Toric Varieties

Outlines:

- Definitions and Examples
- Singularities
- Fundamental Gp, Cohomology Gp.
- Divisors

Field: $\mathbb{C}$.

Def: A quasi-projective variety X is a toric Variety if

- Normal
- $\exists$ a dense open subset $\mathbb{T} \cong$ Torus, s.that: the natural action $T \times \mathbb{T} \to \mathbb{T}$ extend to: $\mathbb{T} \times X \to X$.

Rmk: 1. Torus: $\mathbb{C}^* \times \mathbb{C}^* \times \cdots \times \mathbb{C}^*$ as algebraic group.

2. X Irreducible variety. $\because T$ irreducible $\Rightarrow X$ irreducible

Example: 1. Torus is a toric variety. After takig the S^{-1} it is still integral closed: P62

$\therefore$ Normal

2. $\mathbb{A}^n_k = \mathbb{C}^n$ is a toric variety

$$\begin{cases} (t_1 \cdots t_n) \cdot (a_1 \cdots a_n) \longrightarrow (t_1 a_1, t_2 a_2, \cdots, t_n a_n) \\ \mathbb{C}[x_1 \cdots x_n] \text{ integraly closed. Yes} \end{cases}$$

3. Affine variety: Y is normal $\Leftrightarrow$ $A(Y)$ is integrally closed.

coordinate ring

Construction of Toric variety.

Let N be a lattice ($N \simeq \mathbb{Z}^n$, $\exists n$), $M = \mathrm{Hom}(N, \mathbb{Z})$ be the dual Lattice

Def: A strongly convex rational polyhedral cone $\sigma \subset N_{\mathbb{R}} = N \underset{\mathbb{Z}}{\otimes} \mathbb{R} \cong \mathbb{R}^n$ is:

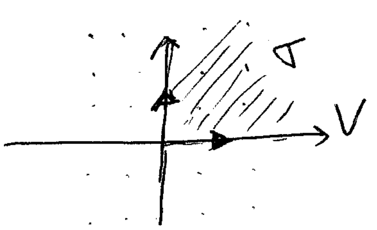
- A cone: $\forall v \in \sigma, \forall \lambda \in \mathbb{R}_{\geq 0}, \Rightarrow \lambda v \in \sigma$
- Polyhedral: σ is the intersection of finite many half spaces: (Generated by on finite many vectors)
- Rational: The half spaces are defined by equations with rational coefficients; (It is generated by vectors in the lattices)
- Strongly Convex: σ contains no linear spaces other than the origin.

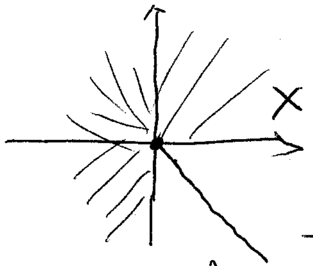
~~Def: Face τ of σ: $\tau = \sigma \cap u^{\perp} = \{ v \in \sigma : \langle u, v \rangle = 0 \}$ for some u in $\sigma^{\vee}$~~

Def: Fan in $N_{\mathbb{R}}$ is a set F of finitely many cone, s.t.:

- Every face of a cone in F is a cone in F
- The intersection of any two cones in F is a face of each cone.

Ex:





× Not a cone

but a fan

Aim: $\{\text{Fans}\} / \text{action of } SL(n,\mathbb{Z}) \xleftrightarrow{\text{bij}} \{\text{Toric varieties.}\} / \text{Isomorphsim}$

$\sigma \subset N_{\mathbb{R}}$ cone, $\sigma^{\vee} = \{ v \in M_{\mathbb{R}} \mid \langle u, v \rangle \geq 0, u \in \sigma \}$ dual cone

Taking the integral points:

$S_\sigma = \sigma^{\vee} \cap M$, S_σ f.g. semigp (Gordan's lemma)

Form the (semi) gp. algebra:

$A_\sigma = \mathbb{C}[S_\sigma]$, $\mathbb{C}$-vector spce with basis $\{ \chi^u, u \in S_\sigma \}$

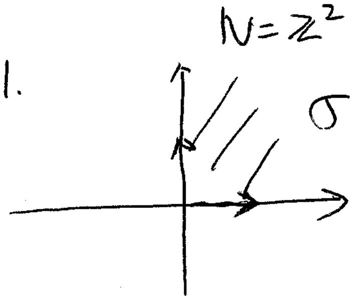
product: $\chi^u \cdot \chi^v = \chi^{u+v}$

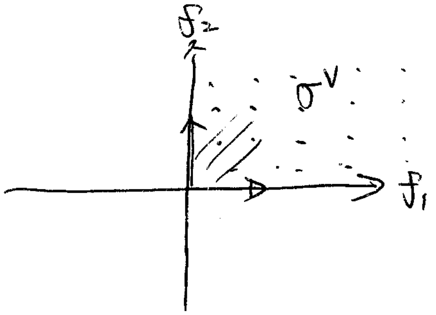
A_σ is f.g. $\mathbb{C}$-algebra.

Form the affine variety:

$U_\sigma = \operatorname{Spec} A_\sigma$.

Ex: 1.





$S_\sigma = \sigma^\vee \cap M = \mathbb{N}^2$

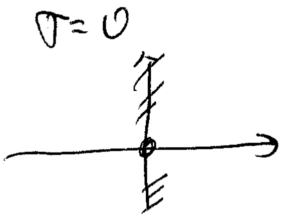
generated by: f_1 f_2

$A_\sigma = \mathbb{C}[S_\sigma]$ $\quad \underset{X}{\overset{\shortparallel}{\chi^{f_1}}}$ $\quad \underset{Y}{\overset{\shortparallel}{\chi^{f_2}}}$

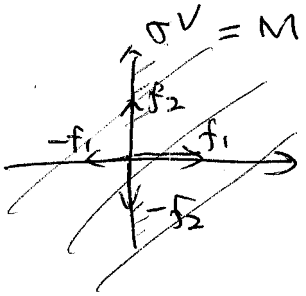
$= \mathbb{C}[X, Y]$

$U_\sigma = \mathbb{C}^2$.

2.



Taking <u>$N = \mathbb{Z}$</u>

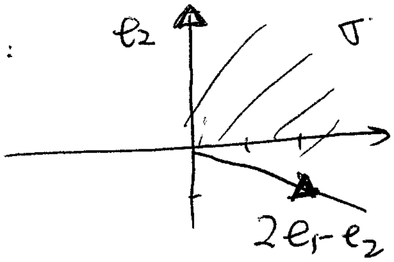


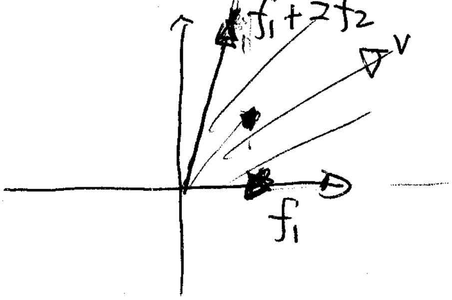
$S_\sigma = \sigma^\vee \cap M = \mathbb{Z}^2$.

$A_\sigma = \mathbb{C}[\mathbb{Z}^2] = \mathbb{C}[X, Y, X^{-1}, Y^{-1}]$

$U_\sigma = \mathrm{Spec}\, A_\sigma = \mathbb{C}^* \times \mathbb{C}^*$

3:





$S_\sigma = \sigma^\vee \cap M$ generator:

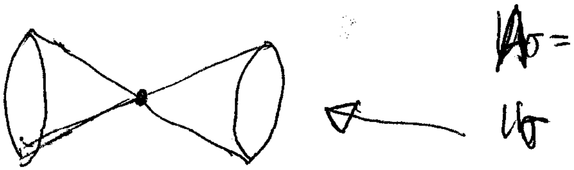
f_1, $f_1 + 2f_2$, $f_1 + f_2$.

$A_\sigma = \mathbb{C}[S_\sigma] = \mathbb{C}[X, XY, XY^2]$

$u = X \quad v = XY \quad w = XY^2$

$v^2 = uw$

$A_\sigma = \mathbb{C}[u, v, w] \big/ (v^2 - uw)$.



Now from a fan Δ, the toric variety $X(\Delta)$ is constructed by: taking the disjoint union of the affice toric varieties: U_σ, $\sigma\in\Delta$.

and gluing as follows:

- $\forall \sigma,\tau\in\Delta$, $\overset{=\delta}{\sigma\cap\tau}$ is a face of σ,τ.

lemma: $\delta\subset\sigma$ face, $U_\delta\subset U_\sigma$ is a principal open subset of U_σ.

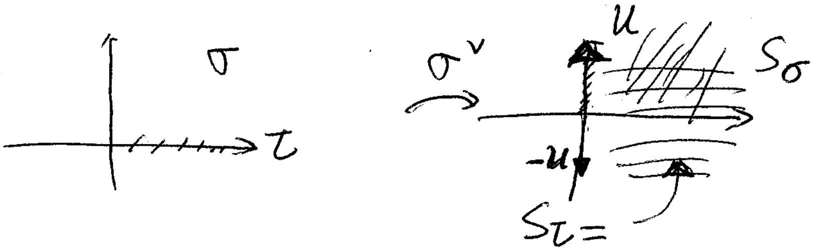
Pf: $\exists u\in S_\sigma$, s.that: $\delta=\sigma\cap u^\perp$

$\Rightarrow$ $S_\tau = S_\sigma + \mathbb{Z}^+(-u)$

$\Rightarrow$ A_τ is a localization of A_σ along χ^u.

$\Rightarrow$ U_τ is a principal open subset of U_σ.

Ex:



$A_\sigma = \mathbb{C}[x\,y]$

$A_\tau = \mathbb{C}[x, y^{\pm 1}]$

$A_\tau = \mathbb{C}[x,y]_{(y, y^2, y^3 \dots)}$

- $U_{\sigma\cap\tau} \subset \underset{U_\tau}{U_\sigma}$ principal open subset.

- Glue U_σ to U_τ by identification on the open subvarieties

[Glueing lemma]: $\varphi_{ij}: U_{\sigma_i\cap\sigma_j} \to U_{\sigma_j\cap\sigma_i}$

s.that ① $\varphi_{ji} = \varphi_{ij}^{-1}$

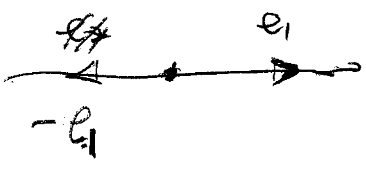
② $\forall i,j,k$ $\varphi_{ij}(U_{\sigma_i\cap\sigma_j}\cap U_{j,k}) = U_{ji}\cap U_{jk}$

$\varphi_{ik} = \varphi_{jk}\circ\varphi_{ij}$

check: Compatible: The order-preserving nature of the correspondence from cones to affine varieties.

Ex: 1. $N=\mathbb{Z}$

dual

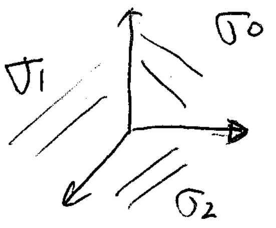


$\mathbb{C}[x^{-1}] \hookrightarrow \mathbb{C}[x,x^{-1}] \hookleftarrow \mathbb{C}[x]$

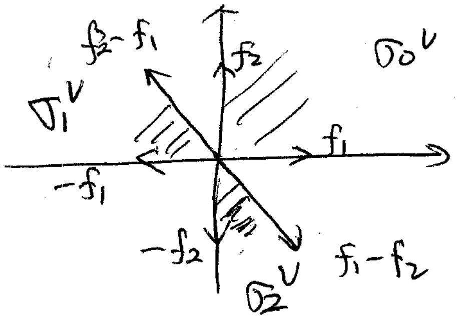
$\mathbb{C} \hookleftarrow \mathbb{C}^* \hookrightarrow \mathbb{C}$

$\mathbb{C}^* \to \mathbb{C}^*$
$x \to x^{-1}$ $\Rightarrow$ $\mathbb{P}^1$

$n=2$



dual: $\longrightarrow$



$\sigma_0^\vee$: $\mathbb{C}[X\,Y]$

$\sigma_1^\vee$: $\mathbb{C}[X^{-1}, X^{-1}Y]$

$\sigma_2^\vee$: $\mathbb{C}[XY^{-1}, Y^{-1}]$

Gluing them:

$\mathbb{P}^2$ $[T_0;T_1;T_2]$

$T_0\neq 0$: $[1; \frac{T_1}{T_0}, \frac{T_2}{T_0}]$ $\quad$ $T_1\neq 0$ $[\frac{T_0}{T_1}; 1, \frac{T_2}{T_1}]$ $\quad$ $T_2\neq 0$ $[\frac{T_0}{T_2}, \frac{T_1}{T_2}, 1]$

$[1; \overset{\shortparallel}{X}, \overset{\shortparallel}{Y}]$ $\quad$ $\overset{\shortparallel}{[X^{-1}, YX^{-1}]}$ $\quad$ $=[Y^{-1}, XY^{-1}; 1]$

$\sigma_0^\vee$ $\quad$ $\sigma_1^\vee$ $\quad$ $\sigma_2^\vee$

Torus action:

σ cone in $N_{\mathbb{R}}$. • $T_N \times U_\sigma \longrightarrow U_\sigma$

$t \quad X \longmapsto tX$

t identified with a map: $t: M \to \mathbb{C}^*$ of gp.

$X\in U_\sigma$ - - - $X: S_\sigma \to \mathbb{C}$ of semigp

$tX: S_\sigma \to \mathbb{C}$ of semigp

$u \mapsto t(u)\,x(u)$

Restriction of $\mathbb{C}[M] \to \mathbb{C}[M]\otimes\mathbb{C}[M]$

• $\mathbb{C}[S_\sigma] \longrightarrow \mathbb{C}[S_\sigma]\otimes\mathbb{C}[M]$

$\chi^u \longmapsto \chi^u\otimes\chi^u.$

$u\in S_\sigma$

$\mathrm{Hom}(\mathbb{C}[S_\sigma], \mathbb{C}] \cong \underset{\text{semigp}}{\mathrm{Hom}}(S_\sigma, \mathbb{C})$

natural bij.

$\mathbb{C}=\{0\}\cup\mathbb{C}^*$

multi. sub semigp of $\mathbb{C}$

$X(\Delta)$: These maps are compatible with inclusions of open subsets corresponding to faces of σ. $\Rightarrow$ They extend the action of T_N on itself.

$X(\Delta)$ is seprated: (Hausdorff):

Lemma:

Q: Which toric varieties are smooth?

- $\boxed{\sigma \text{ spans } N_{\mathbb{R}},\ \sigma^{\perp}=\{0\}. \quad A_\sigma \text{ coordinate ring of } U_\sigma.}$

Lemma: $\mathrm{Hom}(\mathbb{C}[S_\sigma], \mathbb{C}] \cong \underset{\text{semigp hom}}{\mathrm{Hom}(S_\sigma, \mathbb{C})}$ $\quad \mathbb{C}=\{0\}\cup\mathbb{C}^*$ with multip

$$f \longmapsto \begin{array}{l} g: S \to \mathbb{C} \\ u \to f(\chi^u) \end{array}$$

$$f(\chi^u)=g(u) \longleftarrow g$$

Now: $S_\sigma \longrightarrow \{0,1\}$

$$u \longrightarrow \begin{cases} 1 & u\in\sigma^{\perp} \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore \quad \underset{\substack{\uparrow \\ \text{maximal ideal}}}{m} \longrightarrow \mathbb{C}[S_\sigma] \underset{\text{surjective}}{\longrightarrow} \mathbb{C}$$

Corresponding closed pt x_σ of U_σ.

$$\sigma^{\perp}=\{0\} \quad \therefore \quad u \longrightarrow \begin{cases} 1 & u=0 \\ 0 & u\neq 0 \end{cases}$$

$m = $ generated by $\{\chi^u \mid u\neq 0 \ u\in S_\sigma\}$

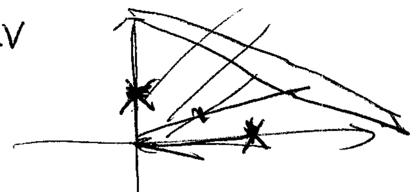
$m^2 = $ generated by $\{\chi^u \mid u=v+w \ v\neq 0 \ w\neq 0\}$

$m/m^2 = \{\chi^u \mid u\neq 0,\ u$ ~~are~~ not the sum of two such vectors$\}$

$$\Updownarrow \quad \mapsto$$

$\{$ generators of $S_\sigma\}$ $\quad \supseteq \{$minimal generators along the edges of $\sigma\}$ $= n$

$\sigma^\vee$



Note: The first elements in M lying along the edges of $\sigma^\vee \in m/m^2$

(Always the case: $\dim m/m^2 \geqq \dim U_\sigma = \dim T_N = n$)

U_σ nonsingular mean: $\dim m/m^2 = n \quad \therefore \sigma^\vee$ cannot have more than n edges ✓

? $\Rightarrow$ The minimal generators along these edges generate $S_\sigma \overset{\checkmark}{\Rightarrow}$ basis of M.

$\Rightarrow$ σ must be generated by a basis of N. $U_\sigma \cong \mathbb{C}^n$

- $\dim \sigma = k < n$

$N = N_\sigma \oplus N''$ $\sigma = \sigma' \oplus \{0\}$ σ' generate N_σ.

$\uparrow$

N_σ is lattice generated by $\sigma \cap N$

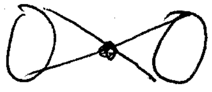
$S_\sigma = (\sigma'^\vee \cap M_\sigma) \oplus M''_c$ dim $n-k$

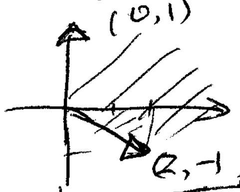
$U_\sigma \cong U_{\sigma'} \times T_{N''}$

$= U_{\sigma'} \times (\mathbb{C}^*)^{n-k}$

U_σ nonsingular $\Longleftrightarrow$ $U_{\sigma'}$ nonsingular.

Prop: U_σ nonsingular $\Longleftrightarrow$ σ is generated $\boxed{\text{by part of a basis for the Lattice}}$ σN

Ex 



$\det \begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix} \neq 1$

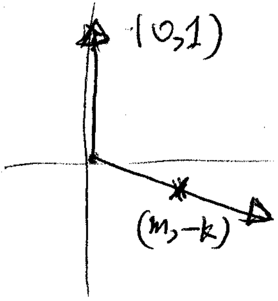
$\boxed{\begin{array}{l}\text{prop: } \Delta' \text{ is a refinement of } \Delta \\ X(\Delta') \longrightarrow X(\Delta) \text{ is birational and proper.}\end{array}}$

Idea:

Resolution of singularities:

$N = \mathbb{Z}^2$.

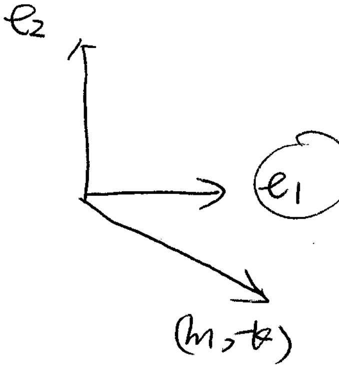
Assume σ has the form



$(m,k) = 1$

$0 < k < m$

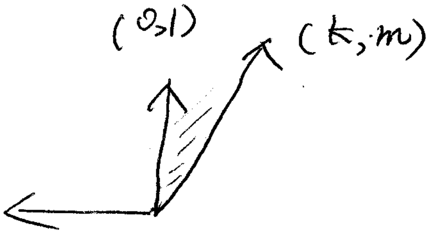
$\left(\because \text{ Change of basis: } \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} m & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} m & 0 \\ cm+x & 1 \end{pmatrix} \right)$



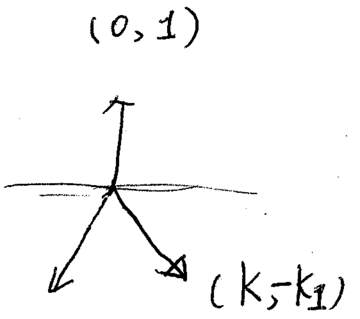
$\boxed{\text{insert } e_1}$

Rotation

rotate the picture by 90°



change of basis $\xrightarrow[\begin{pmatrix} 1 & 0 \\ -c_1 & 1 \end{pmatrix}]{}$



$$\begin{pmatrix} 1 & 0 \\ -c_1 & 1 \end{pmatrix} \begin{pmatrix} k & 0 \\ m & 1 \end{pmatrix} = \begin{pmatrix} k & 0 \\ -c_1 k + m & 1 \end{pmatrix}$$

$-k_1 = -c_1 k + m$

$m = c_1 k - k_1$ $0 \leq k_1 < k$

$$(m,-k) \longrightarrow (k,-k_1) \longrightarrow (k_1,-k_2) \longrightarrow \cdots$$

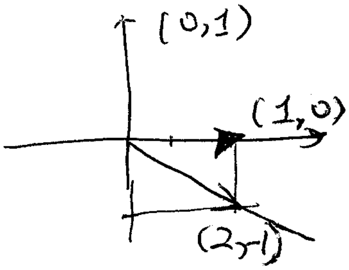
$0 \le k < m$

$\exists\, C_1 \geq 2$, $m = C_1 k - k_1$, $0 \le k_1 < k$

$\exists\, C_2 \geq 2$. $k = C_2 k_1 - k_2$, $0 \le k_2 < k_1$

Use Euclidean algorithm $\Rightarrow \exists$. $(k_n, -k_{n+1}) = (1, 0)$ st $k_{n+1} = 0$ $\square$

Ex:



Fundamental Gps:

Prop: Δ fan that contains an n-dim cone, $\Rightarrow X(\Delta)$ simply connected.

Cor: σ is k-dim cone, $\Rightarrow \pi_1(U_\sigma) \cong \mathbb{Z}^{n-k}$.

Pf: $U_\sigma = U_{\sigma'} \times (\mathbb{C}^*)^{n-k}$

$\pi_1(U_\sigma) = \pi_1(\mathbb{C}^*)^{n-k} = \mathbb{Z}^{n-k}$

$\pi_1(U_\sigma) = N/N_\sigma$ $N_\sigma =$ Lattice generated by $\sigma \cap N$

Cor: $\pi_1(X(\Delta)) = N/N'$, N' is the subgp of N generated by all $\sigma \cap N$ σ varies over Δ.

pf: $\pi_1(X(\Delta)) = \varinjlim \pi_1(U_\sigma) = \varinjlim N/N_\sigma = N/\sum N_\sigma = N/N'$.

Ex: $\mathbb{P}^n$ $\pi_1(P^n) = 0$.

Cohomology Gps:

Lemma: σ n-dim cone, U_σ is contractle.

prop: $H^i(U_\sigma, \mathbb{Z}) \approx \Lambda^i(M(\sigma))$. $M(\sigma) = \sigma^\perp \cap M$

Pf: • σ n-dim cone, U_σ contractble.

$$H: U_\sigma \times [0,1] \longrightarrow U_\sigma$$

$$x \quad t \longmapsto H(x,t): S_\sigma \to \mathbb{C}$$

$$u \longmapsto t^{\langle u, v \rangle} \cdot x(u).$$

$t>0$

check: $\begin{cases} ① & \text{Continus} \\ ② & \begin{cases} H(x,0) = x_\sigma \\ H(x,1) = \text{identity} \end{cases} \end{cases}$

- $\dim \sigma = k$

$$U_\sigma = \boxed{\cancel{U_{\sigma'} \times T_{n-k}}} = U_{\sigma'} \times (\mathbb{C}^*)^{n-k}$$

$U_{\sigma'}$ contractible $\therefore H^i(U_\sigma) = H^i((\mathbb{C}^*)^{n-k}, \mathbb{Z})$

From Top: $H^*(T^n, \mathbb{Z}) = \Lambda_{\mathbb{Z}}[\alpha_1 \cdots \alpha_n]$

? ~~$H^1(T, \mathbb{Z}) = M(\sigma), M(\sigma) = \sigma^\perp \cap M$~~

$H^i(T, \mathbb{Z}) = \Lambda^i(M(\sigma))$ $\quad \underline{\dim(\sigma^\perp) = n-k}$

$\underline{U_{i_0 \cdots i_p} = U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_p}}$

Cohomology of $X(\Delta)$: $\boxed{H^i(X(\Delta), \mathbb{Z}) = ?}$

$\mathcal{F}$ sheaf, $\quad \underset{\text{open}}{V \subseteq X} \quad f: V \hookrightarrow X \quad \mathcal{U} = \{U_\sigma \mid \sigma \in \Delta\}$

Def: Čech resolution

Complex of sheaves on X: $\mathcal{C}^p(\mathcal{U}, \mathcal{F}) := \prod_{i_0 < \cdots < i_p} f_*(\mathcal{F}|_{U_{i_0 \cdots i_p}})$

↖ Čech resolution of $\mathcal{F}$

$d: \mathcal{C}^p \to \mathcal{C}^{p+1}$

$$(d\alpha)_{i_0 \cdots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \cdot a_{i_0 \cdots \hat{i_k} \cdots i_{p+1}}|_{U_{i_0 \cdots i_{p+1}}}$$

$$\Gamma(X, \mathcal{C}^p(\mathcal{U}, \mathcal{F})) \doteq \boxed{C^p(\mathcal{U}, \mathcal{F}) =} \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0 \cdots i_p})$$

Row = Čech resolution of I^q

$\boxed{E_0^{p,q}}$:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \\
0 \arrow[r] & I^1 \arrow[u] \arrow[r] & I^{1,0} \arrow[u] \arrow[r] & I^{1,1} \arrow[u] \arrow[r] & I^{1,2} \arrow[u] \arrow[r] & \cdots \\
0 \arrow[r] & I^0 \arrow[u] \arrow[r] & I^{0,0} \arrow[u] \arrow[r] & I^{0,1} \arrow[u] \arrow[r] & I^{0,2} \arrow[u] \arrow[r] & \cdots \\
0 \arrow[r] & \mathcal{F} \arrow[u] \arrow[r] \arrow[d] & \mathcal{C}^0(x) \arrow[u] \arrow[r] \arrow[d] & \mathcal{C}^1(x) \arrow[u] \arrow[r] \arrow[d] & \mathcal{C}^2 \arrow[u] \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

Injective resolution of $\mathcal{F}$

Taking the global section:

Total complex: $K^q = \bigoplus_{p+l=q} I^{p,l}$

Differential: ?

$\boxed{d = d^h + d^v}$ ~~The complex has exact rows.~~ $\Rightarrow H^q(X, \mathcal{F}) = H^q(K^\cdot(X))$

$0 \to \mathcal{F} \longrightarrow K^0 \longrightarrow K^1 \longrightarrow K^3 \to \cdots$

flasque resolution of $\mathcal{F}$.

Taking the global section:

convergence to ~~Convergences to~~

$$E_1^{p,q} = \bigoplus_{i_0<\cdots<i_p} H^q(U_{i_0} \cap \cdots \cap U_{i_p}) \Longrightarrow H^{p+q}(X)$$

$$= \bigoplus_{i_0<\cdots<i_p} \Lambda^q M(\sigma_{i_0} \cap \cdots \cap \sigma_{i_p}) \overset{\cancel{\text{Converge to}}}{\Longrightarrow} H^{p+q}(X(\Delta))$$

Cor: <u>Euler Characteristic</u>: $\chi(X(\Delta))$

$\underset{k}{\overset{\shortparallel}{\dim \sigma}} < n$ $\quad \Sigma(-1)^q \cdot \text{rank}(\Lambda^q M(\tau))$

$= \Sigma(-1)^q \binom{n-k}{q} = 0$

$= (1-1)^{n-k}$

$\dim \sigma = n. \quad \Sigma(-1)^q \text{rank}(\Lambda^q(0)) = 1$ when $\boxed{q=0}$

$\therefore \chi(X(\Delta)) = \Sigma(-1)^{p+q} \text{rank} E_1^{p,q} = \#n\text{-dim cones in } \Delta$

$= \Sigma(-1)^{p+q} \sum_{i_0<\cdots<i_p} \text{rank} \Lambda^q M(\sigma_{i_0} \cap \cdots \cap \sigma_{i_p})$ $\quad$ when $p \neq 0$ $\dim(\sigma_{i_0} \cap \cdots \cap \sigma_{i_p}) < n$

$= \underline{\Sigma(-1)^q \text{rank} \Lambda^q M(\sigma_i)}$

- If all maximal cones in Δ are n-dim. $\therefore$ <u>U_{σ_i} Contractible</u>

$\therefore$

$E_1^{p,q}$:

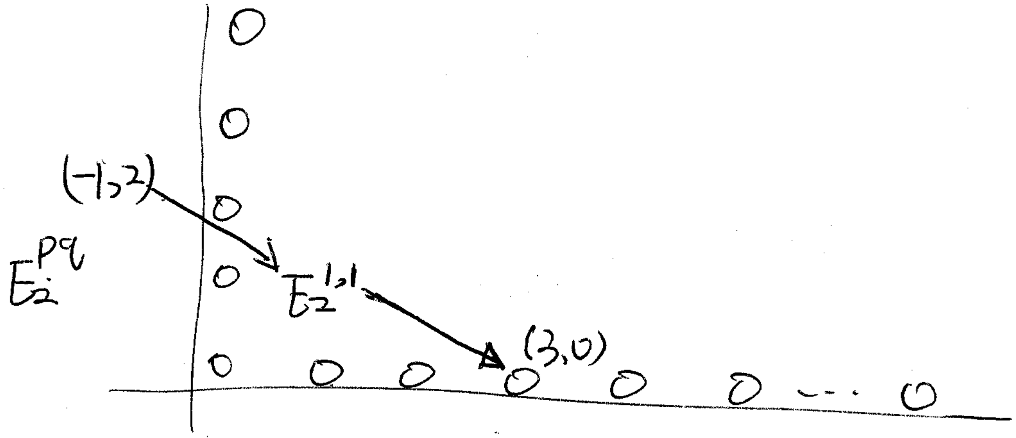
$0 \to \cdot \to \cdot \to$

$0 \to \cdot \to \cdot \to \cdot$

$0 \to \cdot \to \cdot \to \cdot$

$0 \to E_1^{1,1} \to E_1^{2,1} \to$

$\mathcal{F} = \mathbb{Z}$ $\quad 0 \to \oplus \mathbb{Z}_{\sigma_i} \to \oplus \mathbb{Z}_{\sigma_i \cap \sigma_j} \to \oplus \mathbb{Z}_{\sigma_i \cap \sigma_j \cap \sigma_k} \to \cdots \therefore$ Koszul comple



$E_2^{11} = \ker(E_1^{11} \longrightarrow E_1^{2,1})$

$= \ker(\bigoplus_{i<j} M(\sigma_i \cap \sigma_j) \longrightarrow \bigoplus_{i<j<k} M(\sigma_i \cap \sigma_j \cap \sigma_k))$

$(p,q) \xrightarrow{d_r} (p+r, q-r+1)$

$\cdots = E_3^{p,q} = E_2^{1,1}$

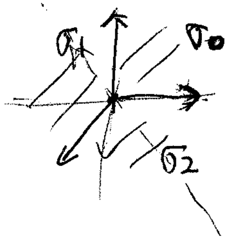
$r=2$ $(p,q) \longrightarrow (p+2, q-1)$

$(1,1) \longrightarrow (3,0)$

$(-1,2) \longrightarrow (1,1)$

$\therefore H^2(X(\Delta)) = E_\infty^{1,1} = E_2^{1,1} = \ker(E_1^{11} \longrightarrow E_1^{2,1})$

Ex: $H^2(\mathbb{P}^2) = \ker(\quad M(\sigma_0 \cap \sigma_1) \oplus \underset{\|}{M(\sigma_0 \cap \sigma_2)} \oplus M(\sigma_1 \cap \sigma_2) \longrightarrow M(\sigma_0 \cap \sigma_1 \cap \sigma_2)$



$\underset{\mathbb{Z}^3}{\langle e_1 \rangle \oplus \langle e_2 \rangle \oplus \langle e_1 - e_2 \rangle} \longrightarrow \mathbb{Z}^2$

$(n_1 e_1, n_2 e_2, n_3 (e_1 - e_2)) \longmapsto \boxed{\cancel{n_1 = n_2 + n_3 = 0}}$

$n_1 e_1 - n_2 e_2 + n_3(e_1 - e_2) = 0$

$\Updownarrow$

$n_1 + n_3 = 0$

$n_2 + n_3 = 0$

$\cong \mathbb{Z}$

$\therefore H^2(\mathbb{P}^2) \cong \mathbb{Z}$

Divisors: (X regular in codim 1. if every local ring of dim 1 is regular, i.e. m/m^2 one dim.
m unique maximal ideal of the corresponding local ring.

Weil Divisor: finite formal sum $\sum a_i V_i$, $a_i \in \mathbb{Z}$, V_i irreducible closed subvarieties of codim 1.

$\mathrm{Div}\,X$ = free abelian gp. generated by ←

Principal Divisor:

Y prime divisor on X, $\eta \in Y$ be generic point. $\Rightarrow$ $\mathcal{O}_{\eta, X}$ DVR. with quotient field K.

Valuation $v_Y : \underset{\underset{f}{\Downarrow}}{K^*} \longrightarrow \mathbb{Z}$ — f nonzero function on X.

$$(f) \doteq \underset{\underset{\text{lemma: finite sum}}{\uparrow}}{\sum} v_Y(f)\cdot Y$$

$\therefore$ (f) is a divisor, called principal divisor.

Def: $\underset{\underset{\text{Divisor class Group}}{\uparrow}}{Cl(X)} = \mathrm{Div}\,X / \sim$, $D \sim D' \Leftrightarrow D - D' = (f)$

Cartier Divisor: Open covering $\{U_i\}$ of X, $\forall i$, an element $f_i \in \Gamma(U_i, \mathcal{K}^*)$ — nonzero rational functions.

($\mathcal{K}$ sheaf: $U \xrightarrow{\text{presheaf}} \underset{\wedge\ \text{not zero divisor}}{S(U)}^{-1}\,\Gamma(U, \mathcal{O}_X)$

$\mathcal{K}^*$ sheaf of invertible elements in the sheaf of ring $\mathcal{K}$)

$$\mathcal{K}^*(U_i) = \text{invertible in: } S(U_i)^{-1}\,\Gamma(U_i, \mathcal{O}_X) = \left\{ \frac{\text{elements in } \Gamma(U_i,\mathcal{O}_X) \text{ not zero divisor}}{\text{elements in } \Gamma(U_i,\mathcal{O}_X) \text{ not zero divisor}} \right\} \leftarrow \text{Nonzero Rational functions}$$

s.t. $\forall i, j$, $f_i / f_j \in \Gamma(U_i \cap U_j, \mathcal{O}^*)$ nonzero regular functions on $U_i \cap U_j$
now have zero.

Principal Cartier Divisor: if ~~it~~ is in the image of $\Gamma(X, \mathcal{K}^*)$.

$\{\text{Cartier Divisor}\} / \sim$ = Picard Gp.

Compute $Cl(X(\Delta))$:

Prop: If Δ spans $N_{\mathbb{R}}$, then $\exists$ a short exact sequence:

$$0 \to M \to \mathbb{Z}^s \to Cl(X) \to 0$$

s = # of components of $X-T$ which are prime divisors

= # edges of the fan $\{v_i\}$

= # Irreducible subvarieties of codim 1 that are T-stable $\{D_i\}$

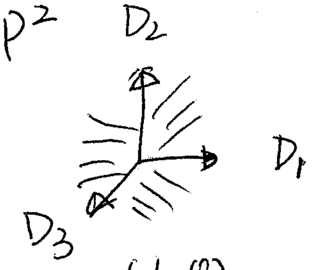
$$u \longrightarrow \boxed{\sum_i \langle u, v_i \rangle D_i} \quad v_i \text{ generators of the } \sim\!\sim \text{ of } \sigma.$$

$\| \; \langle \chi^u \rangle$ Principal $\Rightarrow 0$ in $Cl(X)$.

$(m_1 \cdots m_n) \mapsto \sum m_i D_i$

Note: The fixed subvarieties of codim 1 $\longleftrightarrow$ edges of the fan

Ex: 1: $\mathbb{P}^2$



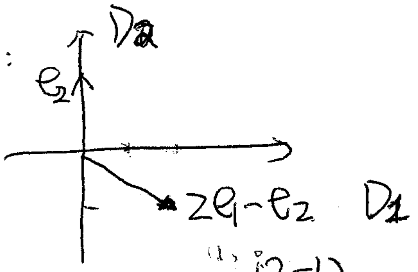
$$0 \longrightarrow \underset{\overset{\|\wr}{M}}{\mathbb{Z}^2} \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}} \underset{\overset{\|}{\langle D_1, D_2, D_3 \rangle}}{\mathbb{Z}^3} \longrightarrow Cl(\mathbb{P}^2) \longrightarrow 0$$

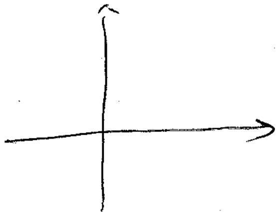
$f_1 \mapsto D_1 - D_3$

$f_2 \mapsto D_2 - D_3$

$\Rightarrow Cl(\mathbb{P}^2) \cong \mathbb{Z}$

Ex2:





$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}} \mathbb{Z}^2 \longrightarrow Cl(X) \longrightarrow 0$$

$f_1 \longrightarrow 2D_1$

$f_2 \longrightarrow D_2 - D_1$

$Cl(X) \cong \mathbb{Z}_2$

Pf of prop:

$U =$ Torus

- $\mathbb{Z}^k \longrightarrow Cl(X) \longrightarrow Cl(U) \longrightarrow 0$, $k = \#$ of components of Z which are prime divisors.

$(m_1 \cdots m_k) \to \sum m_i Z_i$

$$Z = \underset{\uparrow}{Z'} \cup \bigcup_{i=1}^{k} Z_i$$

has codim at least 2.

$\because Cl(\mathbb{A}^n_k) = 0 \Rightarrow Cl(U) = 0$

invariant divisors

$0 \to K \to \mathbb{Z}^k \to Cl(X) \to 0$ Find kernel $K = ?$

$K = \{$ principal divisors which are supported on the invariant divisors $\}$

f rational function, (f) supported on the invariant divisor, no zeroes, no poles on the torus.

$\Rightarrow f = \lambda \chi^u$, $u \in M$

$\Rightarrow$ $M \to \mathbb{Z}^s \to Cl(X) \to 0$ exact.

lemma: $u \to \sum_i \langle u_i, v_i \rangle D_i$

If fan Δ spans $M_\mathbb{R}$ $M \to \mathbb{Z}^s$ injective $\Rightarrow$ Prop. follows. ☐

Compute $Pic(X)$:

prop: If Δ spans $N_\mathbb{R}$. Then $\exists$ a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & Div_T X \arrow[r] \arrow[d, hook] & Pic(X) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & M \arrow[r] & \bigoplus_{i=1}^{d} \mathbb{Z} \cdot D_i \arrow[r] & Cl(X) \arrow[r] & 0
\end{tikzcd}$$