

# GEODESIC EQUIVALENCE PRINCIPLE

NATE BADE

## TRAJECTORY EQUIVALENCE PRINCIPLE

**Lagrangian Systems.** For any smooth manifold  $M$  the tangent bundle  $TM$  has a natural manifold structure with points denoted by  $(x, v)$  for  $x \in M$  and  $v \in T_x M$ . We then call any function  $L \in C^\infty(TM, \mathbb{R})$  a *Lagrangian* and denote it by  $L = L(x, v)$ ,

**Variational Principle.** Let  $\gamma : [0, 1] \rightarrow M$  with  $\gamma(0) = p$  and  $\gamma(1) = q$  be any line connecting the points  $p, q \in M$ . The *Action Function*  $L[\gamma]$  is then define as

$$L[\gamma] = \int_{[0,1]} L(\gamma(t), \gamma'(t)) dt$$

A (smooth) variation of  $\gamma(t)$  is a continuous (smooth) function  $\gamma : (-\epsilon, \epsilon) \times [0, 1] \rightarrow M$  for some  $\epsilon > 0$  denoted by  $\gamma(s, t)$  or  $\gamma_s(t)$  and with the following properties:  $\gamma_0(t) = \gamma(t)$ ,  $\gamma_s(0) = p$  and  $\gamma_s(1) = q$ .

Now, for any fixed  $s \in (-\epsilon, \epsilon)$  we can look at

$$L[\gamma_s] = \int_{[0,1]} L(\gamma_s(t), \gamma'_s(t)) dt$$

We claim without proof that  $L[\gamma_s]$  is a smooth function of  $s$  around 0. We then have the following definition:

**Def:** If  $\frac{d}{ds}|_{s=0}(L[\gamma_s]) = 0$  for all variations of  $\gamma$  then  $\gamma$  is called an extremal of  $L$ . It is clear that if  $\gamma$  is a minimum/maximum curve between  $p$  and  $q$  with respect to  $L$  then  $\gamma$  must be extremal.

## GEODESICS

Let  $(M, g)$  be a Riemannian Manifold and let  $L = \sqrt{g_x(y, y)}$ ; then the extremals are geodesics. We will need the following:

**Eular-Lagrange Equation.** The following is the Euler-Lagrange Equation:

$$\frac{d}{ds}|_{s=0}(L[\gamma_s]) = 0 \iff \left[ \frac{\partial L}{\partial u}(\gamma(t), \dot{\gamma}(t)) \right]' = \frac{\partial L}{\partial x}(\gamma(t), \dot{\gamma}(t))$$

The Euler-Lagrange Equation is important to us since it states, amongst other things, that the left side holding for all variations of  $\gamma$  is some invariant property of  $\gamma$ .

**Geodesics.** Let  $L$  and  $(M, g)$  be as above and let's write  $g$  in coordinates as  $g_{ij} dx^i dx^j$ . Take  $L(x, v) := g_x(v, v)/2$ . Using the Euler-Lagrange Equation and denoting the Christoffel Symbols of the Levi-Civita connection by  $\Gamma_{\alpha, \beta}^k$  we get the following:

$$(1) \quad \ddot{\gamma}^k + \Gamma_{\alpha, \beta}^k(\gamma(t)) \dot{\gamma}^\alpha(t) \dot{\gamma}^\beta(t) = 0$$

$$(2) \quad \text{For } H(x, p) := \frac{1}{2} g^{ij}(x) p_i p_j, \quad \dot{p}_k = \frac{\partial H}{\partial x^k}(x, p), \quad \dot{x}_k = \frac{\partial H}{\partial p_k}(x, p) \text{ and } p_k = g_{kl}(x) v^l$$

where  $H \in C^\infty(T^*M, \mathbb{R})$  is the Hamiltonian function on the cotangent bundle. Or equivalently,

$$(1') \quad \nabla_{\dot{\gamma}(t)} \dot{\gamma}(t)$$

(2') Invariant Form of (2): For  $y \in T^*M$ ,  $\dot{y} = X_H(y)$  where  $X_H$  is a vector field on the cotangent bundle  $T(T^*M)$  and  $d_y H(\cdot) = \omega_y(\cdot, X_H(y))$  where  $\omega = dp \wedge dq$  is the canonical symplectic structure on  $T^*M$ .

**Examples.** Here are two examples of equations that are Hamiltonian: First for  $q(t, x)$ :

$$q_t + 69q_x + q_{xxx} = 0$$

and the nonlinear Schrodinger Equation:

$$i\psi_t + \psi_{xx} = \pm |\psi|^2 \psi = 0$$

#### COMPLETELY INTEGRABLE FLOWS

Now, assume that we are given two Lagrangians  $L_1$  and  $L_2$  such that they have the same Extremals (considered as non-parameterized curves eg as sets on a manifold). Also assume that  $L_1$  and  $L_2$  are different:  $L_1 \neq cL_2$  where  $c$  is a constant (and some other unspecified conditions). Then what can be said about the extremals of  $\gamma(t)$ ?

**Theorem by Matrver-Topolov.** If  $g_1$  and  $g_2$  are Riemannian metrics on  $M$  and if  $L_1 = g_1(v, v)/2$  and  $L_2 = g_2(v, v)/2$  and  $L_1, L_2$  trajectory equivalent. Then the geodesic flows corresponding to  $g_1$  and  $g_2$  are completely integrable (eg solvable using exact forms).

**Open Question.** There is an open problem of describing the general form of Lagrangians and seeing if similar theorems hold. For example, is the above theorem true for Finsler Metrics?