Pairs of Commuting Nilpotent Matrices

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Joint with R. Basili and A. Iarrobino

- k alg. closed field, V a k-VS of dim n

- For any nilpotent $T \in End_K(V)$, there is a basis for V in which T can be written as

$$\begin{vmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_r \end{vmatrix} \quad \text{with } J_i = \begin{vmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{vmatrix}_{p_i \times p_i}$$

Jordan Canonical form of T

$p_1 \geq \ldots \geq p_r$ with $p_1 + \cdots + p_r = n$ $\quad [p_1,\ldots,p_r]$ a partition of n

Ex. n=4, J is the 4×4 Jordan block

$$J=\begin{vmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{vmatrix} \quad J^2=\begin{vmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{vmatrix} \quad J^3=\begin{vmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{vmatrix} \quad J^4=0$$

[4] [2,2] [2,1,1] [1,1,1,1]

$$\begin{vmatrix}0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{vmatrix} \quad \begin{vmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{vmatrix}$$

$$I_n = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ & 1 & 0 & \ddots & \vdots \\ & & 1 & \ddots & 0 \\ & 0 & & \ddots & 0 \\ & & & & 1 \end{vmatrix}_{n \times n} \qquad J_n = \begin{vmatrix} 0 & 1 & 0 & \cdots & 0 \\ & 0 & 1 & \ddots & \vdots \\ & & 0 & \ddots & 0 \\ & 0 & & \ddots & 1 \\ & & & & 0 \end{vmatrix}_{n \times n}$$

$$J^2 = \begin{vmatrix} 0 & 0 & 1 & \cdots & 0 \\ & 0 & 0 & \ddots & \vdots \\ & & 0 & \ddots & 1 \\ & 0 & & \ddots & 0 \\ & & & & 0 \end{vmatrix} \quad \cdots \quad J^{n-1} = \begin{vmatrix} 0 & 0 & 0 & \cdots & 1 \\ & 0 & 0 & \ddots & \vdots \\ & & 0 & \ddots & 0 \\ & 0 & & \ddots & 0 \\ & & & & 0 \end{vmatrix} \qquad J^n = 0$$

$$A = \begin{vmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_r \end{vmatrix}$$

J_i is $p_i \times p_i$, $p_1 \geq \ldots \geq p_r$

$$A^{p_1 - 1} \neq 0$$

$$A^{p_1} = 0$$

Defn. For a nilp $n \times n$ matrix B with Jordan partition $P=[p_1, \ldots, p_r]$

$$\mathcal{C}_B = \{A \in Mat_n(k) \mid AB=BA\}$$

$$\mathcal{N}_B = \{A \in Mat_n(k) \mid AB=BA,\ A^n = 0\}$$

Q. Jordan partition of a general A in $\mathcal{N}_B$?

- [Basili '03] $\mathcal{N}_B$ is an irreducible algebraic variety.

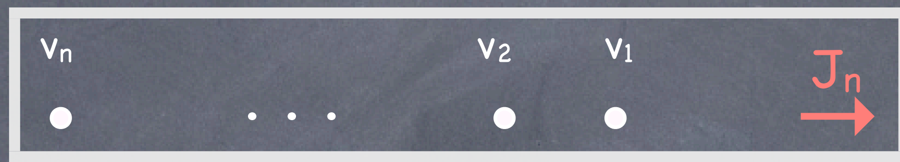
- $Q(P)$: the Jordan partition of a generic element of $\mathcal{N}_B$. Goal: Understand the map $P \rightarrow Q(P)$

- $Q(P)$ the biggest partition in $\mathcal{N}_B$ (Order using orbit closure inclusion)

If $P=[n]$. Take $\{v_1, \cdots, v_n\}$ basis in which B is a Jordan block

$$B=\begin{vmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & 0 \end{vmatrix}$$

$Bv_1=0,\ Bv_2=v_1,\ \ldots,\ Bv_n=v_{n-1}$



If $AB=BA$ then $0=Bv_1=ABv_1=BAv_1 \Rightarrow Av_1=a_0v_1$.

$ABv_2=Av_1=a_0v_1 \Rightarrow Av_2=a_0v_2+a_1v_1$. ...

$$A=\begin{vmatrix} a_0 & a_1 & \ldots & a_{n-1} \\ & a_0 & \ddots & \vdots \\ & & \ddots & a_1 \\ & & & a_0 \end{vmatrix} \qquad a_i \in k$$

$A \in \mathcal{C}_J \Leftrightarrow A=a_0I+a_1J+\cdots+a_{n-1}J^{n-1}$

$A \in \mathcal{N}_J \Leftrightarrow A=a_1J+\cdots+a_{n-1}J^{n-1}$

$k[x]/(x^n)$

Prop. [B] If B has Jordan partition $P=[p_1,\cdots,p_r]$, i.e. $B=\begin{pmatrix} J_{p1} & & \\ & \ddots & \\ & & J_{pr} \end{pmatrix}$

$A \in \mathcal{C}_B$ iff $A=\begin{pmatrix} A_{11} & \cdots & A_{1r} \\ \vdots & \ddots & \vdots \\ A_{r1} & \cdots & A_{rr} \end{pmatrix}$ s.t. A_{ij} is $p_i \times p_j$ in one of the following form

i=j

$$\begin{pmatrix} a^{ii}_0 & a^{ii}_1 & \cdots & a^{ii}_{pi-1} \\ & a^{ii}_0 & \ddots & \vdots \\ & & \ddots & a^{ii}_1 \\ & & & a^{ii}_0 \end{pmatrix}$$

i<j ($p_i \geq p_j$)

$$\begin{pmatrix} a^{ij}_0 & a^{ij}_1 & \cdots & a^{ij}_{pj-1} \\ & a^{ij}_0 & \ddots & \vdots \\ & & \ddots & a^{ij}_1 \\ & & & a^{ij}_0 \\ \hline 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \end{pmatrix}$$

i>j ($p_i \leq p_j$)

$$\left(\begin{array}{c|cccc} 0 \dots 0 & a^{ij}_0 & a^{ij}_1 & \cdots & a^{ij}_{pi-1} \\ \vdots & & a^{ij}_0 & \ddots & \vdots \\ \vdots & & & \ddots & a^{ij}_1 \\ 0 \dots 0 & & & & a^{ij}_0 \end{array}\right)$$

$p \le q$

$$\alpha_{qp} := \left|\begin{array}{c} I_p \\ \hline 0 \end{array}\right|_{q\times p} \in \mathrm{Hom}_k(k^p, k^q)$$

$$\beta_{pq} := \left|\begin{array}{c|c} 0 & I_p \end{array}\right|_{p\times q} \in \mathrm{Hom}_k(k^q, k^p)$$

$$\left|\begin{array}{cccc} a_0 & a_1 & \cdots & a_{p-1} \\ & a_0 & \ddots & \vdots \\ & & \ddots & a_1 \\ & & & a_0 \\ & & & \\ & & & \end{array}\right|_{q\times p}$$

$$\left|\begin{array}{cccccc} & & a_0 & a_1 & \cdots & a_{p-1} \\ & & & a_0 & \ddots & \vdots \\ & & & & \ddots & a_1 \\ & & & & & a_0 \end{array}\right|_{p\times q}$$

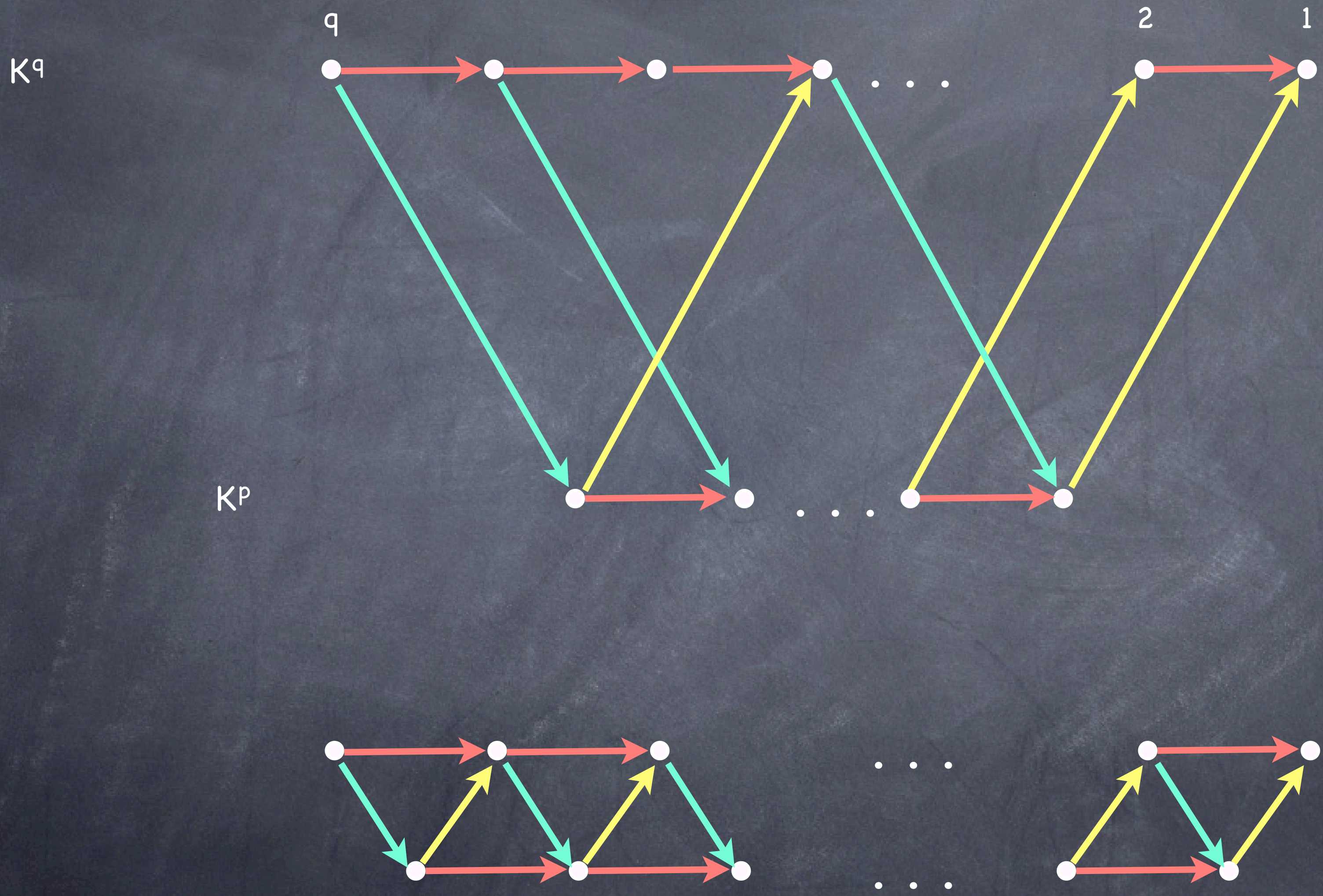
$$\alpha_{qp}\,(a_0I_p + a_1J_p + \cdots + a_{p-1}J_p{}^{p-1}) = (a_0I_q + a_1J_q + \cdots + a_{p-1}J_q{}^{p-1})\ \alpha_{qp}$$

$$\beta_{pq}\,(a_0I_q + a_1J_q + \cdots + a_{p-1}J_q{}^{p-1}) = (a_0I_p + a_1J_p + \cdots + a_{p-1}J_p{}^{p-1})\ \beta_{pq}$$

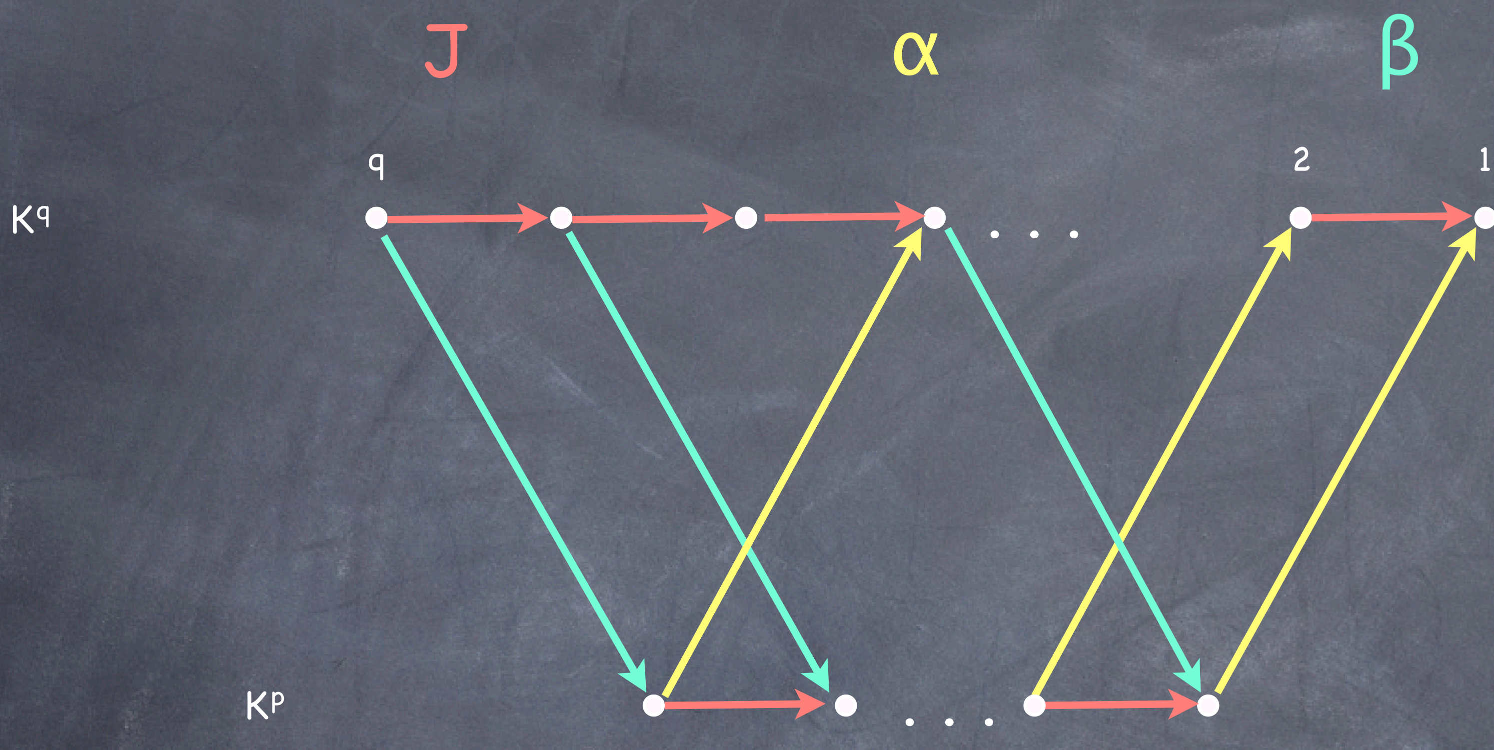
If $p=q$ then $\alpha_{qp} = \beta_{pq} = I$

q>p

J α β



$\alpha\beta = J$ $\beta\alpha = J$

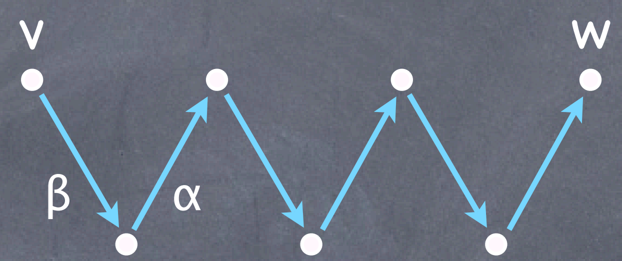


$$\alpha\beta = J_q^{q-p} \qquad \beta\alpha = J_p^{q-p}$$

$$J\beta = \beta J \qquad J\alpha = \alpha J$$

- $P=[4,3]$ $V=V_4 \oplus V_3$

$A \in \mathcal{N}_B$ generic



$A^6 v = sw$, s a scalar

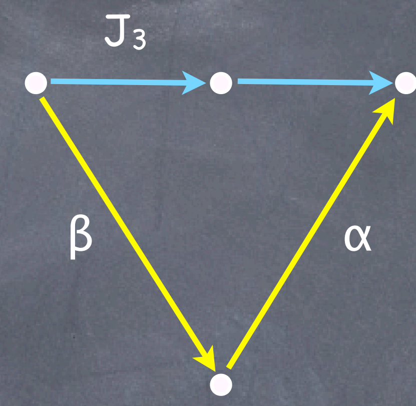
$A^7 = 0$

$Q(P) = [7]$

highest non-vanishing power of A ↔ **longest path**

size of the **biggest part of Q(P)**

- P=[3,1] $V = V_3 \oplus V_1$



$A \in \mathcal{N}_B$ generic

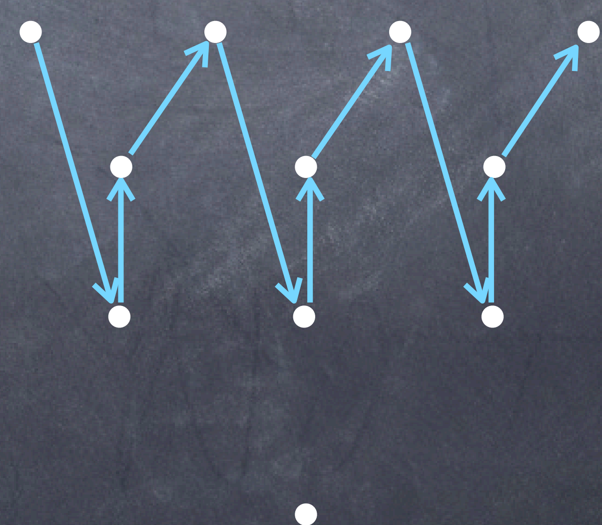
$A^2 \neq 0$

$A^3 = 0$

Q(P)= [3,1]

- P=[4,3,3,1]

$V = V_4 \oplus V_{31} \oplus V_{32} \oplus V_1$



$A \in \mathcal{N}_B$ generic

$A^9 \neq 0$

$A^{10} = 0$

Q(P)= [10,1]

Number of parts of Q(P)

$P=[p_1,\dots,p_r]$ is called "**almost rectangular**" if $p_1 - p_r \le 1$

Ex. [3,2] , [3,3,3,2,2,2,2,2], ...

Thm. [B] Any partition P can be written as a union of almost rectangular sub-partitions. The **minimum number of almost rectangular sub-partitions** needed is equal to the **number of parts of Q(P)**.

Ex. P=[5,4,3,3,2,1]

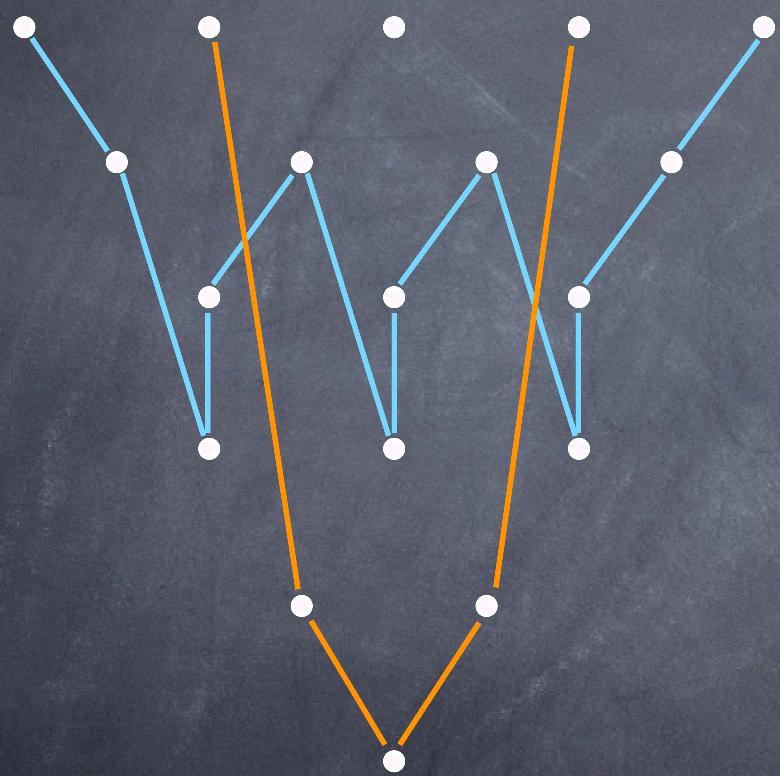
P=[5,4,3,3,2,1]

P=[5,4,3,3,2,1]

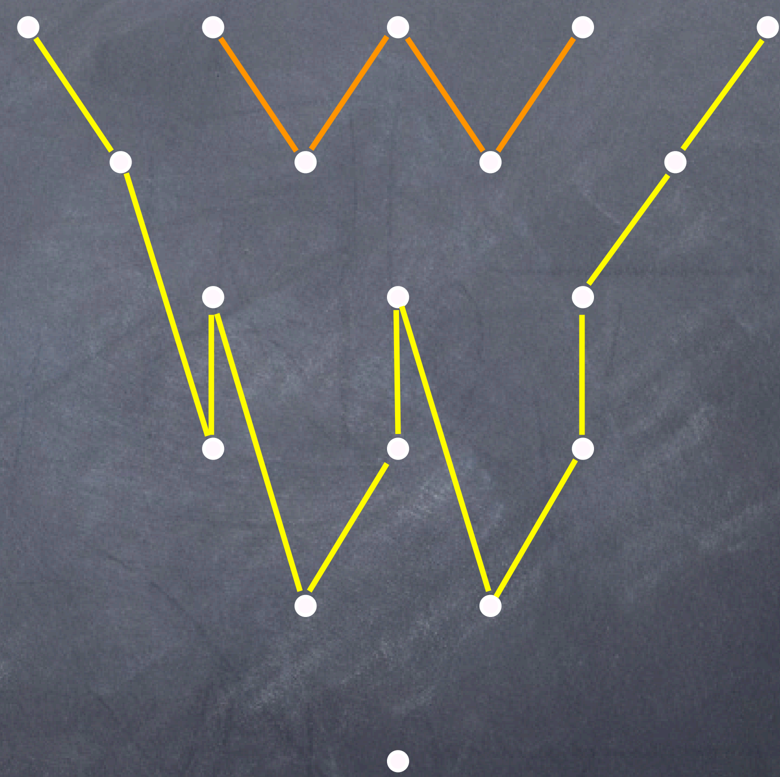
So Q(P) has **3** parts.

Oblak Conjecture

[5,4,3,3,2,1]



[5,4,3,3,2,1]

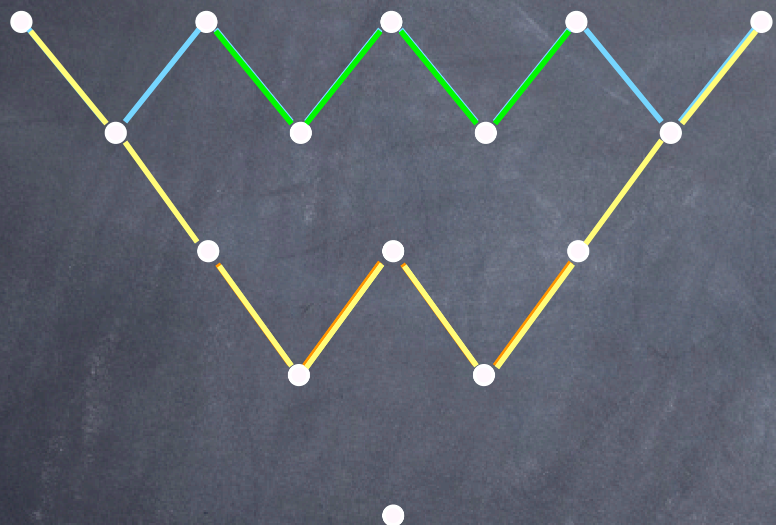


- Q(P) has three parts.
- Q(P)=[12,–,–]

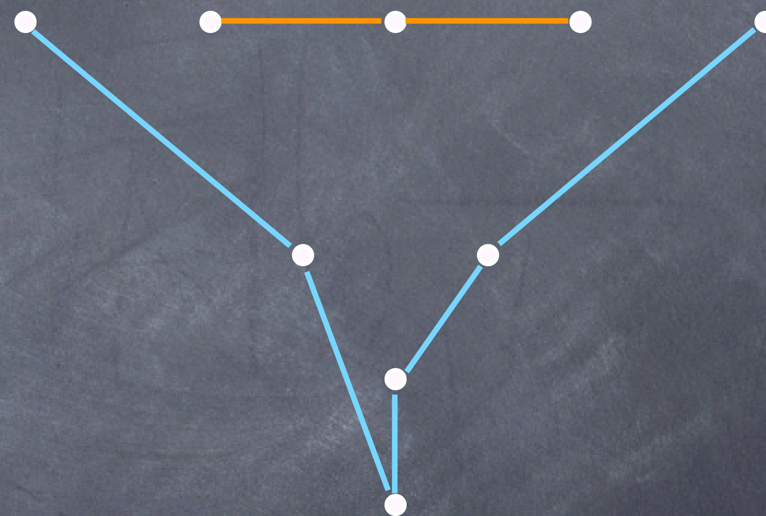
Oblak Conj: Q(P)=[12,5,1]

Ex (Oblak Conj. for Q(P))

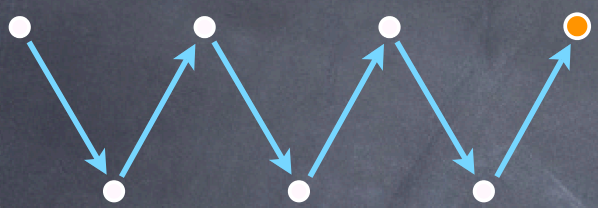
- P=[5,4,3,2,1] Q(P)=[9,5,1]



- P=[5,2,1,1] Q(P)=[6,3]



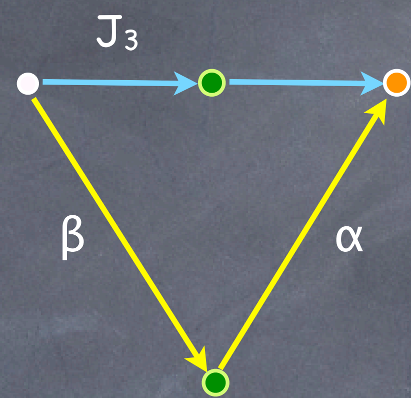
- $P=[4,3]$, $Q(P)=[7]$



$$A_{gen}=\left|\begin{array}{c:c} J_4\ f(J_4) & \alpha\ h(J_3) \\ \hdashline l(J_3)\ \beta & J_3\ g(J_3) \end{array}\right|$$

$$A_{ad}=\left|\begin{array}{c:c} 0 & c\alpha \\ \hdashline d\beta & 0 \end{array}\right|$$

- $P=[3,1]$, $Q(P)=P$



dim Ker A =2

A an n×n matrix with Jordan partition Q

dim Ker A= #parts of Q

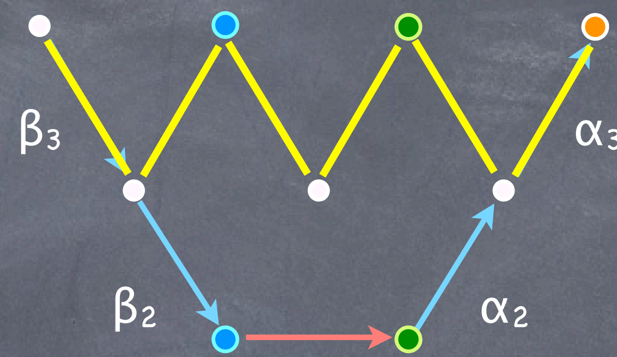
$X \in V_4$, $Y \in V_3$

$$A_{ad}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} c\alpha Y \\ d\beta X \end{bmatrix}$$

$\begin{bmatrix} X \\ Y \end{bmatrix} \in$ Ker A_{ad} iff

$$Y=0\ \&\ X=\left|\begin{array}{c} s \\ 0 \\ 0 \\ 0 \end{array}\right|$$

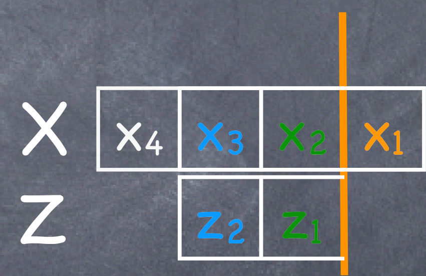
- $P=[4,3,2]$ Q(P): 2 parts ([7,2])



$$A_{ad} = \left|\begin{array}{c:c:c} & \alpha_3 & \\ \hdashline \beta_3 & & \alpha_2 \\ \hdashline & \beta_2 & J_2 \end{array}\right|$$

$X \in V_4$, $Y \in V_3$, $Z \in V_2$

$$A \begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = \begin{vmatrix} \alpha Y \\ \beta X+\alpha Z \\ \beta Y \end{vmatrix}$$



dim Ker A=3

A "special"

$$\begin{vmatrix} X \\ Y \\ Z \end{vmatrix} \in \text{Ker } A \text{ iff } Y=0 \text{ and } \beta X+\alpha Z=0 \text{ and } \beta Y+JZ=0$$

$\Rightarrow JZ=0 \Rightarrow z_2=x_3=0$

Fix B with Jordan partition P, for any $A \in \mathcal{N}_B$

A and B commute and are both nilp.

$k[A,B]=k[\![x,y]\!]/I_{A,B}$ ($I_{A,B}$: ideal of relations)

Artinian algebra

- [BI] If A is generic then the Hilbert function of k[A,B] is the dual partition of Q(P).

- [KO] If A generic then k[A,B] is Gorenstein.

Macaulay's characterization of Hilb of such rings.

- [BI] Q(P)=P iff any two parts of P differ by at least two.

- Q(P) is stable, i.e. Q(Q(P)) = Q(P).

Currently: For A_{ad}, $k[A_{ad},B]$ is Gorenstein with Hilbert function dual to the expected partition according to Oblak conjecture. (standard basis technique)

References

[Bar] V. Baranovsky: The variety of pairs of commuting nilpotent matrices is irreducible, Transform. Groups 6 (2001), no. 1, 3–8.

[B1] R. Basili: On the irreducibility of varieties of commuting matrices, J. Pure Appl. Algebra 149(2) (2000), 107–120.

[B2] : R. Basili: On the irreducibility of commuting varieties of nilpotent matrices. J. Algebra 268 (2003), no. 1, 58–80.

[BI] R. Basili and A. Iarrobino: Pairs of commuting nilpotent matrices, and Hilbert function. J. Algebra 320 # 3 (2008), 1235–1254. ArXiv math.AC: 0709.2304.

[BI2] Basili and A. Iarrobino: An involution on N_B , the nilpotent commutator of a nilpotent Jordan matrix B, preprint 2009.

[BKO] R. Basili, T. Koˇsir, P. Oblak: Some ideas from Ljubjlana, July, 2008, preprint, 6p.

[Br] J. Brian ¸con: Description de Hilbn C{x, y}, Invent. Math. 41 (1) (1977) 45-89.

[HW1] T. Harima and J. Watanabe: The commutator algebra of a nilpotent matrix and an application to the theory of commutative Artinian algebras, J. Algebra 319 (2008), no. 6, 2545–2570.

[I1] A. Iarrobino : Punctual HIlbert schemes AMS Memoir Vol 10, #188 (1977), Amer. Math. Society, Providence.

[KO] T. Kosir and P. Oblak: On pairs of commuting nilpotent matrices, Transform. Groups 14 (2009), no. 1, 175–182.

[Mac] F. H. S. Macaulay: On a method for dealing with the intersection of two plane curves, Trans. Amer. Math. Soc. 5 (4) (1904), 385-410.

[McN] G. McNinch : On the centralizer of the sum of commuting nilpotent elements, J. Pure and Applied Alg. 206 (2006) # 1-2, 123–140.

[Ob1] P. Oblak: The upper bound for the index of nilpotency for a matrix commuting with a given nilpotent matrix, Linear and Mul- tilinear Algebra (electronic 9/2007). Slightly revised in ArXiv: math.AC/0701561.

[Pan] D. I. Panyushev: Two results on centralisers of nilpotent elements, J. Pure and Applied Algebra, 212 no. 4 (2008), 774–779.

[Pol] S. Poljak: Maximum Rank of Powers of a Matrix of Given Pattern, Proc. A.M.S., 106 #4 (1989), 1137–1144.

[Pre] A. Premet: Nilpotent commuting varieties of reductive Lie algebras, Invent. Math. 154 (2003), no. 3, 653–683.

$P = [p_1, \ldots, p_1, \ \ldots \ , p_r, \ldots, p_r]$ $\qquad p_1 > \ldots > p_r$

(n_1 times $\ldots$ n_r times)

$\pi : \mathcal{C}_B \to$ semi-simple part $\subset Mat_{n_1}(k) \times \ldots \times Mat_{n_r}(k)$

$$\mathcal{N}_B = \pi^{-1}(\text{nilp})$$

$$\mathcal{U}_B = \pi^{-1}(\text{strictly upper triangular})$$

- $\mathcal{U}_B \subset \mathcal{N}_B$
- $\forall N \in \mathcal{N}_B \quad \exists C \in \mathcal{C}_B$ s.t. $C^{-1}NC \in \mathcal{U}_B$

Ex. P=[4,3]

$$\left|\begin{array}{cccc:ccc} a & a' & a'' & a''' & c & c' & c'' \\ & a & a' & a'' & & c & c' \\ & & a & a' & & & c \\ & & & a & & & \\ \hdashline & d & d' & d'' & b & b' & b'' \\ & & d & d' & & b & b' \\ & & & d & & & b \end{array}\right|$$

$A \in \mathcal{C}_B$ generic

$$\pi(A) = ([a]\ ,\ [b]) \in Mat_1(k) \times Mat_1(k)$$

$$A \in \mathcal{U}_B\ (=\mathcal{N}_B) \text{ iff } a=b=0$$

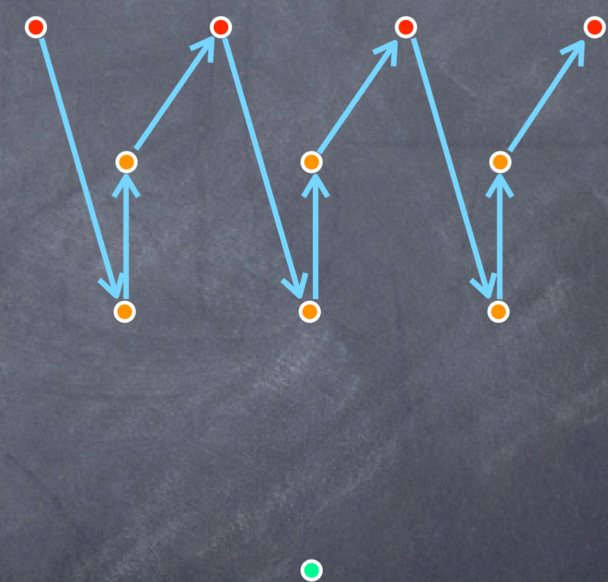
Ex. P=[4,3,3,1]

$A \in \mathcal{C}_B$ has the form

a	*	☆	○	*	☆	○	*	☆	○	*
0	a	*	☆	0	*	☆	0	*	☆	0
0	0	a	*	0	0	*	0	0	*	0
0	0	0	a	0	0	0	0	0	0	0
0	*	☆	○	b	*	☆	b′	*	☆	*
0	0	*	☆	0	b	*	0	b′	*	0
0	0	0	*	0	0	b	0	0	b′	0
0	*	☆	○	c′	*	☆	c	*	☆	*
0	0	*	☆	0	c′	*	0	c	*	0
0	0	0	*	0	0	c′	0	0	c	0
0	0	0	☆	0	0	☆	0	0	☆	d

$A \in \mathcal{U}_B$ iff $a=b=c=b'=d=0$

$V = V_4 \oplus V_{31} \oplus V_{32} \oplus V_1$



$$\pi(A) = \left| a \right|, \begin{vmatrix} b & b' \\ c' & c \end{vmatrix}, \left| d \right|$$

$$\in Mat_1(k) \times Mat_2(k) \times Mat_1(k)$$

$A^9 \neq 0$

$A^{10} = 0$

$Q(P) = [9,1]$