FLAT CONNECTIONS AND QUANTUM GROUPS

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Braid Groups

Intuitive Definition. For $n \geq 2$ let B_n be the group of "braids on n strings" where a braid on n strings is a set of n strings connected two ordered sets of n points and recording at each intersection of the strings whether strings k goes over or under string j. This is a monoid under concatenation of braids (tacking braids end to end). This monoid has identity: the braid the connects each point two its pair with no crossings and it can be seen that taking a braid and replacing each crossing with the reverse crossing give an inverse.

Algebraic Definition. Now, let $B_n =< T_1, \dots, T_{n-1} >$ /some relations where T_i is the identity braid but with a string connecting the point i to $i+1$ crossing over a line connecting point $i+1$ to point i. It is clear that these generate the following relations:

$$T_i T_j = T_j T_i \text{ for } |i-j| \geq 2 \tag{R1}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \tag{R2}$$

By a Theorem due to Artin, $< T_1, \dots, T_{n-1} > /(R1)(R2)$ is a presentation of B_n.

Topological Inclination. To look at the B_n's topologically, we'd really like to find spaces X_n such that $B_n = \pi_1(X_n)$. To this end let C_n be the configuration space of n ordered points $z_1, \dots, z_n \in \mathbb{C}$ so $C_n = \{(z_1, \dots, z_n) \in \mathbb{C}^n | z_i \neq z_j, \forall i \neq j\} = \mathbb{C}^n \setminus \cup \{z_1 = z_2\}$ where the right side is $\mathbb{C}^n$ with the hyperplanes $z_1 = z_2$ removed. Now, the symmetric group S_n acts freely on C_n so let $X_n = C_n / S_n$ be the configuration space of n unordered points in $\mathbb{C}$, then by a theorem of Fox-Neuwith $B_n \cong \pi_1(X_n)$. The proof sketch is highly visual and will be omitted.

Generalized Braid Groups

Algebraic Definition. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. Then we define the Algebraic Braid Group to be the Weyl group of $\mathfrak{g}$ with the relations s_i^2 removed from the presentation:

$$B_{\mathfrak{g}}^{Alg} =< s_1, \dots, s_r >_{s_i s_j \dots = s_j s_i \dots \forall i \neq j}$$

where r is the number of points in the Dynkin diagram of $\mathfrak{g}$ and for each i, j the string on both side of $s_i s_j \dots = s_j s_i \dots$ has length equal to the number associated to the connection between the points i and j in the standard way.

For example let $\mathfrak{g} = \mathfrak{sl}_5$. The Dynkin Diagram is $\circ \xrightarrow{4} \circ$ and so $B_{\mathfrak{sl}_5}^{Alg} =< s_1, s_2 > / s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$.

Topological Inclination Revisited. Let $\mathfrak{h}$ denote the Cartan Subalgebra of $\mathfrak{g}$. Now, $\mathfrak{h}$ acts on $\mathfrak{g}$ by adjoint action and so $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_\alpha \mathfrak{g}_\alpha$ where α are the joint eigenvalues of $\mathfrak{h}$. Then C_n above corresponds to $\mathfrak{h} \backslash \bigcup_\alpha \ker \alpha$. Then let $B_{\mathfrak{g}}^{Top} := \pi_1(\mathfrak{h}_{\text{reg}}/W)$ where W is the Weyl group of $\mathfrak{g}$. By a theorem of Brieskoin $B_{\mathfrak{g}}^{Top} \cong B_{\mathfrak{g}}^{Alg} := B_{\mathfrak{g}}$.

Representations of Braid Groups

Topological Representations. The idea here is to construct representations of $B_{\mathfrak{g}}$ from the monodromy/analytic continouation of solutions of systems of n'th order ODE's on $\mathfrak{h}_{\text{reg}}/W$. We have the following general solution:

Let X be a complex manifold, and lets look at $\frac{\partial f}{\partial z_i} = A_i f$ where $f : X \to \mathbb{C}^n$ and $A_i : X \to \text{End}(\mathbb{C}^n)$. Let Φ be a solution near $x_0 = \gamma(0)$, then Φ_γ is the analytic continuation of Φ along γ so $\Phi : \gamma \to Gl_n(\mathbb{C})$. Let $\mu(\gamma) = \Phi_\gamma^{-1}(1)\Phi_\gamma(0) \in Gl_n(\mathbb{C})$.

Proposition. If the above is integrable eg. $[\partial_i - A_i, \partial_j - A_i] = 0 \equiv \partial_i A_j - \partial_i A_i = [A_i, A_j]$ then $\mu(\gamma)$ only depends on the homotopy class of γ.

Now, let $X = \mathfrak{h}_{\text{reg}}$, $\mathbb{C}^n \to V \in \text{Rep}(\mathfrak{g})$. Then what is A_i? Invariantly we have

$$\nabla = d - \sum \frac{d_\alpha}{\alpha} r_\alpha$$

where $r_\alpha \in \text{End}(V)$.

For example let $a_1, \ldots, a_m$ be a basis of $\mathfrak{h}^*$ and let $a^1, \ldots, a^m$ be the duel basis of $\mathfrak{h}$. Then $f : \mathfrak{h}_{\text{reg}} \to V$ and

$$\frac{\partial f(a)}{\partial a_i} = \left(\sum_\alpha \frac{\alpha(a^i)}{\alpha(a)} \cdot r_\alpha \right) f$$

To each α we associate a 3-dimensional subalgebra $\mathfrak{sl}_2^\alpha$ of $\mathfrak{g}$, eg: if $\mathfrak{g} = \mathfrak{sl}_n$ then $\alpha = z_i - z_j$, $\mathfrak{sl}_2^\alpha =< E_{ij}, E_{ji}, E_{ij} - E_{ji} >$ and $C_\alpha = (e_\alpha f_\alpha + f_\alpha e_\alpha + 1/2h^2\alpha)\alpha/2$. Thus

$$\nabla_C = d - \sum_\alpha \frac{d\alpha}{\alpha} C_\alpha$$

is integrable $\forall h \in \mathbb{C}$.

Corollary. $\mu_h : \pi_i(\mathfrak{h}_{\text{reg}}/W) \to Gl(V)$. These are the topological representations of $B_{\mathfrak{g}}$.

Algebraic Representations. Algebraic representations of $B_{\mathfrak{g}}$ come from the quantum group $\mathscr{U}_\hbar \mathfrak{g}$ which is an algebra depending on a parameter $\hbar$ and is a deformation of $\mathscr{U}\mathfrak{g}$. For example, $\text{Sl}_2 =< e, f, h >$ modulo the relations $[h, e] = 2e$, $[e, f] = h$, $[h, f] = -2f$ where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, g = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

so quantum $\mathrm{Sl}_2 = \mathscr{U}_\hbar \mathrm{Sl}_2 =< E, F, H >$ modulo the relations $[H, E] = 2E$, $[H, F] = -2F$ and

$$[E, F] = \frac{e^{\hbar H} - e^{-\hbar H}}{e^{\hbar} - e^{-\hbar}} = H + \sigma(\hbar^2)$$

Theorem. If V is a representation of $\mathscr{U}_\hbar \mathfrak{g}$ there exists a Weyl group operator $s_1^\hbar \dots s_r^\hbar \in$GL$(V)$ such that $s_i^\hbar s_j^\hbar \dots = s_j^\hbar s_i^\hbar \dots$ for $\forall i \neq j$. As a corollary there exists a map $\lambda : B_{\mathfrak{g}}^{Alg} \to$GL$(V)$.

The Big Theorem. The two representations of $B_{\mathfrak{g}}$ described above are the same.

AFFINE SETTING:

Affine Braid Groups. Let an algebraic affine braid group be define to be $\hat{B}_{\mathfrak{g}}^{Alg} :=< s_0, \dots, s_r >$ /relations depending on the affine Dynkin diagram of $\mathfrak{g}$ in the same way as for non affine braid groups, with s_0 being the added affine vertex. Similarly define a topological affine braid group as follows:

Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra. Let $H \subset G$ be the maximal torus such that Lie$(H) = \mathfrak{h}$. Then $H_{\mathrm{reg}} = H \backslash \bigcup_\alpha \{e^\alpha = 1\}$. Then we define $\hat{B}_{\mathfrak{g}}^{Top} = \pi_1(H_{\mathrm{reg}}/W)$ and note that similar to the non affine case, $\hat{B}_{\mathfrak{g}}^{Alg} = \hat{B}_{\mathfrak{g}}^{Top}$.

Algebraic Representations. Let $L\mathfrak{g} = \mathfrak{g}[ht^{-1}]$ an affine algebra. Then, similar to before, $\mathscr{U}_\hbar(L\mathfrak{g})$ associates to a map $\hat{B}_{\mathfrak{g}}^{Alg} \to$GL$(V)$ for $V \in$Rep$(\mathscr{U}_\hbar(L\mathfrak{g}))$

Topological Representations. Similar to before we have a connection

$$\nabla = d - \hbar \sum_\alpha \frac{d\alpha}{e^\alpha - 1} C_\alpha - d\mu_i A^i$$

where the tail of this equation has values in $Y(\mathfrak{g})$, the Yangians of $\mathfrak{g}$.

Theorem. There exists a ∇ with coefficients in $Y(\mathfrak{g})$ which is flat.

Conjecture. These two representation of affine braid groups are in fact the same.