GEOMETRIC STRUCTURES

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Main Ideas

The idea here is that for a fixed coordinate system a geometric structure is some kind of vector quantity (eg velocity) or scalar quantity (eg temperature) but that most of the time these "quantities" change depending on the choice of coordinates.

Principle of Short Range. The principle of short range states that when changing between two coordinate systems, the change in quantities at a point depends not on the the whole transform but only on the change infinitesimally close to the point in question.

Now, if we tentatively define physical quantities as to be functions from the space of k-frames (defined below) to a variety of ranges it follows that physical quantities have the same value at a point p in coordinates x and y iff $x\&y$ are *tangent of order* k at p. Two coordinate systems are said to be tangent of order k iff $x \circ y^{-1}$ has the same k-jet (all derivatives less than or equal to k are the same) at $y(p)$ as the identity map id: $x^i \to x^i$.

It is then important that we pay attention to the k-infinitesimal structure of coordinates.

Preliminaries and Definitions

0.1. **Basic Definitions.** Let M, N be smooth manifolds and let $f : M \to N$ be a local map if the $\mathrm{Dom}(f) \subseteq M$, $\mathrm{Im}(f) \subseteq N$ are both open. Let $f : U \to N$, $g : V \to N$ be local maps on M and let $p \in U \cap V \neq \emptyset$. Then f and g are tangent of order k (have the same k-jet) if the first k derivatives at p are equal or, alternatively, if the Taylor k-polynomials of f and g (the Taylor polynomials excluding powers higher than k) at p are equal. This is clearly an equivalence relation on on local maps (and germs of local maps) so we mod out by this relation and dente the equivalence class of f by $j^k_p(f)$ or $j^k(f,p)$. Note, that this independent of coordinates and form a smooth bundle over the manifold.

Let $J^k(M,N) = \{j^k_p(f) | p \in M\}$ then by projecting onto the source p and target $f(p)$ of $j^k_p(f)$ we have projections onto M and N respectively.

Now, if $j^k_p(f) = j^k_p(g)$ then it is also true that $j^l_p(f) = j^l_p(g)$ for all $l \leq k$ so $J^k(M,N) \to J^l(M,N)$ for all $l \leq k$.

For some concrete examples, notice that $J^0 * (M,N) = M \times N$ since there is one equivalence class for each map sending $p \mapsto q$ for each $p \in M, q \in N$. Similarly, $J^1(M,N) = \{(p,q,L) | p \in M, q \in N, L : T_pM \to T_qN\}$ since each equivalence class is defined not just by the equality of the source and target but also the equality of the the first derivative eq induced transformation on the tangent spaces.

Now, lets define a smooth structure on J^K. Let (U, x) and (V, y) be atlas's of M, N respectively. Let $T_0^k(m, n) = \{$all polynomials maps from $\mathbb{R}^m \to \mathbb{R}^n$ of order $k\}$. Then since we have coordinates on the derivatives of f, $x(U) \times y(V) \times T_0^k(m, n)$ are coordinate on $J^k(M, N)$ and we state without proof that

$$U \times V \times J^K(U, V) \to x(U) \times y(V) \times T_0^k(m, n)$$

by

$$(p, q, j_p^k(f)) \mapsto (u, \varphi(u), T_m^k(\varphi(u)))$$

are coordinate charts on $J^k(M, N)$ where $u = x(p)$, $\varphi(u) = y \circ f \circ x^{-1}(u)$, and $T_m^k(\varphi(u))$ (also denoted $T_\varphi^k(u)$) is the k'th taylor polynomial of φ at u. It is simple to check that the coordinate changes between charts are smooth. Then $T^k(M, N)$ has a smooth structure and is a smooth M or N bundle under the projections discussed above.

Fibers of $J^k(M, N)$. Let $J_p^k(M, N)$ be the fiber of $J^k(M, N)$ at p. Note that this is a smooth sub-manifold of $J^k(M, N)$. As an example it is simple to see that $J_0^1(\mathbb{R}, M) = TM$. Now, $T_0^k(m, n) \cong \bigoplus_{1 \le i \le k} S^i(m, n)$ where $S^i(m, n)$ are the symmetric i-linear maps taking $\mathbb{R}^m \to \mathbb{R}^n$. Then

$$J^k(U, V) \to x(U) \times y(V) \times \bigoplus_{1 \le i \le k} S^i(m, n)$$

by

$$j^k(f, p) \mapsto (u, \varphi(u), \varphi'(u), \varphi''(u), \ldots, \varphi^{(k)}(u))$$

Sections of $J^k(M, N)$. A local section of $\pi : Z = J^k(M, N) \to M$ is a map from $U \subseteq M$ to $J^K(U, V)$ with $(u, v(u), \varphi(u), \ldots, \varphi^{(k)}(u)\}$ determined entirely by u. A local section s of $\pi : Z = J^k(M, N) \to M$ is called a holonomic section if $s = j^k(f)$ for some local map $f : U \to V$. Similalrly, $L : T_pM \to T_zZ$ is called holonomic if $L = s'(p)$ for some holonomic $s(p) = j_p^k$.

Lemma. If s is a local section of π and $s'(p)$ is holonomic for all p in $\mathrm{Dom}(s)$ then s is holonomic.

Geometric Structures

Frames. Let $Gl^k(m) = \{j_0^k(f) | f : \mathbb{R}^m \to \mathbb{R}^m$ local diffeo, $f(0) = 0\}$ be the degree k local diffeomorphisms of $\mathbb{R}^m$ that fix the origin. This is a Lie Group with respect to the composition: $j^k(\varphi) \cdot j^k(\psi) = j^k(\varphi \circ \psi)$. Then $j_0^k(p) \in T_0^k(m)$ since $\varphi(0) = 0$ and $Gl^k(m) = \{a_1, \ldots, a_k) : a_i \in \mathrm{Sym}^i(m, m)$. As an example, in $Gl^2(m)$ this is given by

$$(a_1, a_2)(b_1, b_2) = (a_1 b_1, a_1 b_2 + a_2(b_1, b_2))$$

For $p \in M$ a k-frame at p is the k-jet $j_0^k(f)$ of a local diffeomorphism $f : \mathbb{R}^m \to M$ such that $f(0) = p$. Then $GL_p^k(M)$ is the set of k-frames into (around?) p and $Gl^k(M) = \bigcup_{p \in M} Gl_p^k(M)$ is a principle bundle with structure group $Gl^k(M)$ acting on the right by $j(\varphi) \cdot j^k(f) = j^k(\varphi \circ f)$.

Geometric Structure. Finally, we define geometric structure. Intuitively a geometric structure is function on frames that behaves "nicely" with respect to coordinate changes. Formally, let Σ be a smooth manifold with an action of $Gl^k(m)$ on it (eg there exists $\lambda : Gl^k(m) \times \Sigma \to \Sigma$). Then a *Geometric Structure* of type λ is an equavarient map $\sigma : Gl^k(M) \to \Sigma$, ie $\sigma(x \cdot g) = g^{-1} \cdot \sigma(x)$.

Examples of Geometric Structures. Parallelization is a geometric structure: let $\Sigma = Gl^2(m)$. Then σ is a parallelization since if $\sigma(\Phi(x)) = \mathrm{id} \in \Sigma$ then we have a unique frame at each point of M and so we have a parallelization.

Similarly, connections are structures of order $k = 2$.