0. Operator in divergence form, ignore

$\bigstar\ \partial_i(a_{ij}\partial_j u) + \text{lower order} = f \quad \text{in } \Omega \underset{\text{bounded}}{\overset{\text{smooth}}{\subset}} \mathbb{R}^n$

These arise in applications.
Electrostatics: $\Delta u = \operatorname{div}(\nabla u) = -4\pi\rho$ (ρ = charge density)
(u = electric potential)

$\vec{E} = -\nabla u$ = electric field induced by ρ.

This assumes a homogeneous medium, where conductivity is constant and independent of direction. Otherwise, for anisotropic media, we encounter conductivities $a_{ij}(x)$.

Ellipticity: $a_{ij}(x)\xi_i\xi_j \geq \varepsilon|\xi|^2$ for $x\in\Omega, \xi\in\mathbb{R}^n$.

Weak solution: for $\varphi\in C_0^1(\Omega)$, i.e. C^1 with compact support in Ω, we get $-\int_\Omega a_{ij}\partial_j u\,\partial_i\varphi\,dx = \int_\Omega f\varphi\,dx$.

This makes sense for $a_{ij}\in L^\infty$ and ∇u integrable (actually, want $\nabla u\in L^2$).

I. Regularity Theory

If u is a weak solution of ($\bigstar$) and f is "smooth" then we expect u to be "smoother" than f. But even for very smooth f, e.g. $f\equiv 0$, the smoothness of u could depend on the a_{ij}s.

Classical result: If the a_{ij}s are Hölder continuous then $u\in C^1$.

'a' is Hölder-continuous $:=$ $|a(x)-a(y)|\leq C|x-y|^\alpha$ for some $0<\alpha<1$: C^α. Lipschitz continuous: $\alpha=1$, but we write Lip instead of C^1.

Hartmann-Wintner: If a_{ij} are "Dini continuous" then $u\in C^1$.

Mazya-McOwen: Want conditions weaker than Dini under which a weak solution is Lip or differentiable, at a point. WLOG we assume that this point x_0 is 0 and $a_{ij}(0)=\delta_{ij}$.

1) <u>Dini Continuity</u>: $|a_{ij}(x)-a_{ij}(y)| \le \omega(|x-y|)$, where ω is the "modulus of continuity". For Hölder, we want $\omega(r)=r^\alpha$ for $0<\alpha<1$. For Dini continuity, we want $\int_0^\varepsilon \frac{\omega(r)}{r}\,dr < \infty$, so Hölder $\Rightarrow$ Dini. <u>Square Dini</u>:

$$\int_0^\varepsilon \frac{\omega^2(r)}{r}\,dr < \infty, \text{ so Dini} \Rightarrow \text{Square Dini (aka } D^2)$$

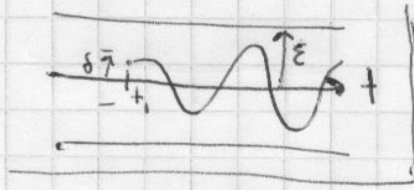
2) <u>Stability Conditions</u>: Let $\theta_i = x_i/|x| \in S^{n-1}$ and $r=|x|$, R an $n\times n$ matrix so that $R_{ij}(r) = \underset{\text{mean integral}}{\fint_{S^{n-1}}} \left(a_{ij}(r\theta) - n\,a_{ik}(r\theta)\theta_k\theta_j\right) d\theta$

$= \fint_{S^{n-1}} \left(A(r\theta) - n A(r\theta)\,\theta\otimes\theta\right) d\theta$. In general, R is not symmetric, but we can show $|R(r)| \le c\,\omega(r)$. Let $r=e^{-t}$, so $r\to 0 \Leftrightarrow t\to\infty$.

$$(\dagger)\quad \frac{d\varphi}{dt} + \tilde{R}(t)\varphi = 0, \quad \text{where } \tilde{R}(t) = R(e^{-t}).$$

Stability of $(\dagger)$ as $t\to\infty$ is related to regularity of weak solutions as $r\to 0$.

a) <u>Uniform stability</u>: $\forall \varepsilon>0\ \exists \delta(\varepsilon)>0$ s.t. $\forall \varphi$ solution $\forall t_1$, $|\varphi(t_1)|<\delta \Rightarrow |\varphi(t)|<\varepsilon$ for all $t>t_1$.



b) <u>Limits at infinity</u>: any solution of $(\dagger)$ satisfies $\varphi(t)\to\varphi(\infty)$ as $t\to\infty$.

Example: $\dot\varphi = g\varphi \Rightarrow \varphi(t) = \varphi(0)\exp\left[\int_0^t g(\tau)\,d\tau\right]$

(a) means $\int_s^t g(\tau)\,d\tau < K$ for all $t>s>0$.

(b) means $\int_0^\infty g(\tau)\,d\tau$ converges to $[-\infty,\infty)$.

If $\tilde{R}\in L^1(0,\infty)$, i.e. $\int_0^\infty |\tilde{R}(t)|\,dt < \infty$, then (a) and (b) hold.

Main theorem: Assume ω satisfies (D^2) and u is a weak solution of $(\star)\ \partial_i(a_{ij}\partial_j u)=0$. Then:

i) If $(\dagger)$ is uniformly stable, then u is Lip at $x=0$:
$u(x)=u(0)+O(|x|)$ as $|x|\to 0$.

ii) If in addition $(\dagger)$ has limits at infinity then u is differentiable at $x=0$:
$u(x)=u(0)+a_jx_j+O(|x|)$ as $|x|\to 0$.

The idea of the proof is to write $u(x)$ as
$u(x)=u_0(|x|)+\vec{v}(|x|)\vec{x}+\mathbf{w}(x)$, where
$u_0(r)=⨍_{S^{n-1}}u(r\theta)d\theta,\quad v_k(r)=\frac{n}{r}⨍_{S^{n-1}}u(r\theta)\theta_k\,d\theta.$

Then we plug into $(\star)$. Can express u_0' in terms of $\vec{v}'$, $\vec{v}$, and ∇w. We get a second-order dynamical system for $\vec{v}$ dependent on w and a second-order PDE for w dependent on $\vec{v}$. The former can be reduced to a first-order system; the latter will be a perturbation of the Laplacian.

Example: $a_{ij}(r\theta)=\delta_{ij}+g(r)\theta_i\theta_j$ where $|g(r)|\le\omega(r)$, where ω is (D^2). Then $R_{ij}=\frac{1-n}{n}g(r)\delta_{ij}$ and $(\dagger)\ \dot{\varphi}=\frac{n-1}{n}g(r)\varphi$.
$\int_s^t\tilde{g}(\tau)d\tau<K\Rightarrow$ all weak solutions are Lip at $x=0$.
$\int_0^\infty\tilde{g}(\tau)d\tau\in[-\infty,\infty)\Rightarrow$ all weak solutions are differentiable at $x=0$.

II. Inverse Problems

Recall (forward) existence theory: if $a_{ij}\in L^\infty(\Omega)$ are known and we are given g on $\partial\Omega$ then we can solve the Dirichlet problem $Lu=0$ in Ω, $u=g$ on $\partial\Omega$. But if we know u in Ω then we can compute $h=\partial u/\partial\nu$ on $\partial\Omega$. We then get the Dirichlet-Neumann map $u|_{\partial\Omega}\to\frac{\partial u}{\partial\nu}\Big|_{\partial\Omega}$. If a_{ij}s are conductivities then u is the voltage, and this is the voltage $\to$ current map, which is experimentally measurable. <u>Question</u>: Does the D-N map uniquely determine the a_{ij}s?

The simplest case is isotropic: $a_{ij} = \gamma(x)\,\delta_{ij}$ where $\gamma \in L^\infty$, $\gamma \geq \varepsilon > 0$. The D-N map Λ_γ is then $u|_{\partial\Omega} \to \gamma \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}$

$$\int_{\partial\Omega} g\,\Lambda_\gamma(g)\,ds = \int_\Omega \gamma\,|\nabla u|^2\,dx.$$

Calderon (1980): Does Λ_γ uniquely determine γ?

Sylvester - Uhlmann (1987): Yes, if $\gamma \in C^\infty$, $n \geq 3$.

P-P-Uhlmann (2003): Yes, if $\gamma \in C^{3/2}$, $n \geq 3$.

Astala - Paivarinta (2006): Yes, if $\gamma \in L^\infty$, $n = 2$.

This is still open for $\gamma \in L^\infty$, $n \geq 3$.