Projective Quotient

$G \subset GL(n)$

G algebraic gp. X variety $G \overset{\text{action}}{\curvearrowright} X$

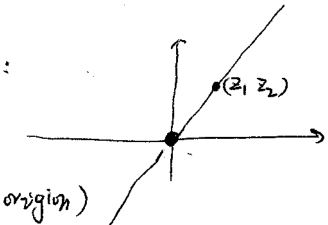
Q: How to define the quotient X/G in alg. geo? Want the quotient still be algebraic variety

$\pi : X \longrightarrow X/G$

Ex: $\mathbb{C}^* \times \mathbb{C}^2 \longrightarrow \mathbb{C}^2$

$t \quad (z_1, z_2) \longmapsto (tz_1, tz_2)$

the orbit space:



The orbits (not origin) are not closed.

The orbit space is not a variety!

Throw away the origin $(0,0)$, then: $\mathbb{C}^2 \setminus \{0\} \curvearrowleft \mathbb{C}^*$ $\quad \mathbb{C}^2\setminus\{0\} / \mathbb{C}^* = \mathbb{P}^1$.

$$\pi : \mathbb{C}^2 \setminus \{0\} \longrightarrow \mathbb{C}^2 \setminus \{0\} / \mathbb{C}^* = \mathbb{P}^1(\mathbb{C}) \triangleq \mathbb{C}^2 /\!/ \mathbb{C}^*$$

Want: invariant regular function here $\qquad$ regular function here $\qquad$ but not true!

Note: $\mathbb{P}^1$ cpt subset, regular function on $\mathbb{P}^1$ is constant.

$$\pi : \mathbb{C}^2 \setminus \{0\} \longrightarrow \mathbb{P}^1(\mathbb{C}) = U_1 \cup U_2$$

$\pi(V_1) = U_1$

$\pi(V_2) = U_2$

$\{(z_1, z_2) \mid z_1 \neq 0\} = V_1 \longrightarrow U_1 = D_+(z_1) = \{(z_1 : z_2) \mid z_1 \neq 0\} \subset \mathbb{P}^1$

$\{(z_1, z_2) \mid z_2 \neq 0\} = V_2 \longrightarrow U_2 = D_+(z_2) = \{(z_1 : z_2) \mid z_2 \neq 0\} \subset \mathbb{P}^1$

$\mathcal{O}(V_1) = \mathbb{C}[z_1, z_2, z_1^{-1}]$

$\mathcal{O}(U_1) \leftarrow$ regular function on U_1

$= \mathbb{C}[\frac{z_2}{z_1}] \leftarrow \mathbb{C}^*$-invariant. (Precise: G-invariant sections of G-linearised invertible sheaf).

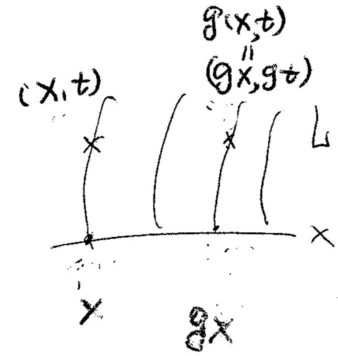
$\mathcal{O}(U_2) = \mathbb{C}[\frac{z_1}{z_2}]$

$$\pi^* \mathcal{O}(U_i) \cong \mathcal{O}(V_i)^G$$

$= \mathbb{C}[z_1, z_2, z_i^{-1}]$

<u>General</u>; What's the quotient should be?

X variety. $\begin{array}{c} L \\ p \downarrow \\ X \end{array}$ line bundle. $\Gamma(X, L)$ sections



$G \curvearrowright X$

Extend the action $G \curvearrowright X$ to $G \curvearrowright L$ that compatible with the action $G \curvearrowright X$

Def: A <u>linearization</u> of the action G with respect to L is an action $G \curvearrowright L$

(i) $p(g^{(x,t)}) = g.p^{(x,t)} = g.x$, $\forall g \in G$, $y \in L$

(ii) $L_x \longrightarrow L_{gx}$ is linear.
$y \longmapsto gy$

So: $G \curvearrowright \Gamma(X, L^{\otimes r})$, $\forall r$

$\forall f \in \Gamma(X, L^{\otimes r})^G$ $\quad X_f = \{x \in X \mid f(x) \neq 0\}$ is open set of X. invariant under G.

L ample bundle $\Rightarrow$ X_f affine open subset of X, $\exists f, \exists r > 0$

<u>Def</u>:

$X^{ss} = \{x \in X \mid \exists r > 0,\ \exists f \in \Gamma(X, L^{\otimes r})^G,\ f(x) \neq 0$ and X_f affine$\} \underset{\text{open}}{\subset} X$

semi-stable pt.

$X^s = \{x \in X \mid x$ semistable, $\dim \mathcal{O}(x) = \dim G$, $G \curvearrowright X_f$ is closed, $\} \underset{\text{open}}{\subset} X^{ss} \subset X$

stable pt.

<u>Construct quotient for $X^{ss} \overset{\text{open}}{\subset} X$</u>

$$X^{ss} = \bigcup_{\substack{f \in \Gamma(X, L^{\otimes r})^G \\ \neq 0}} X_f$$

induced by: $\mathcal{O}(X_f)^G \hookrightarrow \mathcal{O}(X_f)$

$$X /\!/ G :\overset{\Delta}{=} \bigcup_{f \text{ gluing}} (X_f / G),$$

$X_f / G := \mathrm{Spec}\left(\mathcal{O}(X_f)^G\right)$

Note: Must be f.g.

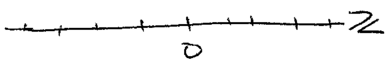
Ex: $\mathbb{C}^2 \circlearrowleft \mathbb{C}^*$

$$t \quad ((z_1,z_2),\lambda) \longmapsto ((tz_1,tz_2), t^a\lambda) \quad \forall a \in \mathbb{Z}$$

$$\begin{array}{c} \mathbb{C}^2\times\mathbb{C} \\ p\downarrow \\ \mathbb{C}^2 \end{array} \qquad \begin{array}{ccc} \mathbb{C}^*\times\mathbb{C}^2\times\mathbb{C} & \longrightarrow & \mathbb{C}^2\times\mathbb{C} \\ \downarrow & & \downarrow \\ \mathbb{C}^*\times\mathbb{C}^2 & \longrightarrow & \mathbb{C}^2 \\ t \quad (z_1, z_2) & \mapsto & (tz_1, tz_2) \end{array}$$

$$f\in \Gamma(\mathbb{C}^2, L)^G;\ f: \mathbb{C}^2 \longrightarrow \mathbb{C} \qquad f\in \mathbb{C}[z_1,z_2]$$
$$(z_1,z_2) \longmapsto f(z_1,z_2)$$

$$\forall t\in\mathbb{C}^*,\quad t\cdot f(z_1,z_2) = t^a f(t^{-1}z_1, t^{-1}z_2) = f(z_1,z_2)$$
$$t^{-a}f(tz_1,tz_2) = f(z_1,z_2),\ \forall t\in\mathbb{C}^*$$
$$\Rightarrow f(tz_1,tz_2) = t^a f(z_1,z_2) \Rightarrow f \text{ homo. poly of degree } a.$$



$a<0$ No such section.

$a=0$ f constant.

when $a=1$, f homo. poly of degree 1.

$$f_1 = z_1 \qquad f_2 = z_2$$

$$X_{f_1} = \{(z_1,z_2) \mid z_1\neq 0\} \qquad X_{f_2} = \{(z_1,z_2)\mid z_2\neq 0\}$$

$$X_{f_1}/G = \operatorname{Spec}\mathbb{C}[\tfrac{z_2}{z_1}] = \mathbb{C} \qquad X_{f_2}/G = \operatorname{Spec}\mathbb{C}[\tfrac{z_1}{z_2}] = \mathbb{C}$$

Get: $\mathbb{C}^2 /\!/_{L_{a=1}} \mathbb{C}^* = \mathbb{P}^1$

Rmk:
$$\begin{array}{ccc} X & \overset{\text{rational}}{\dashrightarrow} & X/\!/_L G \\ \cup & & \\ X^{ss} & \xrightarrow{\pi} & X/\!/_L G \\ \cup & & \cup \\ \pi': X^s & \longrightarrow & Y^s \end{array}$$

Y^s open $\pi^{-1}(Y^s) = X^s$

Y^s orbit space also variety. $\Rightarrow$ <u>Geo. quotient</u>

Rmk 2: when X is projective variety

Simple version of quotient: Choose: $L = H$ hyperplane bundle

$G \curvearrowright X^{\subset \mathbb{P}^n}$ can be induced from $G \curvearrowright \mathbb{A}^{n+1}$ (linear action)

linearization:

$f \in P(X, L^{\otimes r})^G$. inv. homogeneous poly of degree r.

$X^{ss} = \{ x \in X \mid \exists f$ hom. ply. $\deg \geq 1,\ f(x) \neq 0 \}$

$X^s = \{ x \in X \mid x$ semi stable, $\dim O(x) = \dim G.\ G \curvearrowright X_f$ closed $\}$

$X//G = \bigcup (X_f/G) \qquad X_f/G := \operatorname{Spec}(\mathcal{O}(X_f)^G)$

$X//G$ projective $\qquad X//G = \operatorname{Proj}\left(\bigoplus_{r=0}^{\infty} P(X, L^{\otimes G})\right)$

Examples to compute quotient: $G = SL(n)$ reductive

Ex 1: $SL(n) \curvearrowright \mathbb{P}^{n-1}$ induced from: $SL(n) \curvearrowright k^n$ by mult. of matrices.

Invariant polys are constant $\Rightarrow X^{ss} = \phi$

Ex 2. $SL(n) \curvearrowright \{$ Quadric hypersurfaces in $\mathbb{P}^{n-1} \}$

Quadric hypersurfaces defined by the polynomial $\{ \sum_{1 \leq i \leq j \leq n} a_{ij} x_i x_j \}$

$\therefore \{$ Quadric hypersurfaces $\} \leftrightarrow \{ \sum_{1 \leq i \leq j \leq n} a_{ij} x_i x_j \} \leftrightarrow \{ (a_{ij})_{n \times n} \mid$ symmetric matrix $\}$

$\updownarrow$

Let S = Vector space of symmetric matrix $\qquad \mathbb{P}(S)$

$SL(n) \curvearrowright \mathbb{P}(S)$ given by:
$$SL(n) \times S \to S$$
$$g \quad A \mapsto g^T A g$$

$$SL(n) \times k[a_{ij}] \to k[a_{ij}]$$
$$g^{-1} \quad f \quad \mapsto \quad g^{-1} . f(A) = f(g^T A g)$$

Clearly: $\Delta = \det A$ is an invariant for this action.

$$(\mathbb{P}(S))^{ss} = \{A \mid \Delta(A) \neq 0\} = \{A \mid \underset{A\text{ symmetric}}{\det A \neq 0}\} = \mathbb{P}(S)_\Delta \leftarrow \text{Semi-stable pts}$$

$$\mathbb{P}(S)/\!/G = \mathbb{P}(S)_\Delta / G = \operatorname{Spec}\big(\underset{\substack{\uparrow \\ \text{homogenous localization}}}{(K[\Delta])_{(\Delta)}}\big) = \operatorname{Spec}\left(\frac{k\Delta}{\Delta}\right) = \operatorname{Spec} k = * \leftarrow \text{one point.}$$

Note: $SL(n) \curvearrowright (\mathbb{P}(S))^{ss}$ has just one orbit

$\qquad\qquad\quad \| $

$\qquad\qquad \mathbb{P}(S)_\Delta$

$\therefore \dim(\mathbb{P}(S)^{ss}) \neq \dim SL(n) \Rightarrow$ No stable pt.

$\therefore\ X^s = \phi.$

<u>Ex3</u>:

$$SL(2) \curvearrowright \{f = a_0x_0^3 + a_1x_0^2x_1 + a_2x_0x_1^2 + a_3x_1^3\}$$

$$\updownarrow$$

$$\mathbb{P}^3$$

Action:

$$\underset{\mathbb{P}^3}{SL(2) \times \{f\}} \longrightarrow \underset{\mathbb{P}^3}{\{f\}}$$

$$g \qquad f(x_0,x_1) \longmapsto f(g^{-1}(x_0,x_1)) \qquad \forall f,\ \forall g \in SL(2)$$

$$g = \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}^{-1}, \qquad f(g^{-1}(x_0,x_1)) = f(\alpha x_0 + \beta x_1, \gamma x_0 + \delta x_1).$$

$$SL(2) \times k[a_0,a_1,a_2,a_3] \longrightarrow k[a_0,a_1,a_2,a_3]$$

$$g \qquad h(a_0,a_1,a_2,a_3) \longmapsto h(g^{-1}(a_0,a_1,a_2,a_3))$$

$$h(f(x_0,x_1)) \qquad\qquad h(f(g(x_0,x_1)))$$

Prop: $k[a_0,a_1,a_2,a_3]^{SL(2)} = k[\Delta]$ $\qquad \Delta = 27a_0^2a_3^2 - a_1^2a_2^2 - 18a_0a_1a_2a_3 + 4a_0a_2^3 + 4a_1^3a_3$

$$(\mathbb{P}^3)^{ss} = \{f(x_0,x_1) \mid \Delta(f) \neq 0\}$$

$$\mathbb{P}^3/\!/SL(2) = \operatorname{Spec}(k[\Delta]_{(\Delta)}) = \operatorname{Spec} k = * \leftarrow \text{one pt.}$$

Note: $\{\mathbb{P}^3\} = \{f\} \overset{\text{Natural}}{\underset{\text{bij}}{\longleftrightarrow}} \{3 \text{ pts with multi. in } \mathbb{P}^1\}$

$f \longleftrightarrow f(x) = 0$

$\Delta \neq 0 \longleftrightarrow 3$ distinct pts in $\mathbb{P}^1$

$SL(2) \curvearrowright \mathbb{P}^3_\Delta$ has just one orbit. $\left(\because\ \forall\, \underset{\text{distinct}}{x_1, x_2, x_3},\ \underset{\text{distinct}}{y_1, y_2, y_3} \in \mathbb{P}^1,\ \exists!\, g \in PGL(2) \text{ s.that } g x_i = y_i;\ i = 1,2,3\right)$

The action of $SL(2)$ corresponds to the action of $SL(2) \curvearrowright \mathbb{P}^1$ natural

$\dim \mathbb{P}^3_\Delta = \dim SL(2) = 3$

$\therefore (\mathbb{P}^3)^s = \mathbb{P}^3_\Delta \leftarrow$ stable pt.

Computation of invariants is difficult, establish a <u>criterion</u> which avoids computing invariants.

Def: A 1-parameter subgp (1-PS) of G is a non-trivial homo. $\lambda : \mathbb{C}^* \longrightarrow G$

$G \curvearrowright X \subset \mathbb{P}^n$ proj. variety induced by $G \curvearrowright \mathbb{C}^{n+1}$.

$\forall \lambda : \mathbb{C}^* \to G$ we have: $\mathbb{C}^* \times \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^{n+1}$

$t \quad \underline{x} \mapsto \lambda(t)\underline{x}$

Any such action can be diagonalised, ie: $\begin{pmatrix} t^{r_0} & & \\ & t^{r_1} & \\ & & \ddots \; t^{r_n} \end{pmatrix}$

$\exists$ basis $e_0, e_1, \dots e_n$ of $\mathbb{C}^{n+1}$ s.that: $\lambda(t)e_i = t^{r_i} e_i$

$\forall x \in X$, $\hat{x} \in \mathbb{C}^{n+1}$ $\quad \hat{x} = \sum_{i=0}^{n} \hat{x}_i e_i$

$\lambda(t)\hat{x} = \sum t^{r_i} \hat{x}_i e_i$

Define: $\mu(x, \lambda) = \max\{ -r_i \mid \hat{x}_i \neq 0 \}$

Thm: (Hilbert Criterion).

Let G be a reductive gp acting linearly on a proj. variety X in $\mathbb{P}^n$.

Then: x semi-stable $\iff$ $\mu(x, \lambda) \geq 0$, $\forall$ 1-PS λ of G

x stable $\iff$ $\mu(x, \lambda) > 0$, $\forall$ 1-PS λ of G.

Use criterion:

Ex: $SL(2) \curvearrowright \{$binary form of degree $n\}$

$\{ f = \sum_{i=0}^{n} a_i x_0^{i} x_1^{n-i} \} \longleftrightarrow \mathbb{P}^n$

action: $g.f(x_0, x_1) = f(g^{-1}(x_0, x_1))$

Note: $\mu(gx, \lambda) = \mu(x, \underline{g^{-1}\lambda g})$ replace λ by a conjugate in making calculations

Every 1-PS is conjugate to one of the form

$\forall r > 0$, $\lambda_r(t) = \begin{pmatrix} t^r & 0 \\ 0 & t^{-r} \end{pmatrix}$

Using Criterion: f non-stable $\iff$ $\exists r$, $\mu(f, \lambda_r) \leq 0$.

$$\lambda_r(t)\left(\sum_i a_i X_0^{n-i}X_1^i\right) = \sum_i a_i \left((t^r X_0)^{n-i}(t^{-r}X_1)^i\right) = \sum_i t^{r(2i-n)} a_i X_0^{n-i} X_1^i$$

$\mu(f, \lambda_r) = r(n-2i_0)$, i_0 is the smallest value of i, for which $a_i \neq 0$

$\mu \leq 0 \iff i_0 \geq \frac{n}{2} \iff \forall\, i < \frac{n}{2},\ a_i = 0 \iff$ The pt $(1,0)$ occurs as a pt of multiplicity $\geq \frac{n}{2}$ for the given form

$\mu < 0 \iff i_0 > \frac{n}{2} \iff \forall\, i \leq \frac{n}{2},\ a_i = 0 \iff$ The pt $(1,0)$ occures as a pt of multiplicity $> \frac{n}{2}$ for the given form

(A pt (a,b) multiplicity s, means:

$f(a,b) = 0$

$\left.\frac{\partial^{s-1} f}{\partial x_i^{s-1}}\right|_{(a,b)} = 0 \qquad \frac{\partial^s f}{\partial x_i^s}(a,b) \neq 0.$

simple root: Multi 1, mean: (a,b) is root of $f(z_0, z_1)$

Multi 2, mean: (a,b) root of $f(z_0 z_1)$

$\left.\frac{\partial f}{\partial z_i}\right|_{(a,b)} = 0$)

For: $f(x,y) = \sum_{i=0}^{d} \xi_i \binom{d}{i} x^{d-i} y^i$

Define: discriminant: $\Delta(\xi) := \begin{vmatrix} \xi_0 & (d-1)\xi_1 & \cdots & \xi_{d-1} & & \\ & \xi_0 & \cdots & & \xi_{d-1} & \\ & & \ddots & & & \ddots \\ & & & \xi_0 & (d-1)\xi_1 \cdots & \xi_{d-1} \\ \xi_1 & (d-1)\xi_2 & \cdots & \xi_d & & \\ & \xi_1 & (d-1)\xi_2 & \cdots & \xi_d & \\ & & \ddots & & & \ddots \\ & & & \xi_1 & (d-1)\xi_2 \cdots & \xi_d \end{vmatrix}$

Coeff. of $\frac{1}{d}\frac{\partial f}{\partial x}$

coeff. of $\frac{1}{d}\frac{\partial f}{\partial y}$

So $f(z_0 z_1)$ has multiple root $\iff \Delta(z_0 z_1) = 0$

prop: A binary form of degree n is stable (semi-stable) $\iff$ NO point of $\mathbb{P}^1$ occures as a pt of multiplicity $\geq \frac{n}{2}$ $\left(> n/2\right)$ for the given form.

$n=3$ Semistable $\iff$ No pt of $\mathbb{P}^1$ has multi > 1.5

$\iff$ All roots of $f(z_0, z_1)$ simple roots

$\iff \Delta \neq 0$ $\quad \therefore\ X^{ss} = (\mathbb{P}^3)_\Delta$

stable $\iff$

$n=4$. semistable $\iff$ No pts of $\mathbb{P}^1$ has multi > 2. stable $\iff$ No pts of $\mathbb{P}^1$ has multi ≥ 2

$n=4$ semi-stable $\Leftrightarrow$ All roots of $f(z_0, z_1)$ has multi ≤ 2

stable $\Leftrightarrow$ All roots of $f(z_0, z_1)$ has multi < 2

$$\Updownarrow$$

simple roots

$$\therefore \quad (\mathbb{P}^4)^s = \mathbb{P}^4_\Delta$$

$$SL(2) \curvearrowright (\mathbb{P}^4)^s = (\mathbb{P}^4_\Delta)$$

$$\{\mathbb{P}^4\} \xleftrightarrow[\text{by } f]{\text{Natural}} \{4 \text{ pts with multip. in } \mathbb{P}^1\}$$

$$\{f\}$$

$$f \longleftrightarrow f(z)=0$$

$$\Delta \neq 0 \longleftrightarrow 4 \text{ distinct pts in } \mathbb{P}^1$$

Action: $SL(2)^{\circlearrowleft} \longleftrightarrow SL(2)^{\circlearrowleft}$ by natural action on $\mathbb{P}^1$

$$\therefore \quad (\mathbb{P}^4)^s / SL(2) = \mathbb{C}$$

$$(\mathbb{P}^4)^{ss} = \mathbb{P}^4_\Delta \cup \left\{ \begin{array}{l} \text{two orbits:} \\ \cdot \text{ one double root} \\ \cdot \text{ two double root} \end{array} \right\}$$

$$(\mathbb{P}^4)^{ss} / SL(2) = \mathbb{P}^1(\mathbb{C}).$$