YANGIANS AND QUANTUM LOOP ALGEBRAS

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Contents

1. Definitions

Let $\mathfrak{sl}_2$ be the Lie algebra of 2×2 traceless matrices with complex entries. $\mathfrak{sl}_2$ has the following basis:

$$f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

The following commutation relations can be easily verified:

$$[h, e] = 2e \quad [h, f] = -2f \quad [e, f] = h$$

Definition 1.1. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$. $\mathfrak{g}[z, z^{-1}] := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$ is called the loop algebra, with bracket:

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg$$

The same formula defines a structure of Lie algebra on $\mathfrak{g}[u] := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[u]$ which is called the current algebra. The current algebra has natural $\mathbb{N}$ grading by defining $deg(u) = 1$. We have the following Lie algebra homomorphism:

$$\exp : \mathfrak{g}[z, z^{-1}] \to \mathfrak{g}[[u]]$$

which maps $X.z^r$ to $X.e^{ru}$ for every $r \in \mathbb{Z}$.

Proposition 1.2. *Let I be the ideal in $U(\mathfrak{g}[z, z^{-1}])$ defined as the kernel of the following algebra homomorphism:*

$$U(\mathfrak{g}[z, z^{-1}]) \xrightarrow{z=1} U\mathfrak{g}$$

Then $\exp : U(\mathfrak{g}[z, z^{-1}]) \to U(\mathfrak{g}[[u]])$ *respects the following filtrations on the algebras:*

a) *Filtration on $U(\mathfrak{g}[z, z^{-1}])$ defined by the ideal I.*
b) *Filtration on $U(\mathfrak{g}[[u]])$ defined by the $\mathbb{N}$ grading on $\mathfrak{g}[u]$.*

Moreover the extended exponential map on the level of completed algebras is an isomorphism.

The following are the main objects of study in this talk:

Definition 1.3. The quantum loop algebra of $\mathfrak{sl}_2$, denoted by $U_qL\mathfrak{sl}_2$ (or just U), is a unital associative algebra over $\mathbb{C}(q)$ generated by $K^{\pm 1}$, $H_r (r \in \mathbb{Z} \setminus \{0\})$ and E_k, F_k for $k \in \mathbb{Z}$, subject to the following relations:

(QL1) The generators $K^{\pm 1}, H_r (r \in \mathbb{Z}\setminus\{0\})$ commute. Let U^0 be the commutative algebra generated by these generators. Define $\psi_k, \phi_{-k} \in U^0$ for each $k \in \mathbb{N}$, another system of generators, defined by the following equations:

$$\psi(z) = \sum_{k\geq 0} \psi_k z^{-k} = K \exp\left((q - q^{-1}) \sum_{r\geq 1} H_r z^{-r} \right)$$

$$\phi(z) = \sum_{k\geq 0} \phi_{-k} z^{k} = K^{-1} \exp\left(-(q - q^{-1}) \sum_{r\geq 1} H_{-r} z^{r} \right)$$

(QL2) For each $k, l \in \mathbb{Z}$ we have:

$$[E_k, F_l] = \frac{\psi_{k+l} - \phi_{k+l}}{q - q^{-1}}$$

where we employ the convention that $\psi_{-k} = \phi_k = 0$ for each $k > 0$.

(QL3) For each $r \in \mathbb{Z} \setminus \{0\}$ and $k \in \mathbb{Z}$ we have:

$$KE_kK^{-1} = q^2 E_k \quad KF_kK^{-1} = q^{-2} F_k$$

$$[H_r, E_k] = \frac{[2r]}{r} E_{r+k} \quad [H_r, F_k] = -\frac{[2r]}{r} F_{k+r}$$

(QL4) For each $k, l \in Z$ we have:

$$E_{k+1}E_l - q^2 E_l E_{k+1} = q^2 E_k E_{l+1} - E_{l+1} E_k$$
$$F_{k+1}F_l - q^{-2} F_l F_{k+1} = q^{-2} F_k F_{l+1} - F_{l+1} F_k$$

Definition 1.4. The Yangian of $\mathfrak{sl}_2$, denoted by $Y_\hbar\mathfrak{sl}_2$ (or just Y) is a unital associative algebra over $\mathbb{C}[\hbar]$ generated by h_r, e_r, f_r $(r \in \mathbb{N})$, subject to the following relations:

(Y1) The generators $h_r : r \in \mathbb{N}$ commute. Let Y^0 be the commutative subalgebra generated by these generators.

(Y2) For each $r \in \mathbb{N}$ we have:

$$[h_0, e_r] = 2e_r \quad [h_0, f_r] = -2f_r$$

(Y3) For each $r, s \in \mathbb{N}$ we have:

$$[e_r, f_s] = h_{r+s}$$

(Y4) For each $r, s \in \mathbb{N}$ we have:

$$[h_{r+1}, e_s] - [h_r, e_{s+1}] = \hbar(h_r e_s + e_s h_r)$$

$$[h_{r+1}, f_s] - [h_r, f_{s+1}] = -\hbar(h_r f_s + f_s h_r)$$

(Y5) For each $r, s \in \mathbb{N}$ we have:

$$[e_{r+1}, e_s] - [e_r, e_{s+1}] = \hbar(e_r e_s + e_s e_r)$$

$$[f_{r+1}, f_s] - [f_r, f_{s+1}] = \hbar(f_r f_s + f_s f_r)$$

The Yangian has natural $\mathbb{N}$ grading by: $deg(\hbar) = 1$ and $deg(x_r) = r$ for each $x \in \{e, f, h\}$.

Proposition 1.5. *At $q = 1$ $U_qL\mathfrak{sl}_2$ is isomorphic to $U(\mathfrak{sl}_2[z, z^{-1}])$. Similarly at $\hbar = 0$ the Yangian $Y_\hbar\mathfrak{sl}_2$ is isomorphic to $U(\mathfrak{sl}_2[u])$*

Proof

Let us write the defining relations of $U(\mathfrak{sl}_2[z, z^{-1}])$ and $U(\mathfrak{sl}_2[u])$. These algebras are generated by $x.z^k$ and $x.u^r$ respectively, where $x \in \{e, f, h\}$ and $k \in \mathbb{Z}$, $r \in \mathbb{N}$. We claim that at the classical limit $X_k \to x.z^k$ (for the case of quantum loop algebra) and $x_r \to x.u^r$ (for the case of the Yangian). This claim can be easily verified using the definition of these algebras, which proves the required assertion.

CQFD

Definition 1.6. Let $\varepsilon \in \mathbb{C}^\times$. The specialization of U_q at $q = \varepsilon$ is denoted by U_ε. Similarly $Y\mathfrak{sl}_2$ denotes the specialization of $Y_\hbar\mathfrak{sl}_2$ at $\hbar = 1$.

When we work with $U_qL\mathfrak{sl}_2$ and $Y_\hbar\mathfrak{sl}_2$, both q and $\hbar$ are considered as formal variables related by $q^2 = e^\hbar$.

2. Representations of quantum loop algebra and Yangian

Definition 2.1. Let V be a finite dimensional representation of $U_\epsilon L\mathfrak{sl}_2$. Let $\Lambda = (l_k : k \in \mathbb{Z})$ be a collection of complex numbers. A non zero vector $v \in V$ is said to be *highest weight vector* of highest weight Λ if:

(1) $E_rv = 0$ for every $r \in \mathbb{Z}$.
(2) $\psi_kv = l_kv$ and $\phi_{-k}v = l_{-k}v$ for every $k \geq 1$ and $i \in I$. Also $\psi_0v = l_0v$ and $\phi_0v = l_0^{-1}v$.

We say that V is a highest weight representation if there exists a highest weight vector $v \in V$ which generates V as a $U_\epsilon L\mathfrak{sl}_2$ module.

Similar to the construction of Verma modules for semisimple Lie algebras, one can construct a universal highest weight module $M(\Lambda)$ for any collection of complex numbers Λ. Let $V(\Lambda)$ denote its unique irreducible quotient. The following theorem classifies all finite dimensional irreducible representations of $U_\epsilon L\mathfrak{sl}_2$:

Theorem 2.2. *Every finite dimensional irreducible representation of $U_\epsilon L\mathfrak{sl}_2$ is a highest weight representation. Moreover given $\Lambda = (l_k : k \in \mathbb{Z})$, the irreducible representation $V(\Lambda)$ is finite dimensional if and only if there exists a polynomial (called* Drinfeld polynomial*) $(P(u) \in \mathbb{C}[u])_{i\in I}$, normalized so as to have $P(0) = 1$ such that:*

$$\sum_{r\geq 0} l_r z^{-r} = \epsilon^{deg(P)} \frac{P(\epsilon^{-1}/z)}{P(\epsilon/z)} = l_0^{-1} + \sum_{r\geq 1} l_{-r} z^r$$

This polynomial determine the finite dimensional irreducible representation uniquely.

Definition 2.3. Let V be a finite dimensional module over $Y\mathfrak{sl}_2$. Given a collection $\mathbf{h} = \{\xi_r \in \mathbb{C} : r \in \mathbb{N}\}$ of complex numbers and a non zero vector $v \in V$, we say v is highest weight vector of highest weight $\mathbf{h}$ if:

(1) $e_rv = 0$ for every $r \in \mathbb{N}$.
(2) $\xi_rv = h_rv$ for every $r \in \mathbb{N}$.

Again let $M(\mathbf{h})$ be the Verma module over $Y\mathfrak{sl}_2$ corresponding to the weight $\mathbf{h}$, and let $V(\mathbf{h})$ be its irreducible quotient. The finite dimensional irreducible representation of $Y\mathfrak{sl}_2$ are classified in the following theorem.

Theorem 2.4. *Every finite dimensional irreducible representation of $Y\mathfrak{sl}_2$ is a highest weight representation. Moreover $V(\mathbf{h})$ is finite dimensional if and only if there exists a monic polynomial (again called* Drinfeld polynomial*) $P(u) \in \mathbb{C}[u]$ such that*

$$1 + \sum_{r \geq 0} \xi_r u^{-r-1} = \frac{P(u+1)}{P(u)}$$

This polynomial uniquely determine the representation $V(\mathbf{h})$.

In order to summarize the results stated above we have the following:

$$\begin{array}{ccc} Irr(Y) & \longrightarrow & Irr(U) \\ \Vert & & \Vert \\ \bigsqcup_{N\geq 0} \mathbb{C}^N/\mathfrak{S}_N & \longrightarrow & \bigsqcup_{N\geq 0} (\mathbb{C}^\times)^N/\mathfrak{S}_N \end{array} \tag{2.1}$$

3. Statement of problem

In this section I will state the main problem(s) which form the foundation of the bridge between Yangians and quantum loop algebras.

Formal setting: Does there exist an algebra homomorphism $\Phi : U_qL\mathfrak{sl}_2 \to Y_\hbar\mathfrak{sl}_2$ (completed with respect to $\mathbb{N}$ grading), such that $\Phi|_{\hbar=0} = \exp$ (see Proposition 1.5)?

Numerical setting: Does there exist a functor $F : Rep(Y) \to Rep(U)$ which enlarges the diagram (2.1) in the following sense?

$$\begin{array}{ccc} Rep(Y) & \dashrightarrow & Rep(U) \\ \uparrow & & \uparrow \\ Irr(Y) & \longrightarrow & Irr(U) \\ \Vert & & \Vert \\ \bigsqcup_{N\geq 0} \mathbb{C}^N/\mathfrak{S}_N & \longrightarrow & \bigsqcup_{N\geq 0} (\mathbb{C}^\times)^N/\mathfrak{S}_N \end{array} \tag{3.1}$$

Some remarks are in order before we go to the solution of these problems.

Remark 3.1. (1) The solution to the counterpart of the formal problem in finite dimensional setting is constructed using cohomological methods. More explicitly, using the machinery of Hochschild cohomology, one can show that there exists an algebra isomorphism between the quantum group $U_\hbar\mathfrak{g}$ and the enveloping algebra $U\mathfrak{g}[[\hbar]]$ for semisimple Lie algerba $\mathfrak{g}$. However this isomorphism is very hard to write down.

(2) There is yet another (and better) analogue of the formal problem, which arises in the study of affine and degenerate affine Hecke algebras. In this analogous setting an algebra homomorphism was constructed by G. Lusztig in early 1980's.

(3) The formal problem also has a geometric viewpoint (which also exists for the Hecke algebras setting). Namely both the quantum loop algebra and the Yangian admit geometric realizations (as defined by V. Ginzburg) using the quiver varieties. This result was proved by H. Nakajima (for the quantum loop algebra case) and M. Varagnolo (for the Yangian case). We will not have enough time to go into the details of this beautiful area of mathematics during this talk, however (commercial break!) V. Toledano Laredo will be supervising a reading course on geometric representation theory in Fall 2010.

(4) An affirmative answer to the numerical problem is believed to exist by several mathematicians. As a matter of fact, the existence of F is a "folklore theorem". However no explicit functor F exists in literature.

4. Solution for the formal case

In this section we state (and hopefully prove) an affirmative answer to the formal question. More precisely we aim at constructing an algebra homomorphism Φ : $U_qL\mathfrak{sl}_2 \to \widehat{Y_\hbar\mathfrak{sl}_2}$ which satisfies the following two natural constraints:

- $\Phi|_{\hbar=0} = \exp$ as given in Proposition 1.5.
- $\Phi|_{U^0} : U^0 \to \widehat{Y^0}$ respects the Drinfeld polynomials (see below).

Let us begin by making precise the second constraint. Define the Drinfeld homomorphisms:

Definition 4.1. Let $N \in \mathbb{N}$ be a fixed positive integer. Define:

$$S(N) := \mathbb{C}[q^{\pm 1}, T_1^{\pm 1}, \cdots, T_N^{\pm 1}]^{\mathfrak{S}_N}$$

$$R(N) := \mathbb{C}[\hbar, t_1, \cdots, t_N]^{\mathfrak{S}_N}$$

$\mathcal{D}_N^U : U^0 \to S(N)$ is define by the following formal equation:

$$\psi(z) = \prod_{j=1}^{N} \frac{qz - q^{-1}T_j}{z - T_j} = \phi(z)$$

Similarly $\mathcal{D}_N^Y : Y^0 \to R(N)$ is defined by the following formal equation:

$$h(u) = 1 + \hbar \sum_{r \geq 0} h_r u^{-r-1} = \prod_{j=1}^{N} \frac{u - t_j + \hbar}{u - t_j}$$

Now the second constraint can be rephrased as requiring the commutativity of the following diagram for each $N \in \mathbb{N}$.

$$\begin{array}{ccc} U^0 & \xrightarrow{\mathcal{D}_N^U} & S(N) \\ \Big\downarrow & & \Big\downarrow \text{exp} \\ Y^0 & \xrightarrow[\mathcal{D}_N^Y]{} & R(N) \end{array} \tag{4.1}$$

Theorem 4.2. *Define $d_r \in Y^0$ for each $r \in \mathbb{N}$ by comparing the coefficients of the following:*

$$d(u) := \hbar \sum_{r \geq 0} d_r u^{-r-1} = \log\left(h(u)\right)$$

The $\Phi^0 : U^0 \to \widehat{Y^0}$ defined by the following formulae makes the diagram (4.1) commute:

$$K \longmapsto e^{\hbar d_0/2}$$

$$H_r \longmapsto \frac{\hbar}{q-q^{-1}} \sum_{k \geq 0} d_k \frac{r^k}{k!}$$

Geometric Ansatz: In order to extend Φ^0 as defined in Theorem 4.2 to an algebra homomorphism Φ, we impose an additional requirement, to be referred to as *geometric ansatz*. We require that there exist $g^{\pm}(u) \in Y^0[[u]]$ such that $g^{\pm}(0) = 1$ and

$$\Phi(E_0) = \sum_{m \geq 0} g_m^+ \frac{e_m}{m!}$$

$$\Phi(F_0) = \sum_{m \geq 0} g_m^- \frac{f_m}{m!}$$

The origin of this ansatz lies in geometric representation theory (see Remark 3.1).

We are now in position to state the main theorem:

Theorem 4.3. *There exists an algebra homomorphism of geometric type $\Phi : U_q L\mathfrak{sl}_2 \to \widehat{Y_\hbar \mathfrak{sl}_2}$. Moreover we have:*

(1) $\Phi|_{\hbar=0} = \exp$.

(2) *If Φ' is another algebra homomorphism of geometric type, then there exists $\xi \in \left(\widehat{Y^0}\right)^\times$ such that*

$$\Phi'(X) = \xi \Phi(X) \xi^{-1}$$

(3) *Let $J \subset U_q L\mathfrak{sl}_2$ be the kernel of the following algebra homomorphism:*

$$U_q L\mathfrak{sl}_2 \xrightarrow{q=1} U(\mathfrak{sl}_2[z, z^{-1}]) \xrightarrow{z=1} U\mathfrak{sl}_2$$

Then Φ respects the filtration on $U_q L\mathfrak{sl}_2$ defined by J (and the $\mathbb{N}$-filtration on $\widehat{Y_\hbar \mathfrak{sl}_2}$). Moreover Φ extends to an isomorphism of completed algebras (see Proposition 1.2 for the classical counterpart).