1}]$ so that $(\rho_{i-1}\rho_i)^{p_i} = 1$, then $\sigma_i^{p_i} = 1$. Furthermore, $(\sigma_i\sigma_{i+1}\cdots\sigma_j)^2 = (\rho_{i-1}\rho_j)^2 = 1$ if $i < j$. Finally, Γ^+ has an intersection property similar to Γ. In general, if G is a group generated by $\sigma_1, \ldots, \sigma_{n-1}$ such that the generators σ_i satisfy the above properties, we will call G a string χ-group.

Now, if we let ρ_0 act on the set of cosets Γ/Γ^+ in the obvious way, we see that $\rho_0\Gamma^+$ contains all the generators ρ_i, so that $\rho_i\Gamma^+ = \rho_j\Gamma^+$ for all i, j, and so $\rho_i\rho_j G^+ = G^+$ for all i, j. Thus Γ^+ has index at most 2 in Γ.

Given a regular polytope $\mathcal{P}$, we say that it is *directly regular* if the index of Γ^+ in Γ is exactly 2. As an example of a polytope that is not directly regular, consider the hemicube, with group

$$\langle \rho_0, \rho_1, \rho_2 \mid \rho_i^2, (\rho_0\rho_1)^4, (\rho_0\rho_2)^2, (\rho_1\rho_2)^3, (\rho_0\rho_1\rho_2)^3 \rangle.$$

Since Γ^+ is of index at most 2 in Γ, then Γ/Γ^+ is actually a group, so what happens when we kill the rotations σ_i? We can rewrite the last relation as

$$\rho_0(\rho_1\rho_2)(\rho_0\rho_1)(\rho_2\rho_0)(\rho_1\rho_2) = \rho_0\sigma_2\sigma_1(\sigma_1\sigma_2)\sigma_2,$$

so that since this word is killed, and so are the rotations σ_i, we conclude that ρ_0 is killed, and it follows that the whole group is killed. So $\Gamma^+ = \Gamma$. Note that all of the convex polytopes are directly regular.

Earlier, we saw that string C-groups were in 1-to-1 correspondence with regular polytopes. Are string χ-groups in 1-to-1 correspondence with directly regular polytopes?

Consider the group

$$G = \langle \sigma_1, \sigma_2 \mid \sigma_1^4, \sigma_2^4, (\sigma_1\sigma_2)^2, (\sigma_1^{-1}\sigma_2)(\sigma_1\sigma_2^{-1})^2 \rangle.$$

If this is the rotation subgroup of a directly regular polytope, that polytope has full automorphism group

$$\Gamma = \langle \rho_0, \rho_1, \rho_2 \mid \rho_i^2, (\rho_0\rho_1)^4, (\rho_0\rho_2)^2, (\rho_1\rho_2)^4, (\rho_1\rho_0\rho_1\rho_2)(\rho_0\rho_1\rho_2\rho_0)^2 \rangle.$$

However, while the order of G is 20, the order of Γ is 8 – so G can't be its rotation subgroup. Why does this happen? We can write G as W^+/M, where

$$W^+ = [\infty, \infty]^+ = \langle \sigma_1, \sigma_2 \mid (\sigma_1\sigma_2)^2 \rangle,$$

and where M is the normal closure of $(\sigma_1^{-1}\sigma_2)(\sigma_1\sigma_2^{-1})^2$ in W^+. In order for G to be the rotation subgroup of Γ, it must be the case that $\Gamma = W/M$, where

$$W = [\infty, \infty] = \langle \rho_0, \rho_1, \rho_2 \mid \rho_i^2, (\rho_0\rho_2)^2 \rangle.$$

In other words, we need for M to be normal in W. In this example, that fails to be the case.

So, not every string χ-group is the rotation subgroup of a directly regular polytope. However, every string χ-group is either the rotation subgroup of a directly regular polytope or the automorphism group of a chiral polytope, which we now define.

Definition ("global"): A polytope $\mathcal{P}$ is chiral if its automorphism group $\Gamma(\mathcal{P})$ has two orbits on the flags, and such that adjacent flags are in different orbits.

Note: the second half is important! There are two-orbit polytopes that are not chiral.

We can formulate the definition in another way:

Definition ("local"): A polytope $\mathcal{P}$ with base flag Φ is chiral if it is not regular, and there exist automorphisms $\sigma_1, \ldots, \sigma_n$ such that σ_i fixes all faces in $\Phi/\{F_{i-1}, F_i\}$ and cyclically permutes consecutive i-faces of $\mathcal{P}$ in F_{i+1}/F_{i-2}.

Examples:

1. The torus map $\{4,4\}_{(b,c)}$, with $bc(b-c) \neq 0$, is chiral. This is a polytope with $b^2 + c^2$ square faces, that many vertices, and twice as many edges. Similar examples arise from the plane tessellations $\{3,6\}$ and $\{6,3\}$.
2. The locally toroidal polytope $\{\{4,4\}_{(b,c)}, \{4,3\}\}$, with $bc(b-c) \neq 0$, is chiral. It is presently unknown what values of b and c make this finite.

Just as we were able to build regular polytopes from string C-groups, we can build polytopes from string χ-groups in essentially the same way. If a string χ-group G cannot be "lifted" to a string C-group, then when we build a polytope from G, it will be chiral.

Given a chiral polytope $\mathcal{P}$, all of its sections are either chiral or directly regular. In particular, its facets and vertex figures are chiral or directly regular. What is surprising is that its $(n-2)$-faces and edge figures must be directly regular! Why is this important? Given a regular polytope $\mathcal{P}$, it is possible to extend it to a regular polytope $\mathcal{Q}$ having $\mathcal{P}$ as facets, and to repeat this process. What the above remark says is that we can't repeatedly extend a chiral polytope to larger and larger chiral polytopes. So in general, in order to find chiral polytopes in a given rank, we pick a string Coxeter group C and look for subgroups N that are normal in C^+ but not in C.

A chiral polytope occurs in two *enantiomorphic forms*; informally speaking, there is a left-handed and a right-handed version. If $\mathcal{P}$ is chiral, then $\overline{\mathcal{P}}$ will denote its enantiomorphic form.

Up until now, I've described chirality as a binary property, but in fact, there is a useful way to measure chirality. Before we can introduce that idea, we need to know about coverings of polytopes.

Definition. *A map $\varphi : \mathcal{P} \to \mathcal{Q}$ is a homomorphism of polytopes if it is incidence preserving; that is, if $F \leq G$ in $\mathcal{P}$ implies $\varphi(F) \leq \varphi(G)$ in $\mathcal{Q}$.*

Definition. *A homomorphism $\varphi : \mathcal{P} \to \mathcal{Q}$ is a rap-map if it is rank and adjacency preserving. That is, φ is a rap-map if the rank of $\varphi(F)$ in $\mathcal{Q}$ is the same as the rank of F in $\mathcal{P}$, and if whenever Φ and Ψ are adjacent flags of $\mathcal{P}$, then $\varphi(\Phi)$ and $\varphi(\Psi)$ are adjacent flags of $\mathcal{Q}$. A surjective rap-map is called a covering, and if there is a covering from $\mathcal{P}$ to $\mathcal{Q}$ we say that $\mathcal{P}$ covers $\mathcal{Q}$ and we write $\mathcal{P} \searrow \mathcal{Q}$.*

Examples:

1. The hexagon covers the triangle.
2. There is no rap-map from a triangle to a hexagon.

Note that once we know where a rap-map sends one flag of $\mathcal{P}$, we actually know the whole map.

Now, let $\mathcal{P}$ and $\mathcal{Q}$ be chiral or directly regular n-polytopes. Let $\mathcal{U}$ be the universal n-polytope

$\{\infty, \ldots, \infty\}$, and let W^+ be its rotation group. Then W^+ has presentation:

$$\langle \sigma_1, \ldots, \sigma_{n-1} \mid (\sigma_i \cdots \sigma_j)^2 = 1, 1 \leq i < j \leq n-1 \rangle.$$

We can write the rotation group of $\mathcal{P}$ as W^+/M and the rotation group of $\mathcal{Q}$ as W^+/K for some normal subgroups M and K. Then I can take both of these groups to be generated by $\sigma_1, \ldots, \sigma_{n-1}$ – they'll just have relations in addition to those of W^+. Then $\mathcal{P}$ covers $\mathcal{Q}$ if and only if the group homomorphism $\varphi : \Gamma^+(\mathcal{P}) \to \Gamma^+(\mathcal{Q})$ sending σ_i to σ_i is surjective. (In fact, if it's not surjective, then it's not well-defined). Furthermore, φ is surjective if and only if $K \leq M$.

Note: As a corollary to the above, note that a chiral polytope does not cover its enantiomorphic form.

Let $\mathcal{P}$ be a chiral polytope, and $\overline{\mathcal{P}}$ its enantiomorphic form. They have groups W^+/M and $W^+/\overline{M}$, respectively. If $\mathcal{Q}$ is a chiral or directly regular polytope that covers both of them, and it has rotation group W^+/K, then by the above, $K \leq M$ and $K \leq \overline{M}$. So $K \leq M \cap \overline{M}$. Furthermore, since M and $\overline{M}$ are normal in W^+, then $M \cap \overline{M}$ is too, so there actually is a rotation subgroup $W^+/(M \cap \overline{M})$. Finally, this is the rotation subgroup of a directly regular polytope, because the enantiomorphic form has group $W^+/(\overline{M} \cap M)$.

Thus, if $\mathcal{P}$ is a chiral polytope with group W^+/M, then the smallest regular polytope $\mathcal{Q}$ that covers it also covers $\overline{\mathcal{P}}$, and $\mathcal{Q}$ has group $W^+/(M \cap \overline{M})$. The kernel of the covering map is $M/(M \cap \overline{M}) \simeq M\overline{M}/M$.

Definition. *Let $\mathcal{P}$ be a chiral or directly regular polytope. Then $X(\mathcal{P})$ is the kernel of the covering map from the minimal regular cover of $\mathcal{P}$ to $\mathcal{P}$. It is also the kernel of the covering map from $\Gamma^+(\mathcal{P})$ to the group of the maximal regular polytope that is covered by $\mathcal{P}$.*

Example: $\mathcal{P} = \{4, 4\}_{(1,2)}$. Then $\overline{\mathcal{P}} = \{4, 4\}_{(2,1)}$. The smallest regular cover of $\mathcal{P}$ is $\{4, 4\}_{(5,0)}$, and the chirality group $X(\mathcal{P})$ is isomorphic to a cyclic group of order 5. (It is the translation subgroup of $\mathcal{P}$, generated by $\sigma_1^{-1}\sigma_2$.)

When $X(\mathcal{P})$ is the trivial group, then $\mathcal{P}$ is directly regular. Otherwise, if $X(\mathcal{P})$ is a nontrivial normal subgroup of $\Gamma^+(\mathcal{P})$, then $\mathcal{P}$ is chiral.

For the last part of the talk, we will define a way of constructing new polytopes from old ones, and our goal is find conditions under which the new polytope is chiral.

Definition. *Let $\mathcal{P}$ be a chiral or directly regular polytope with rotation group $\Gamma^+(\mathcal{P}) = W^+/M$, and let $\mathcal{Q}$ be a chiral or directly regular polytope with rotation group $\Gamma^+(\mathcal{Q}) = W^+/K$. Then we define the mix of $\mathcal{P}$ and $\mathcal{Q}$, denoted $\mathcal{P} \diamond \mathcal{Q}$, to be the poset with group $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q}) := W^+/(M \cap K)$.*

In general, the mix of two polytopes is not necessarily strongly connected, so it is not necessarily a polytope. But if, for instance, $\mathcal{P}$ and $\mathcal{Q}$ have isomorphic facets or vertex figures, then their mix is a polytope. Note that $\mathcal{P} \diamond \mathcal{Q}$ covers both $\mathcal{P}$ and $\mathcal{Q}$ by our previous discussion. In fact, this is one nice characterization of $\mathcal{P} \diamond \mathcal{Q}$: it is the smallest polytope that covers $\mathcal{P}$ and $\mathcal{Q}$.

The previous definition is not very helpful when the groups $\Gamma^+(\mathcal{P})$ and $\Gamma^+(\mathcal{Q})$ are given in terms of their generators σ_i. In that case, the following proposition is helpful.
Proposition. *Let* $\Gamma^+(\mathcal{P})$ *be generated by* $\sigma_1, \ldots, \sigma_{n-1}$*, and let* $\Gamma^+(\mathcal{Q})$ *be generated by* $\sigma'_1, \ldots, \sigma'_{n-1}$*. Then* $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$ *is the subgroup of* $\Gamma^+(\mathcal{P}) \times \Gamma^+(\mathcal{Q})$ *generated by* (σ_i, σ'_i)*,* $1 \leq i \leq n-1$*.*

Even if $\mathcal{P}$ and $\mathcal{Q}$ are chiral, it is not necessarily the case that $\mathcal{P} \diamond \mathcal{Q}$ is chiral. In fact, the smallest regular cover of $\mathcal{P}$ is just $\mathcal{P} \diamond \overline{\mathcal{P}}$. But in many cases, even if $\mathcal{Q}$ is directly regular, the mix of $\mathcal{P}$ and $\mathcal{Q}$ is chiral.

Theorem. *Let* $\mathcal{P}$ *and* $\mathcal{Q}$ *be finite chiral or directly regular polytopes. If either* $|X(\mathcal{P})| \nmid |\Gamma^+(\mathcal{Q})|$ *or* $|X(\mathcal{Q})| \nmid |\Gamma^+(\mathcal{P})|$*, then* $\mathcal{P} \diamond \mathcal{Q}$ *is chiral.*

Proof. Suppose $\mathcal{P} \diamond \mathcal{Q}$ is directly regular. Then since $\mathcal{P} \diamond \mathcal{Q}$ covers $\mathcal{P}$, it also covers its minimal regular cover $\mathcal{P} \diamond \overline{\mathcal{P}}$. The size of $\Gamma^+(\mathcal{P}) \diamond \overline{\Gamma^+(\mathcal{P})}$ is $|\Gamma^+(\mathcal{P})||X(\mathcal{P})|$, so $|\Gamma^+(\mathcal{P})||X(\mathcal{P})|$ divides the order of $\Gamma^+(\mathcal{P}) \diamond \Gamma^+(\mathcal{Q})$, which divides the order of $\Gamma^+(\mathcal{P}) \times \Gamma^+(\mathcal{Q})$ i.e., $|\Gamma^+(\mathcal{P})||\Gamma^+(\mathcal{Q})|$. Thus $|X(\mathcal{P})|$ divides $|\Gamma^+(\mathcal{Q})|$. By symmetry, $|X(\mathcal{Q})|$ divides $|\Gamma^+(\mathcal{P})|$. $\square$