On representations with finitely many orbits

Jerzy Weyman

0. Reps with finitely many orbits: Examples from linear algebra

Example: ① $GL_m \times GL_n \curvearrowright \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^m, \mathbb{C}^n)$

$f: \mathbb{C}^m \to \mathbb{C}^n$ linear map. the natural action corresponds to row and column operations, f can be reduced to the form:

$$\begin{array}{cc} & \begin{array}{cc} r & \end{array} \\ r & \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right) \end{array}$$

where $r = \operatorname{rank} f$.

② $GL_n \curvearrowright S_2(\mathbb{C}^n)$

$$S_2(\mathbb{C}^n) = \left\{ \sum_{i,j} a_{ij} x_i x_j \;\middle|\; a_{ij} = a_{ji} \right\} \leftarrow \text{Symmetric } n\times n \text{ matrices.}$$

$GL_n(\mathbb{C})$ acts by: $(A, X) \mapsto AXA^T$.

Then the qudratic form can be reduced to a sum of squares:

$$f \sim x_1^2 + \cdots + x_r^2$$

③. $GL_n(\mathbb{C}) \curvearrowright \mathrm{Mat}_{n\times n}(\mathbb{C})$ by conjugation:

$$(A, X) \longmapsto AXA^{-1}.$$

The orbits correspondto Jordan forms

ie: $\begin{pmatrix} J_{m_1}^{\lambda_1} & 0 & 0 & \cdots \\ 0 & J_{m_2}^{\lambda_2} & & \\ 0 & & \ddots & \\ \vdots & & & J_{m_r}^{\lambda_r} \end{pmatrix}$, where J_m^λ is an $m\times m$ Jordan block $\begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}_{m\times m}$

How about in other cases?

$$GL_p \times GL_q \times GL_r \curvearrowright \mathbb{C}^p \otimes \mathbb{C}^q \otimes \mathbb{C}^r$$

In general, don't have finitely many orbits: $\dim \mathbb{C}^p \otimes \mathbb{C}^q \otimes \mathbb{C}^r > \dim GL_p \times GL_q \times GL_r$

1. Cartan classification of simple Lie algebra:

Def: Lie algebra: A vector space L over $\mathbb{C}$, with an operation $L \times L \to L$, $(x, y) \mapsto [x, y]$

is called an Lie algebra, if:

① The bracket operation is bilinear

② $[x, x] = 0$, $\forall x \in L$

③ $[x[y, z]] + [y[z, x]] + [z, [x, y]] = 0$ $(\forall x, y, z \in L)$

↑ Jacobi Identity

Def: Semisimple Lie alg: A Lie alg. L is called semisimple, if $\operatorname{Rad} L = 0$,

where $\operatorname{Rad} L$ is the unique maximal solvable ideal of L, called radical of L.

Root space decomposition:

Let L be semi-simple Lie alg. $H \subset L$ be maximal toral subalg (H is abelian).

$$H \times L \to L$$

$$(h, x) \mapsto [h, x]$$

$\because H$ is abelian $\operatorname{ad}_L H = \{\operatorname{ad} h \mid h \in H\}$ is a family of semisimple endomorphisms of L.

$\Rightarrow \operatorname{ad}_L H$ is simultaneously diagonalizable.

$\therefore L = \bigoplus_{\alpha \in H^*} L_\alpha \qquad L_\alpha = \{x \in L \mid [h x] = \alpha(h) x, \ \forall h \in H\}$

$L_0 = C_L H = H$ ← can prove using L semisimple.

Denote: $\Phi = \{\alpha \in H^* \mid \alpha \neq 0, L_\alpha \neq 0\}$ ← such α is called root of L.

Thm: L semisimple Lie alg, H maximal toral $\Phi = \{\text{set of roots}\}$

$E_{\mathbb{R}} =$ Euclidean space spanned by Φ. Then:

inner product is extended by killing form: $(\ ): L \times L \to \mathbb{C}$, $x, y \mapsto Tr(ad\,x\,ad\,y)$

(a). Φ spans E, $0 \notin \Phi$

(b). If $\alpha \in \Phi$, the $-\alpha \in \Phi$, but no other scalar multiple of α is a root

(c) If $\alpha, \beta \in \Phi$, then $\beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \alpha \in \Phi$ $\quad (*)$

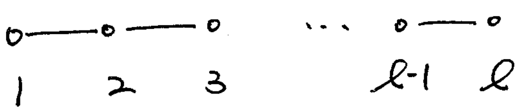
(d) If $\alpha, \beta \in \Phi$, then $\frac{2(\beta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}$.

Def: <u>Root systems</u>: A subset Φ of Euclidean space E is called a <u>root system</u> in E, if Φ satisfied $(*)$.

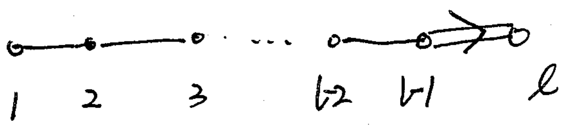
<u>Classification Thm</u>:

The root system can be classified using the Dynkin diagrams.

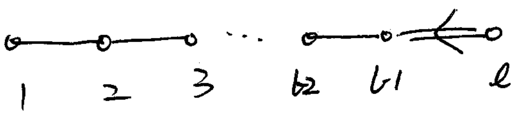
Thm: Φ irreducible root system of rank l, its Dynkin diagram is one of the following:

A_l $(l \geq 1)$


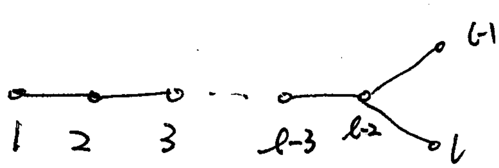
sl_{n+1}

B_l $(l \geq 2)$


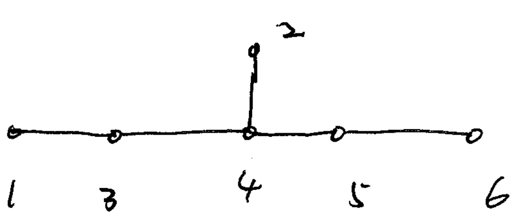
$so(2n+1)$

C_l $(l \geq 3)$


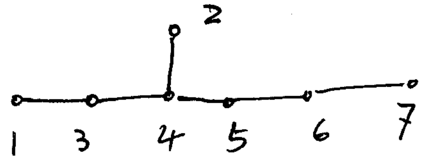
$sp(2n)$

D_l $(l \geq 4)$


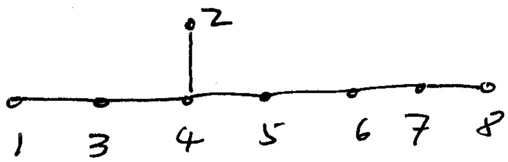
$so(2n)$

E_6


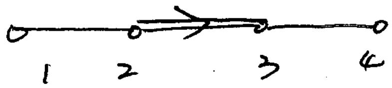
78

E_7 

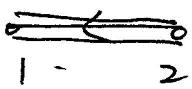
133

E_8 

248

F_4 

52

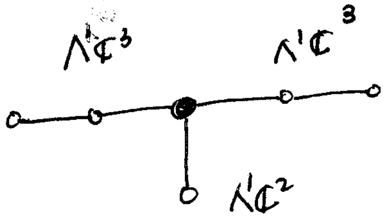
G_2 

14

3. Classify reps with finite many orbits:

Thm (Kac) Except for 2 exceptions, irreducible repn of (reductive) Lie alg. with finitely many orbits correspond to Dykin diagrams with distinguished node.

Example:



remove $\bullet$: 

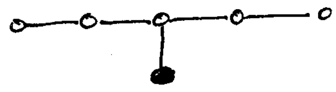
sl_3 sl_3 sl_2 $\mathbb{C}$

$$sl_3 \times sl_3 \times sl_2 \times \mathbb{C}^* = G_0 \curvearrowright V = \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$$

1 of the exceptions is $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^n$ has finitely many orbits under $sl_2 \times sl_3 \times sl_n \times \mathbb{C}^1$ for $n \geq 6$

E.g. $\Lambda^3\mathbb{C}^n$. 

$\Lambda^3\mathbb{C}^6$, $\Lambda^3\mathbb{C}^7$, $\Lambda^3\mathbb{C}^8$ finitely many orbits, $\Lambda^3\mathbb{C}^9$ does not.

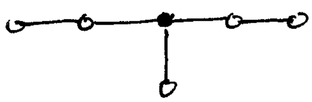
Example: E_6 

$\mathfrak{g}_0 = \mathfrak{gl}(6)$

$36 + 40 + 2 = 78.$

$\mathfrak{g}_1 = \Lambda^3 \mathbb{C}^6$

$\mathfrak{g}_2 = \Lambda^6 \mathbb{C}^6$

Example: E_6 

$\mathfrak{g}_0 = \mathfrak{sl}_2 + \mathfrak{sl}_3 + \mathfrak{sl}_3 + \mathbb{C}$	20	
$\mathfrak{g}_1 = \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$	2×18	$20+36+18+4 = 78$
$\mathfrak{g}_2 = \Lambda^2\mathbb{C}^2 \otimes \Lambda^2\mathbb{C}^3 \otimes \Lambda^2\mathbb{C}^3$	2×9	
$\mathfrak{g}_3 = S_{21}\mathbb{C}^2 \otimes \Lambda^3\mathbb{C}^3 \otimes \Lambda^3\mathbb{C}^3$	2×2	

Example: 

$\mathfrak{g}_0 = \mathfrak{sl}_2 + \mathfrak{sl}_5 + \mathbb{C}$

$4 + 24 = 28$

$\mathfrak{g}_1 = \mathbb{C}^2 \otimes \Lambda^2 \mathbb{C}^5$

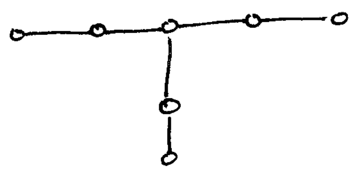
$28 + 2(20+5) = 28 + 50 = 78.$

$\mathfrak{g}_2 = \Lambda^2\mathbb{C}^2 \otimes \Lambda^4 \mathbb{C}^5$

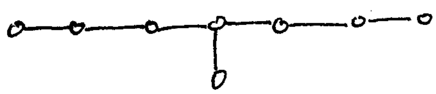
Thm: The reps where orbits can be parametrized (Similar to Jodan Canonical Form). more or less corresponds to ~~extended~~ Dynkin diagram with distinguished node.

Extended Dynkin diagram :

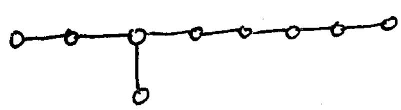
$\tilde{E}_6$



$\tilde{E}_7$:



$\tilde{E}_8$:



<u>Example</u>: $GL_n(\mathbb{C}) \curvearrowright Mat_n(\mathbb{C})$ $\quad (A, X) \mapsto AXA^{-1}$.

The invariants: $\subset \mathbb{C}[x_{ij}]$, which polys are invariant under $GL_n(\mathbb{C})$?

Answer: $\mathbb{C}[P_1, \cdots, P_n]$ $\qquad P_X(T) = \det(X - T\,Id) = \sum_i P_i(X)\, T^{n-i}$

P_i: coeff. of char. polynomials

$\therefore \mathbb{C}[x_{ij}]^{GL_n(\mathbb{C})} = \mathbb{C}[P_1, \cdots, P_n]$ is a polynomial ring.

Zeros of all invariants are just the nilpotents (Finite many orbits).